The three real roots of a cubic can be seen where its graph crosses the *x* axis but is there a way to see where the roots are when two of them are complex?

<u>Answer:</u>

Let's just consider "cubics" and state what the Fundamental Theorem of Algebra says.

i.e. that any cubic equation of the form: $ax^3 + bx^2 + cx + d = 0$ will have three solutions.

It seems quite reasonable to expect the graph of $y = ax^3 + bx^2 + cx + d$ to cross the x axis 3 times because $y = 0$ exactly 3 times.

This does not seem to be true if 2 of the solutions are complex numbers but in fact it is still true!

We find that some <u>special complex x values</u> will still give us <u>real y values</u>.

Consider the equation $y = x^3 - x^2 + 2 = (x + 1)(x^2 - 2x + 2)$

$$= (x + 1)(x - (1 + i))(x - (1 - i))$$

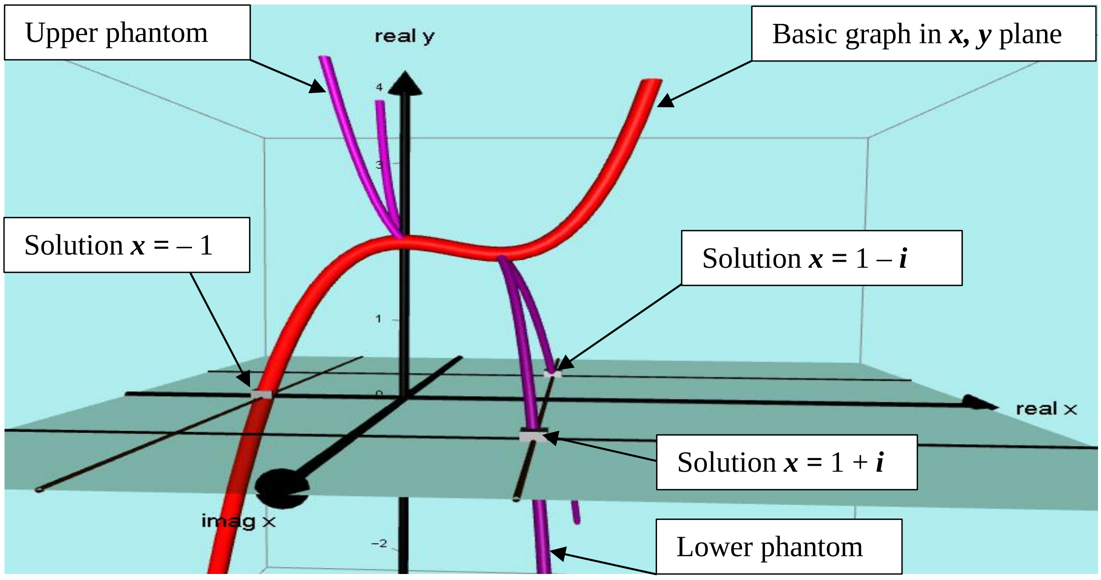
The solutions of $x^3 - x^2 + 2 = 0$ are $x = -1$, $x = 1 + i$ and $x = 1 - i$ so logically the y value of the graph is zero at these 3 places.

The inspirational idea is to create a <u>complex x plane</u> instead of an <u>x axis</u>.

Doing this produces what I call "Phantom Graphs".

These phantom graphs consist of <u>all the complex x values which still have real y values.</u>

In cases like the above equation, these phantom bits are joined to the basic graph at its turning points as you will see on the diagram below.



And we can see that the graph does in fact cross the x plane 3 times at the x values $x = -1$, $x = 1 + i$ and $x = 1 - i$ which are the solutions of the equation $x^3 - x^2 + 2 = 0$

The actual equations of the phantom graphs for the original graph, ***$y = x^3 - x^2 + 2$ can be obtained as follows:***

Recall that I said some <u>special complex x values</u> will still give us <u>real y values</u>. The first step is to allow complex x values by changing x into x + iz where z is the 3rd axis at right angles to the normal x, y plane.
The equation is now $y = (x + iz)^3 - (x + iz)^2 + 2$

Expanding and separating into real and imaginary parts, we get:

$$y = \left[(x^3 - 3xz^2) - (x^2 - z^2) + 2\right] + i\left[(3x^2z - z^3) - 2xz\right] \text{——} \textbf{\textit{Equation A}}$$

but we only want REAL y values so the imaginary part $3x^2z - z^3 - 2xz = 0$
Factorising this expression, we get : $z(3x^2 - z^2 - 2x) = 0$ this means that
<u>either</u> $z = 0$
(which means Equation A just becomes the basic equation $y = x^3 - x^2 + 2$)

<u>or</u> $z^2 = 3x^2 - 2x$
and on substituting this into Equation A we get:

$y = (x^3 - 3x(3x^2 - 2x)) - (x^2 - (3x^2 - 2x)) + 2$ together with $z = \pm(3x^2 - 2x)^{1/2}$ and these two equations give coordinates x, y and z for the two phantom graphs.

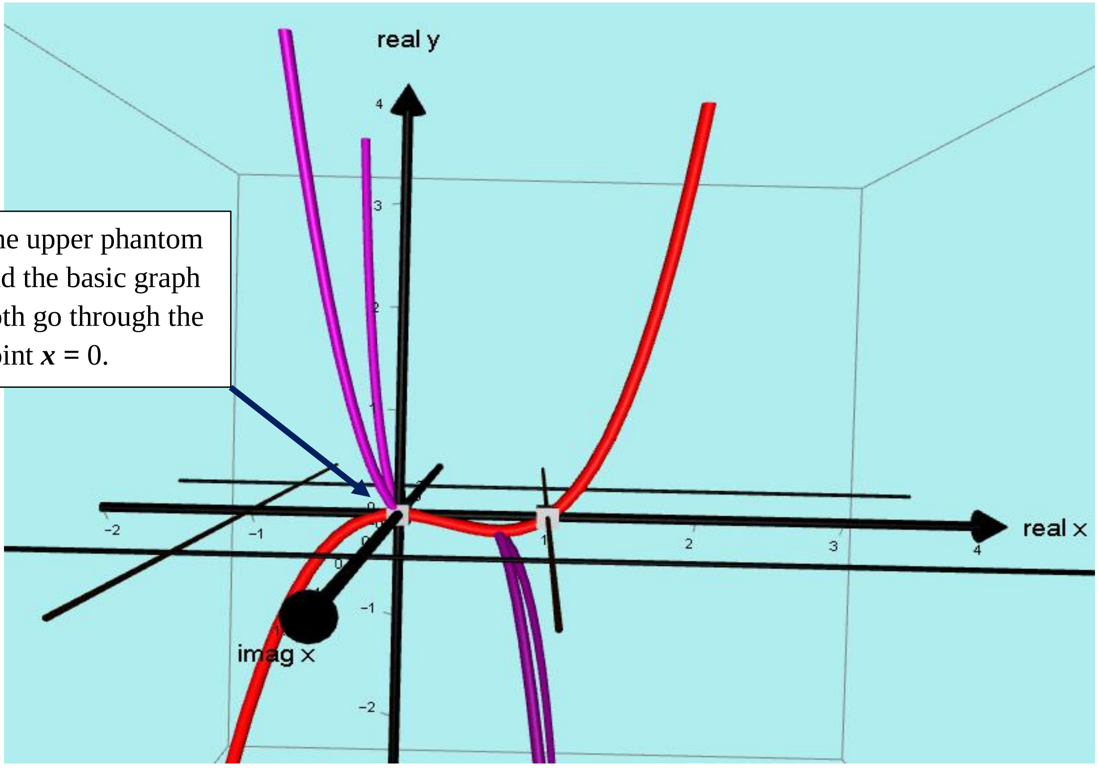
I have used the Autograph program to draw these 3D graphs and to do this the equations needed to be "parametrized" as follows:
Basic graph $y = t^3 - t^2 + 2,\ x = t,\ z = 0$
Phantoms $y = (t^3 - 3t(3t^2 - 2t)) - (t^2 - (3t^2 - 2t)) + 2,\ x = t,\ z = \pm(3t^2 - 2t)^{1/2}$

<u>Special interesting point:</u>

If we simply translate the graph down 2 units the equation becomes $y = x^3 - x^2 = x^2(x - 1)$ which only crosses the x axis at the points $x = 0$ and $x = 1$ so in this case the equation only has real solutions.

Now this <u>seems</u> to violate the fundamental theorem of algebra but the graph does in fact cross the x axis 3 times!

The graph goes once through $x = 1$ and if we consider the point where $x = 0$, the basic graph goes through this point and the upper phantom goes through this point too.



The upper phantom and the basic graph both go through the point $\boldsymbol{x} = 0$.