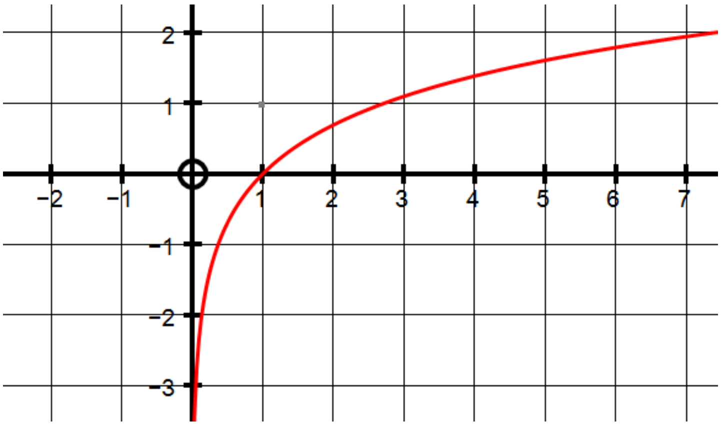
The Graph of *y* = *ln(x)* for <u>positive</u> AND <u>negative</u> values of *x*.

When we restrict ourselves to the **real numbers**, ln(– 1) does not make sense because the graph only seems to exist for $x > 0$



However, if we allow ***complex y values*** we can actually find values of ***ln(x)*** for negative ***x*** values!

We know the three series for e^x, $\sin(x)$ and $\cos(x)$:

$$\left\{\begin{array}{l} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \\[2ex] \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\[2ex] \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \end{array}\right\}$$

So let us consider $e^{i\theta}$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$= \cos(\theta) + i\sin(\theta)$$

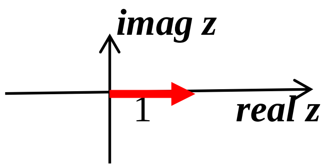
Or $cis(\theta) = e^{i\theta}$

If $z = r\,cis(\theta) = re^{i\theta}$

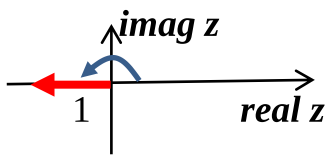
Then $\ln(z) = \ln(re^{i\theta})$
$= \ln(r) + \ln(e^{i\theta})$

So $\ln(z) = \ln(r) + i\theta$

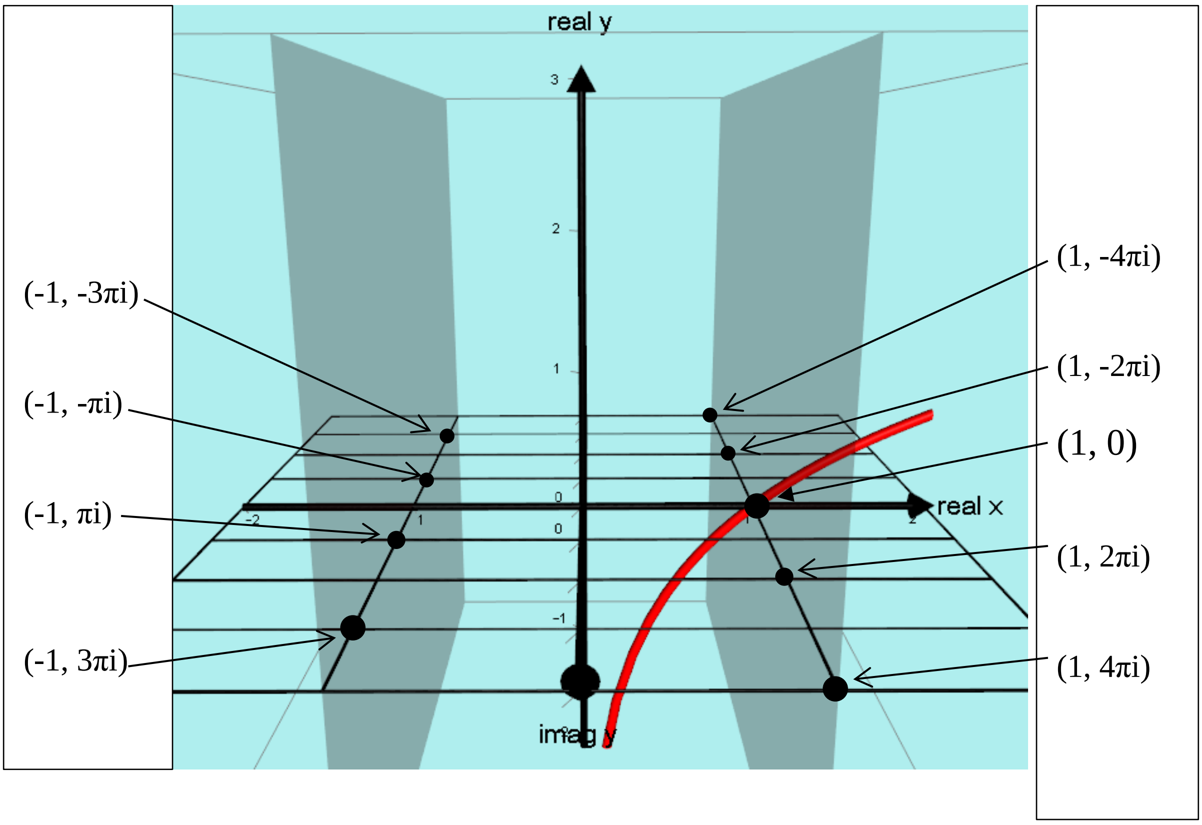
Now this is very interesting because if we let $\boldsymbol{z} = 1 + 0\boldsymbol{i}$
then $\boldsymbol{r} = 1$ and θ is not just 0 but $2n\pi$ where $\boldsymbol{n} \in$ ***Integers***
so $\ln(1) = 0 + 2n\pi i$
This means that for the graph $\boldsymbol{y = ln(x)}$
if $\boldsymbol{x} = 1$ then $\boldsymbol{y}$ could be 0 or $\pm 2\pi\boldsymbol{i}$ or $\pm 4\pi\boldsymbol{i}$ or $\pm 6\pi\boldsymbol{i}$ etc



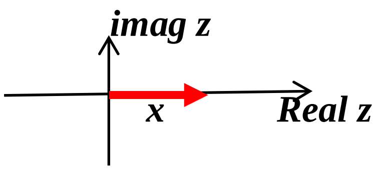
Also if we let $\boldsymbol{z} = -1 + 0i$
then $\boldsymbol{r} = +1$ and $\theta = \pi$
so $\ln(-1) = 0 + (2n + 1)\pi i$
This means that for the graph $\boldsymbol{y = ln(x)}$
if $\boldsymbol{x} = -1$ then $\boldsymbol{y}$ could be $\pm\pi\boldsymbol{i}$ or $\pm 3\pi\boldsymbol{i}$ or $\pm 5\pi\boldsymbol{i}$ etc



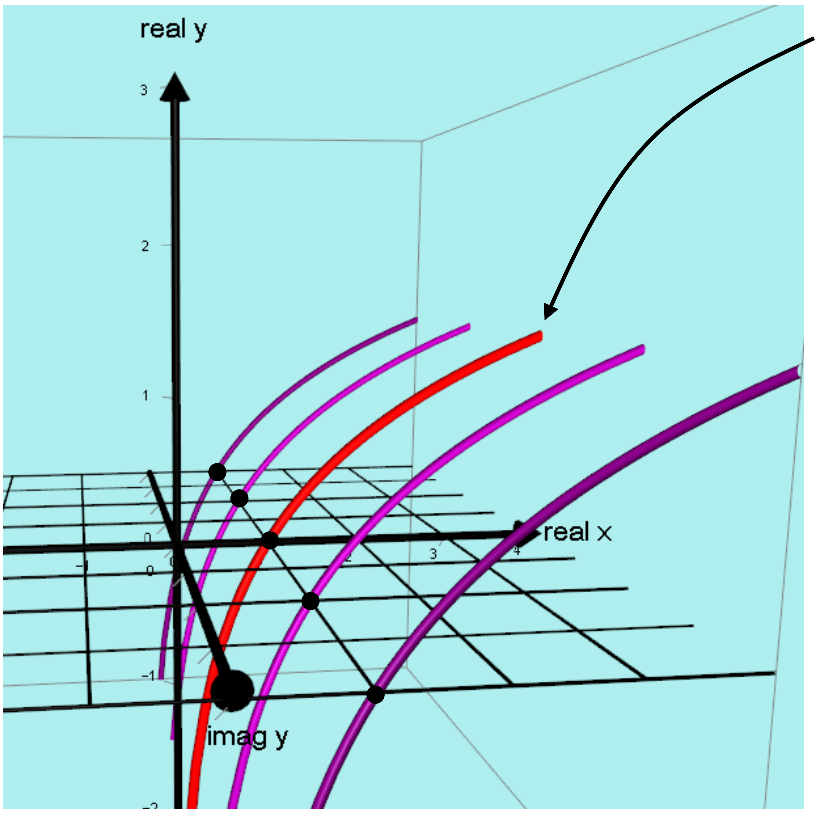
Somehow, the graph of $\boldsymbol{y = ln(x)}$ is not just the **RED** graph below but it also passes through all the marked points!



If z is just <u>any positive</u> real number z = x + 0i



then r = x and θ = 0, ±2π, ±4π etc
This means the graph of y = ln(x) really consists of countless parallel graphs spaced at distances of 2πi to each other.
The equations are y = ln(x) + 2nπi



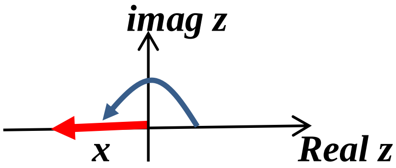
The **RED** graph is the usual
y = ln(x)
It is in the x, y plane and when x = 1 then y = 0

The spacing lines in the imaginary y direction are at intervals of 2π rads

The PURPLE graphs are all parallel to the red graph at distances of ±2π and ±4π
When x = 1 then y = ± ***2πi*** and ± ***4πi***

The equations are:
y = ln(x) ± ***2πi*** and
y = ln(x) ± ***4πi***

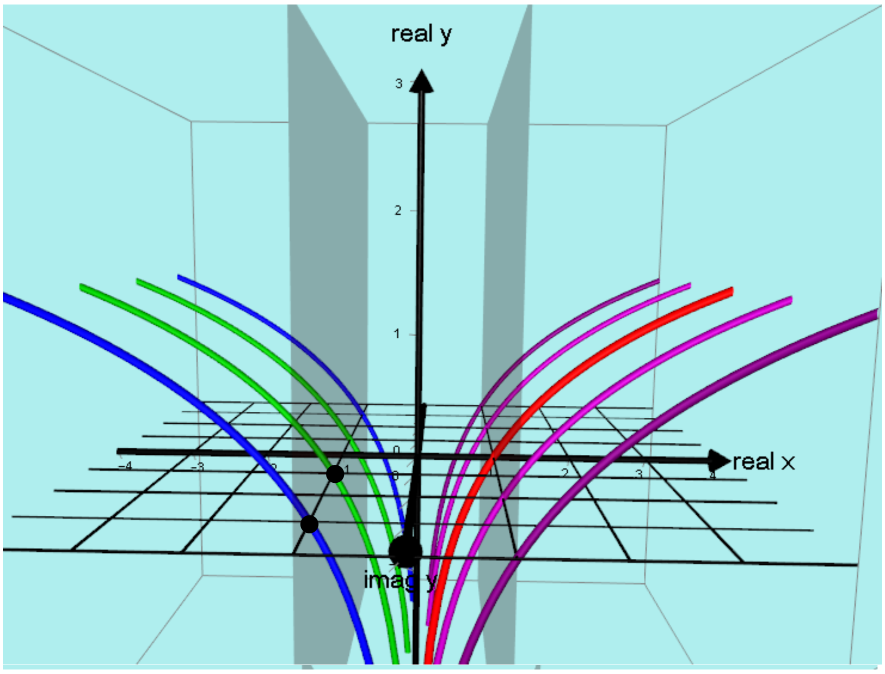
If z is a <u>negative</u> real number $z = -x + 0i$
then $r = |x|$ and $\theta = \pm\pi, \pm 3\pi, \pm 5\pi$ *etc*



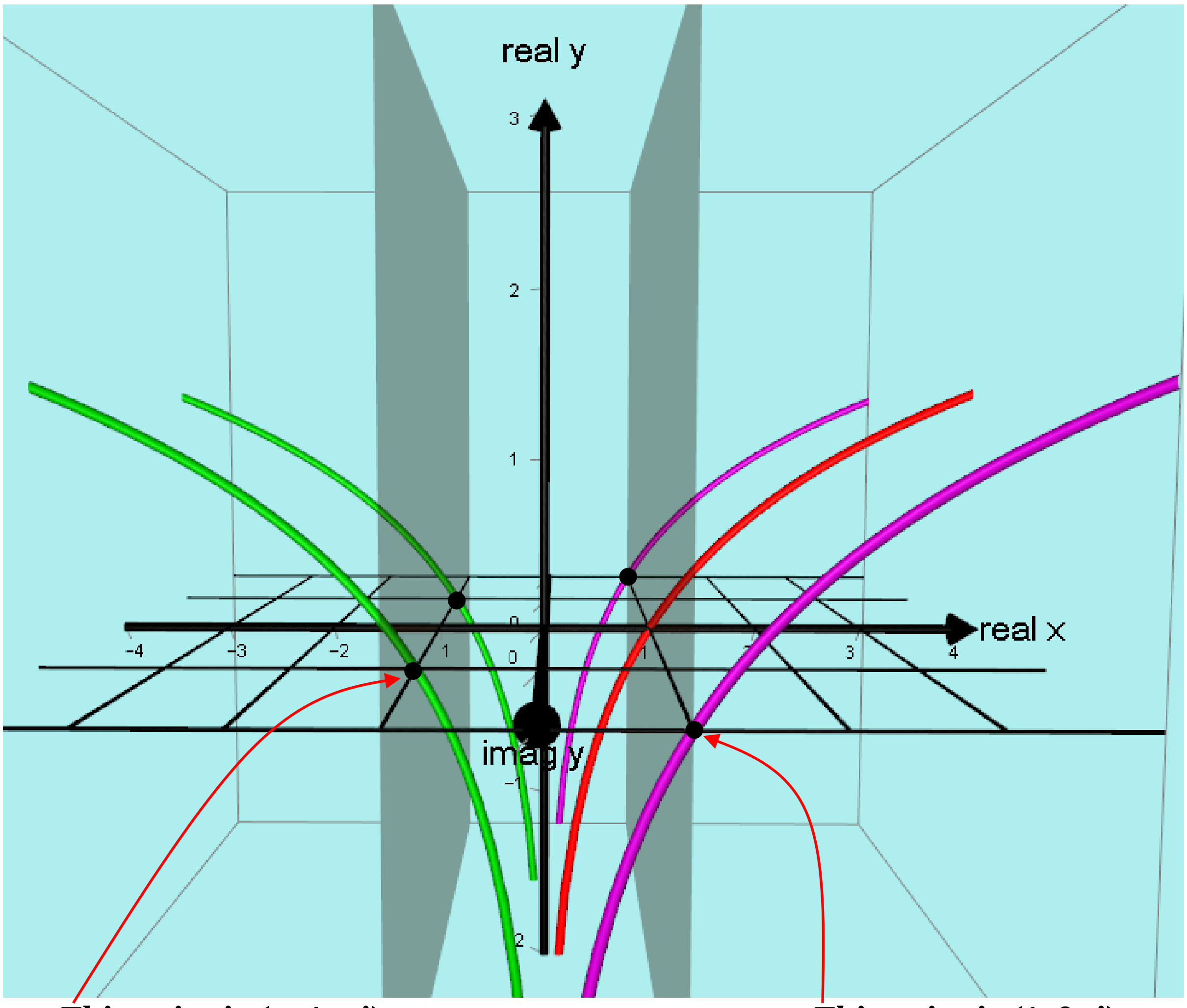
This means the graph of $y = \ln(x)$ where x is a negative real number, really consists of countless parallel graphs also spaced at distances of 2π to each other. The equations are $y = \ln|x| + (2n + 1)\pi i$

The two **GREEN** graphs are:
$y = \ln|x| \pm \pi i$
When $x = -1$
then $y = \pm \pi i$

The two **BLUE** graphs are:
$y = \ln|x| \pm 3\pi i$
When $x = -1$
then $y = \pm 3\pi i$



For better clarity, I will just include y = $\boldsymbol{ln(x) \pm 2\pi i}$ for positive $\boldsymbol{x}$ values and for negative $\boldsymbol{x}$ values I will just include y = $\boldsymbol{ln|x| \pm \pi i}$



This point is (– 1, πi)
which shows ln(– 1) = πi

This point is (1, 2πi)
which shows ln(1) = 2πi