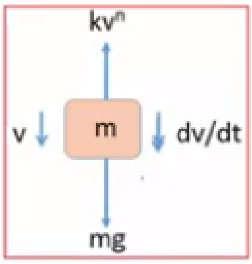
Differential Equations and Mechanics

1st Order D.E. => Terminal Velocity



For a person jumping out of an aeroplane:

Mass = **m** | Acceleration = **mdv/dt**

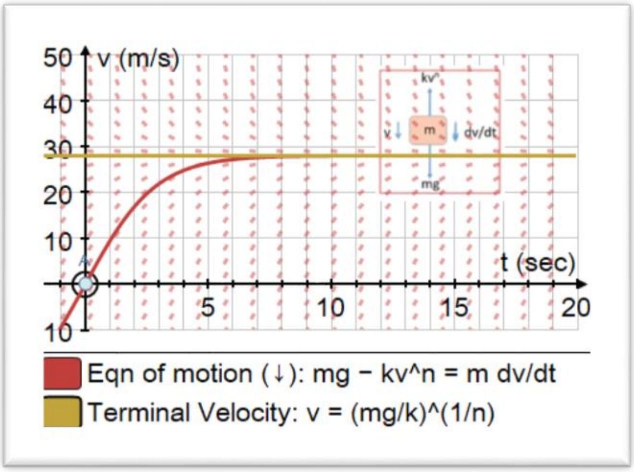
Weight = **mg** | Resistance model: **kv^n**

Equation of motion (↓): **$mg - kv^n = m\,dv/dt$**

In Autograph, using *Axes -> Edit Axes -> Labels*
  Set the variables to (→) **t** and (↑) **v**
  Set the labels to **t (sec)** and **v (m/s)**

Place a **selected point** at **(0,0)** to start the solution
Enter Equation: Name: Eqn of motion (↓)
  Equation: **mg – kv^n = m dv/dt**
Edit constants:
  g = 9.81 m/s² (fixed, unless you go very high!)
  k = 1 (can be varied by posture)
  m = 80 kg (typical mass of an adult)
  n = 2 (giving a squared model for air resistance)
Startup Options: select **Point** – then OK

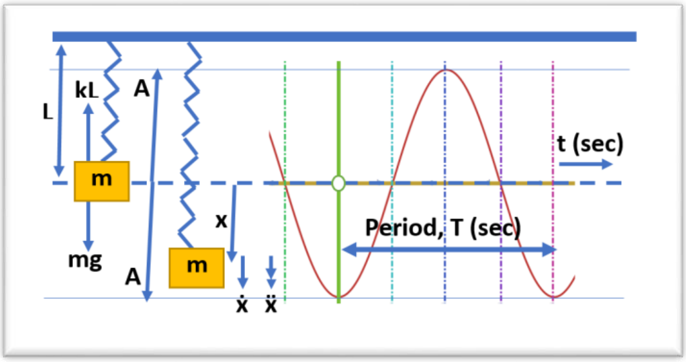
It will show the **slope field** with the default scales.
Go to *Axes -> Edit Axes:* x: -2 to 20, y: -10 to 50



The D.E. solution shows the terminal velocity clearly. It occurs as v increases and **kv^n -> mg**

Enter new equation: **v = (mg/k)^(1/n)**
You can vary 'n' and 'k' and see it all change.

2nd Order D.E. => SHM



A mass on a spring, with constant 'k', drops to its equilibrium position 'L' such that **mg = kL . . . (1)**

Pulled down a further A cm and let go. At position 'x' below equilibrium, the Equation of Motion (↓) is
=> **$mg - k(L + x) = m\ddot{x}$**, using (1) gives: **$-kx = m\ddot{x}$**
=> **$\ddot{x} = -(k/m)x$**, or **$\ddot{x} + \omega^2 x = 0$**, where **$w = \sqrt{(k/m)}$**

=> **$x = a\sin(\omega t + \phi)$** and **$\dot{x} = a\omega\cos(\omega t + \phi)$**
Initial conditions at t = 0: x = A, and $\dot{x}$ = 0 give
 $A = a\sin(\phi)$ and $0 = a\omega\cos(\phi)$ => $\phi = \pi/2$ and a = A
 $\omega = \sqrt{(k/m)} = 2\pi f$ => Period, $T = 1/f = 2\pi/\omega$

Adding in some damping

Damping can be modelled by a force that opposes the motion and is proportional to velocity, **$d\dot{x}$**
The Equation of Motion (↓) is now:
$mg - d\dot{x} - k(L + x) = m\ddot{x}$ => $-d\dot{x} - k(x) = m\ddot{x}$
=> **$\ddot{x} + (d/m)\dot{x} + (k/m)x = 0$**
=> **$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$** where **$2\lambda = d/m$** and **$\omega^2 = (k/m)$**
=> **$x = ae^{-\lambda t}\sin(\omega t + \phi)$**

To explore all this in Autograph:

Set axes to x against t
SHM: Enter: **x" + w²x = 0** (converts to **$\ddot{x} + w^2x = 0$**)
  Click on (0,3) and observe SHM. Vary w.
DAMPED SHM:
  Enter: **x" + 2λx' + w²x = 0** (-> **$\ddot{x}+2\lambda\dot{x}+w^2x=0$**)
  Click on (0,3) and observe damped SHM.
  λ=0 (SHM, undamped), λ=1 (critical damping)

Used in this page, Autograph keyboard: λ, √, π
Others from Character Map: ← ↑ → ↓ , $\dot{x}$, $\ddot{x}$, ω

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