NEW POINTS ON THE GRAPH $y^x = x^y$

Thanks to Marcelo Arruda from BRAZIL for the following ideas.

Using the basic form of De Moivre's theorem: if $z = r cis\theta = (\cos\theta + i\sin\theta)$ then: $z^n = r^n cisn\theta = r^n(\cos n\theta + i\sin n\theta)$, so if z is to be REAL then $\sin n\theta$ must be zero so $\boldsymbol{\theta = n\pi}$ (ie multiples of π rads or 180^0)
Therefore if z is to be real then $z^n = r^n(\cos n\pi + i\sin n\pi)$

Then, considering two **negative** real numbers "***a***" and "***b***", (N.B. the modulus is always positive so the **modulus** of "***a***" is "***– a***") then we can write:

$$z^n = \quad r^n \quad (\cos n\theta + i\sin n\theta)$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$a^b = (-a)^b(\cos b\pi + i\sin b\pi) = \left((-\boldsymbol{a})^{-\boldsymbol{b}}\right)^{-1}(\cos b\pi + i\sin b\pi)$$

$$b^a = (-b)^a(\cos a\pi + i\sin a\pi) = ((-\boldsymbol{b})^{-\boldsymbol{a}})^{-1}(\cos a\pi + i\sin a\pi)$$

Now we will look at the two parts of these expressions and analyse them individually:

$$a^b = ((-a)^{-b})^{-1}(\boldsymbol{\cos b\pi + i\sin b\pi})$$

$$b^a = ((-b)^{-a})^{-1}(\boldsymbol{\cos a\pi + i\sin a\pi})$$

So, if the **positive** numbers $x = -a$ and $y = -b$ satisfy $x^y = y^x$, then the **red** parts of above equations show this result and must be equal to each other.
(ie The equation: $x^y = y^x$ becomes $(-a)^{-b} = (-b)^{-a}$)

The **blue** parts will be equal to each other if:
$\cos \boldsymbol{b}\pi = \cos \boldsymbol{a}\pi$ and $\sin \boldsymbol{b}\pi = \sin \boldsymbol{a}\pi$, which means $\boldsymbol{b\pi = a\pi \pm 2k\pi}$ and therefore $\boldsymbol{b = a \pm 2k}$.(where $\boldsymbol{k}$ is any whole number)

Recall ***a*** and ***b*** are negative so multiplying that last equality by *-1* we get $\boldsymbol{-b = -a \pm 2k}$ (remember "***– b***" and "***– a***" are positive numbers!)

So, if we can find pairs of **<u>positive numbers</u>** ***x*** and ***y*** which differ by $\boldsymbol{2k}$ and which obey $\boldsymbol{x^y = y^x}$, then their **opposite negative numbers** $-x \text{ and } -y$ will also satisfy the equation $(-x)^{(-y)} = (-y)^{(-x)}$.
The simplest example of this is when ***x*** = 4 and ***y*** = 2. These numbers differ by 2 and they satisfy $4^2 = 2^4$ so this means that the **opposites** ***x*** = – 4 and ***y*** = – 2 will also satisfy the equation: $\boldsymbol{x^y = y^x}$ **because** $\boldsymbol{(-4)^{(-2)} = (-2)^{(-4)}}$

To find such numbers, we let $\boldsymbol{y = x - 2k}$
and solve $\boldsymbol{x^{x-2k} = (x - 2k)^x}$ --------------------EQU 1
for $k = 1, 2, 3$ and so on.

<u>Examples:</u>
If **k = 1**, Equ. 1 becomes $x^{x-2} = (x-2)^x$, whose solution is $x = 4$.
(Found by drawing the graphs $f(x) = x^{x-2} and\ f(x) = (x-2)^x$ using the AUTOGRAPH program and finding the intersection point.)
This leads to $x = 4\ and\ y = x - 2 = 2$. The positive solutions are **+4** and **+2** and therefore, $\mathbf{-4}$ and $\mathbf{-2}$ will also satisfy $\boldsymbol{x^y = y^x}$
(In each case, the ***x*** and ***y*** values can be swapped to produce $x = -2\ , y = -4$)
(We already knew these solutions.)

Now, let's explore some new solutions using $\boldsymbol{y = x - 2k}$:

If **k = 2** (so the ***x*** and ***y*** differ by 4) then Equ 1 becomes $x^{x-4} = (x-4)^x$
whose solution is $x = 5.6647143$ (from Autograph)
This leads to $y = x - 4 = 1.6647143$,
so $\boldsymbol{x = -5.6647143}$ and $\boldsymbol{y = -1.6647143}$ will be solutions too.
Testing: $(-5.6647143)^{-1.6647143} = 0.0275738 + 0.048443i$
$(-1.6647143)^{-5.6647143} = 0.0275738 + 0.048443i$
(Again we can say $\boldsymbol{x = -1.6647143}$ and $\boldsymbol{y = -5.6647143}$ are solutions)

If **k = 3** (so the ***x*** and ***y*** differ by 6) then Equ 1 becomes $x^{x-6} = (x-6)^x$,
whose solution is $x = 7.4941717$
This leads to $y = x - 6 = 1.4941717$,
so $\boldsymbol{x = -7.4941717}$ and $\boldsymbol{y = -1.4941717}$ will be solutions too.
Testing: $(-7.4941717)^{-1.4941717} = -0.000903 + 0.049311i$
$(-1.4941717)^{-7.4941717} = -0.000903 + 0.049311i$
(Again $\boldsymbol{x = -1.4941717}$ and $\boldsymbol{y = -7.4941717}$ are solutions too.)

If **k = 4** (so the ***x*** and ***y*** differ by 8) then Equ 1 becomes $x^{x-8} = (x-8)^x$,
whose solution is $x = 9.3944668$
This leads to $y = x - 8 = 1.3944668$,
so $\boldsymbol{x = -9.3944668}$ and $\boldsymbol{y = -1.3944668}$ will be solutions too.
Testing: $(-9.3944668)^{-1.3944668} = -0.014319 + 0.041595i$
$(-1.3944668)^{-9.3944668} = -0.014319 + 0.041595i$
(Again $\boldsymbol{x = -1.3944668}$ and $\boldsymbol{y = -9.3944668}$ are solutions too**.)**

If **k = 5** (so the ***x*** and ***y*** differ by 10) then Equ 1 becomes $x^{x-10} = (x-10)^x$,
whose solution is $x = 11.33$
This leads to $y = x - 10 = 1.33$
so $\boldsymbol{x = -11.33}$ and $\boldsymbol{y = -1.33}$ will be solutions too.
Testing: $(-11.33)^{-1.33} = -0.020 + 0.0340i$
$(\ -1.33)^{-11.33} = -0.020 + 0.0340i$
(Again $\boldsymbol{x = -1.33}$ and $\boldsymbol{y = -11.33}$ are solutions too.)

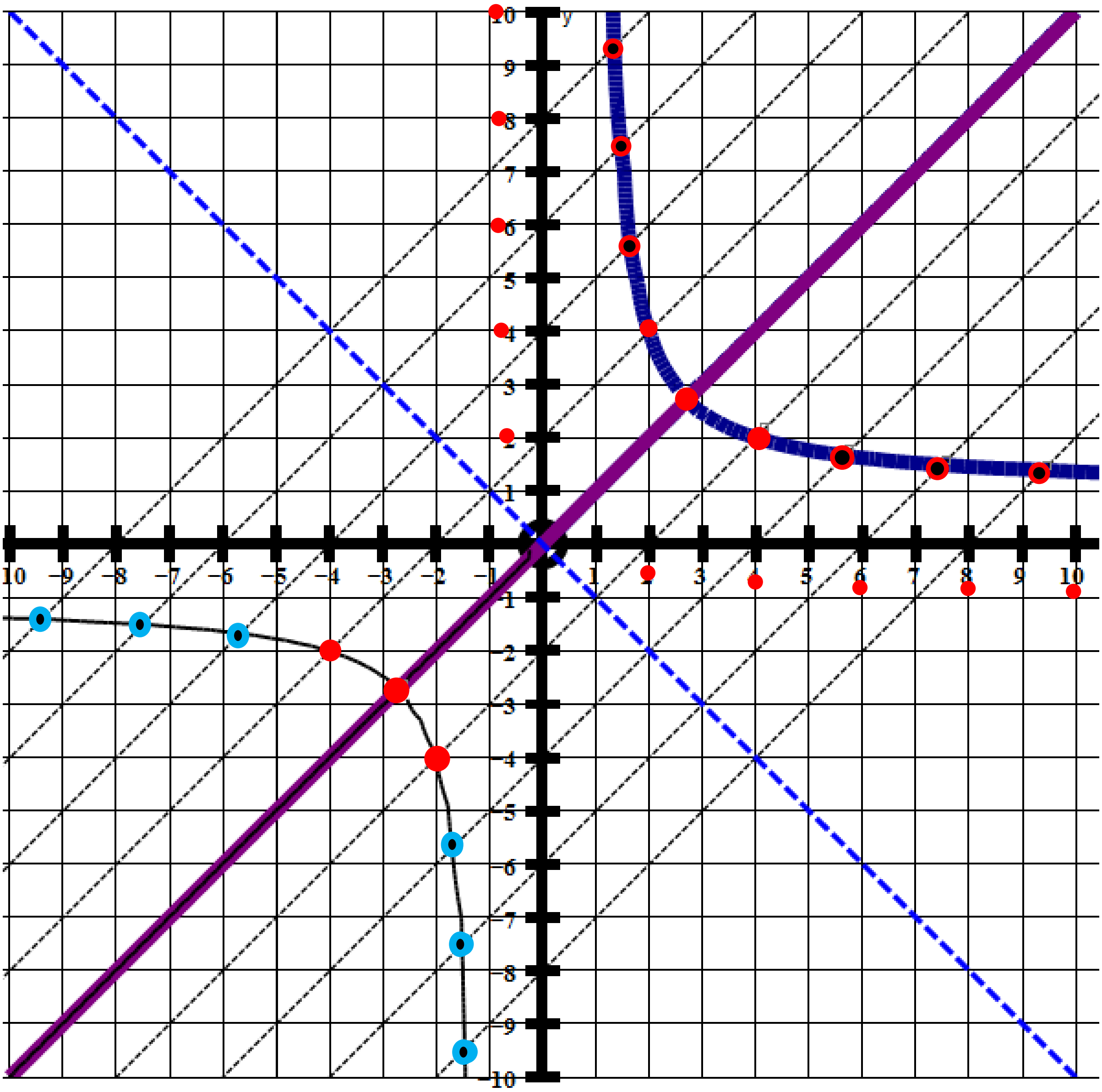
We can continue this as far as we like, but the pattern is better seen graphically.

The **intersection points** (●) of the "*hyperbola-like*" curve in the 1st quadrant, of positive $\boldsymbol{x}$ ***and*** $\boldsymbol{y}$ solutions of $x^y = y^x$, with the lines $\boldsymbol{y = x, y = x \pm 2, y = x \pm 4,}$ $\boldsymbol{y = x \pm 6}$ etc., **are reflected in the line** $\boldsymbol{y = -x}$ **so that they re-appear in the 3rd quadrant but with the negative versions of the coordinates.**

We already knew the points (–4, –2), (–2.718, –2.718) and (–2, –4).

The solutions to $x^y = y^x$ previously known are **all the points on the purple line** $\boldsymbol{y = x}$, **all the points on the blue "hyperbola-like curve"** and all the points denoted by red dots (●)

The LIGHT BLUE points (●) are the new ones.



The solutions to $\boldsymbol{x^y = y^x}$ previously known are all the points on the purple line $\boldsymbol{y = x}$, all the points on the blue “hyperbola-like curve” and all the points denoted by red dots (●)