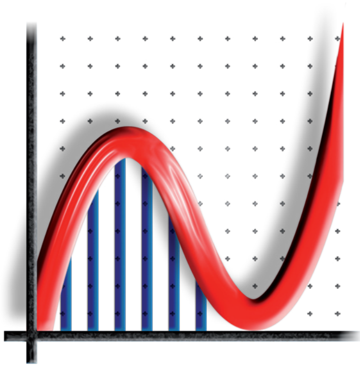
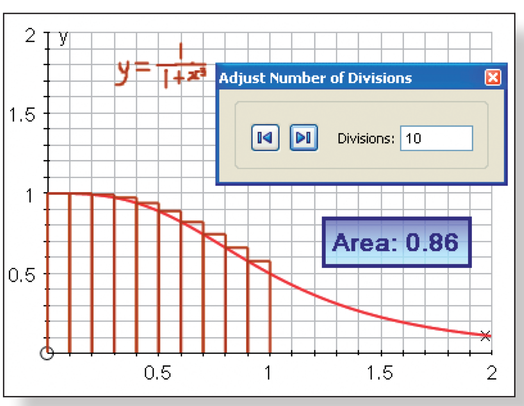
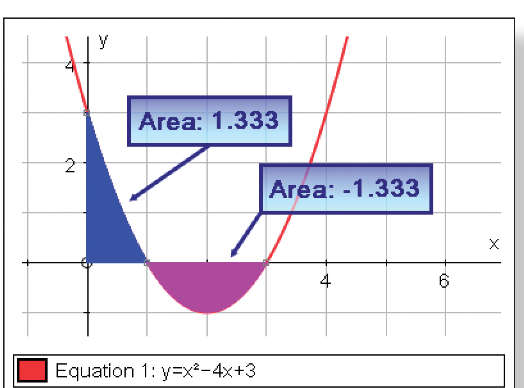
Getting Going



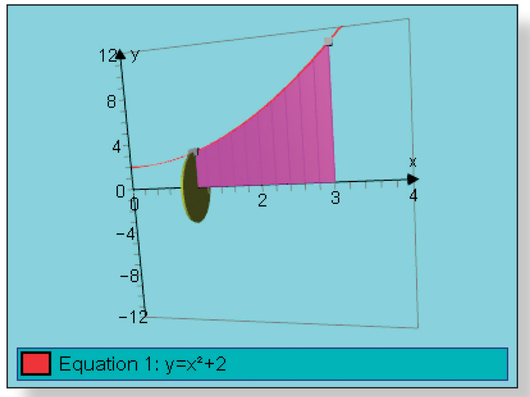
Teacher-Led Demonstrations



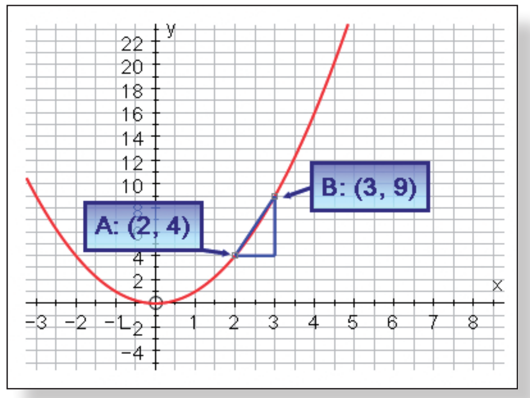
A demonstration which dynamically illustrates and compares two types of numerical integration: Rectangles and the Trapezium Rule. Students are introduced to over and under estimates, as well as issues of the concavity of function.



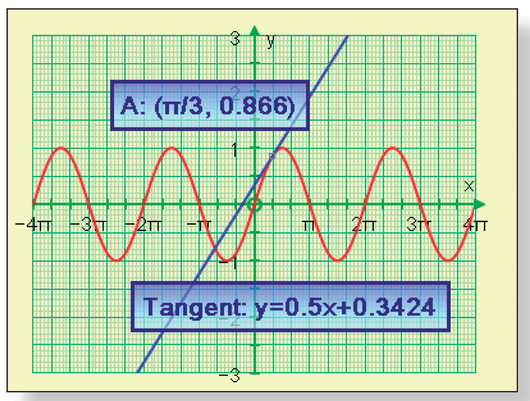
A demonstration which clearly conveys and explains the notoriously misunderstood issues surrounding integration – negative areas, improper integrals, unbounded functions and unbounded integrals.



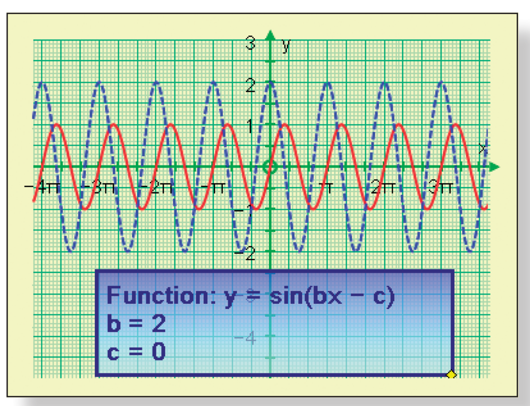
This demonstration utilises Autograph's unique 3D interface to dynamically and interactively introduce students to the concept of the volume of revolution. Students can see solid shapes forming and hence are able



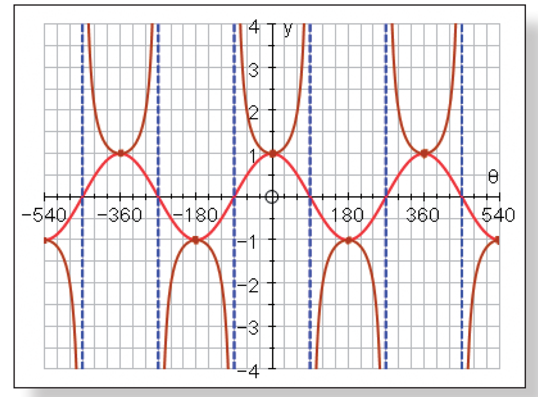
to derive the formula for calculating their volumes.



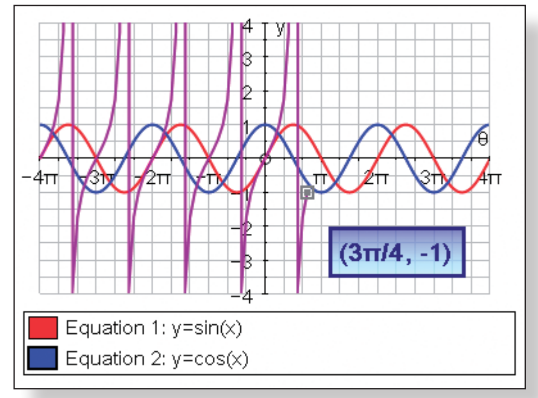
An interactive and dynamic demonstration that allows students to visualise exactly where the concept of differentiation is derived from. This demonstration lays the foundations for differentiation from first princi-



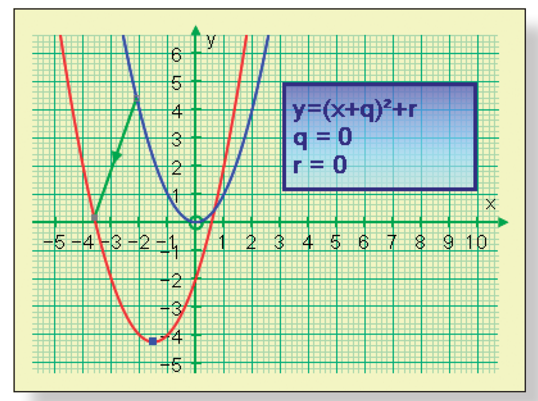
ples.



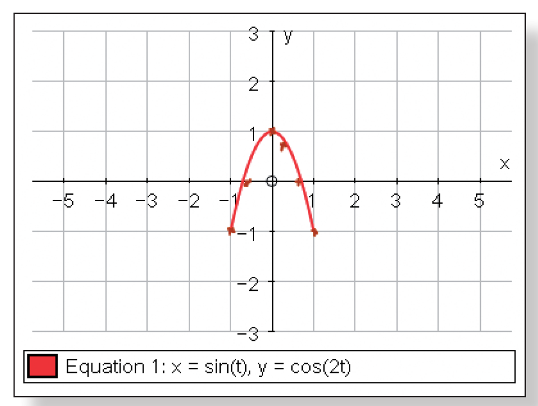
This demonstration allows a class to interactively discover the gradient functions of y = sin(x), y = cos(x) and y = tan(x). There is plenty of opportunity for class participation!



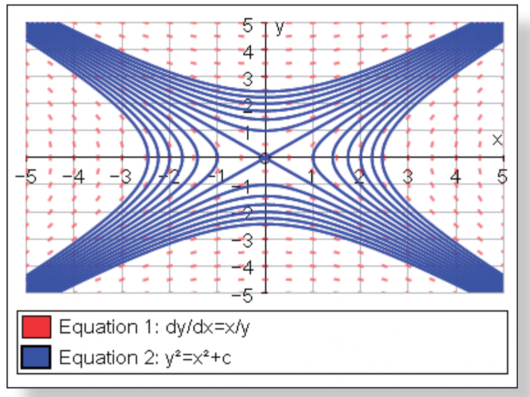
This demonstration dynamically and intuitively allows students to discover the Chain Rule making use of Autograph's constant controller and building upon the prior work on trigonometric functions.



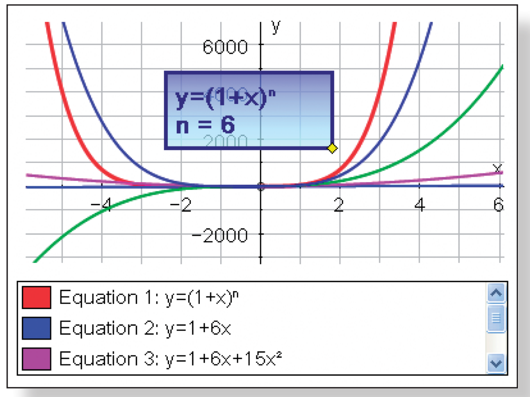
This demonstration allows students to interactively and dynamically build up a picture of the shapes of the graphs y = sec(x), y = cosec(x) and y = cot(x) from the shapes of the graphs y = cos(x), y = sin(x) and y = tan(x).

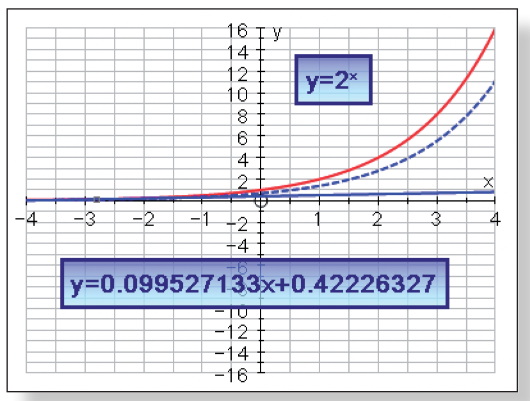


This demonstration allows students to visualise common trigonometric identities, such as tan(x) = sin(x)/cos(x) and sin2(x) + cos2(x) = 1, and understand exactly why they work.

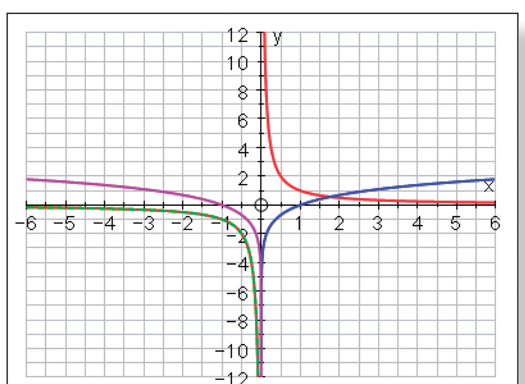


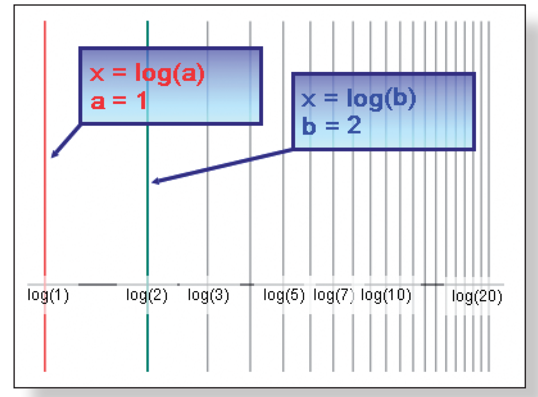
This demonstration is designed to illustrate the students the importance and usefulness of the completed square form of an equation in terms of co-ordinate geometry. There are four challenging questions included to put students to the test and provoke interesting discussion.





In this demonstration, students are introduced to the concept of parametric equations in a visual and dynamic way. They are encouraged to experiment and suggest variations which will help consolidate and deepen understanding of what can be a rather difficult topic.





In this demonstration, students are dynamically introduced to the concept of first-order differential equations using one of Autograph's most powerful functions.

1 Introduction

I remember when I was studying for my Mathematics A Levels. It was unusual enough for boards to be white, let alone interactive. Cutting edge technology was a watch with an inbuilt calculator, and a dynamic geometry lesson would be one in which the teacher suddenly switched from white chalk to red in the process of drawing a wonky circle. The late 1990s were a dark time indeed.

However, the technological revolution that has swept the world over the course of the last few decades has also landed in our classrooms. Overhead projectors have gone digital, scientific calculators are now graphical, static whiteboards have given way to their interactive cousins, computers are faster, mountains of dedicated mathematical software are widely available, and the internet has opened up possibilities far beyond what Pythagorus, Euler and the greatest minds in the history of the world could ever have imagined.

Alas, in many instances, this technology is not being used to its full potential. It either sits rotting away in storeroom cupboards or on the digital dumping ground of a computer's desktop screen, or it is used in an extremely limited and ineffective way, which can often cause frustration on the part of both the teacher and the students, thus causing more harm than good.

This sorry state of affairs could be, as many have argued, because a large proportion of teachers are reluctant to change; unwilling to try new things and break the habits and traditions that have served them perfectly well over the course of their long and distinguished careers. But I disagree. The teachers that I have talked to are ready to embrace this influx of technology, they are ready to welcome it with open arms into their classrooms, but they are just a little scared to do so.

Now, this fear stems from many different sources, and much of it is well founded. It may be fear that their pupils will know far more about each piece of technology than they ever will, thus making the teacher feel vulnerable, exposed and uncomfortable, no longer the all-knowing figure of confidence and authority that they feel they should be. Or it could be because teachers fear that by the time they have got to grips with a certain piece of technology, the world will have once more moved on, making their newly discovered knowledge both redundant and useless.

Or – and this I feel is the most prevalent reason – it is because teachers fear that they simply do not have enough time. With curriculums changing every couple of years, new strategies and initiatives bounding in from every direction, lessons to plan, books to mark, students to teach, and (in the precious few seconds of each day that remain) lives to be led, where is the time to sign up to a course, or read a book, or sit at a computer hoping the ability to use whatever technology it may be might just seep through the pixels on the screen to the pores on the skin?

And so what often happens is this: through no fault of their own, eager teachers try the new technology without having adequately planned for its use, and disaster inevitably follows. During computer-based lessons, students sneak onto other internet sites, to play games, check email, or poke friends. Whilst the teacher is happily clicking through their immensely detailed, perfectly animated PowerPoint presentation, students appear bored, lack focus and seem unmotivated. Using Autograph to draw the line $y = 3x + 4$ does not magically make the students understand straight line graphs any better than had the teacher simply drawn it by hand, and now the teacher cannot remember how to return to the interactive whiteboard's inbuilt flip-chart. The Excel spreadsheet that worked like a dream at home is now threatening to blow up the school's computer system because of the existence of a mysterious thing called a macro. The ideal lesson is suddenly scuppered by the fact that the school network is down, or YouTube videos are blocked, or some delightful little devil has stolen the batteries from the mouse. Having spent four times as long to plan than normal, the lesson is declared a failure by all, and the teacher vows to never use that stupid piece of technology again.

It's a dark place where everyone has been one time or another, and where no-one wants to return.

For a while now I have been a firm believer in what I like to call the *effective use of technology*. Simply turning on an interactive white board does not automatically make the lesson interactive, neither does clicking through a PowerPoint presentation, nor simply using a software package such as Autograph to draw a series of straight lines. The technology should be the facilitator in the lesson, not the driving force. Effective use of technology is when that piece of technology, whatever it may be, genuinely enhances the learning experience of the students and, just as importantly, improves the teacher's experience and enjoyment of the lesson.

What I hope to show both in this book, and in its sister publication *Autograph Activities: Student Investigations*, is how to use one such piece of technology, Autograph, effectively. In my experience Autograph is one of the most under-used pieces of software in schools. Many have used Autograph to do things like draw lines, curves and circles, but few seem to have fully exploited its true potential, and that is a great pity.

Once more I feel the major reason behind this is the time factor. However, what I hope to show in this book is that Autograph is incredibly simple and logical to use. Once you have spent about one hour having a go at the included interactive *Autograph Tutorial*, you will be in a position to take on any of the fifteen *Teacher Demonstrations*. Better still, it will not take long before you start using the software to develop your own activities based on your own ideas. Once again, technology is effective only when it is the facilitator of teaching and learning.

The *Teacher Demonstrations* will allow you to dynamically introduce, review, extend or illustrate important topics or concepts in ways not previously possible. They are intended for use on an Interactive Whiteboard or by means of a digital projector. There is countless opportunity for student interaction, asking and answering probing questions, testing hypotheses and predictions. The topics covered include: introducing the natural logarithm function; deriving trigonometric identities; understanding the binomial approximation; introducing the concept of differentiation; and using Autograph's unique 3D interface to examine volumes of revolution.

What I hope this book will achieve is to offer ways of effectively using a fantastic piece of software without you having to put in hours and hours of your precious time. I hope it will encourage both you and your students to use the software to plan activities and investigate concepts not covered in this book. I hope it will further strengthen your students' enjoyment of learning mathematics, exciting them, enriching their learning experience, further opening their eyes to what a wonderful and fascinating subject mathematics is. Most importantly of all, I hope the book will further help you enjoy teaching mathematics, inspiring your students, embracing technology, and using it in the way it was intended – as a facilitator of excellent teaching and learning, and an incredibly effective one at that.

Also available in this series:

Autograph Activities: Student Investigations

The *Student Investigations* provide the perfect vehicle for independent, dynamic learning. Autograph is an excellent tool for investigation, and mathematics is at its strongest and most appealing when students can embark upon such journeys of self-discovery. Students use Autograph and the accompanying worksheet to discover and examine concepts that would not be viable in the normal classroom setting. They are encouraged to derive things for themselves, to predict, to observe, to trial out hypotheses *to enrich their learning experience and deepen their levels of understanding*. Topics covered include: examining the transformation of functions; understanding vectors in two and three dimensions; and looking at numerical methods to solve equations.

The aim of this book is to help fully utilise Autograph's potential as a tool for interactive demonstration. Below are details about the Teacher Demonstrations contained in this book, including advice regarding timings, set-up and delivery.

Note: Before attempting the Teacher Demonstrations, it is strongly advised that teachers work through the *Autograph Tutorials* on page 12 of this textbook. The first interactive tutorial should help familiarise the teacher with the controls of Autograph, which will help boost confidence and allow far greater progress throughout the demonstrations. The second tutorial has been specifically designed for unlocking the full potential of Autograph on the interactive whiteboard, a graphics tablet or by means of a digital overhead projector.

The Teacher Demonstrations

These demonstrations are designed for use in the classroom, either to dynamically introduce, review, extend or illustrate an important concept or topic. They are designed to be both interactive and engaging. Usually the demonstrations will last between five and twenty minutes, although you may be tempted to spend longer on them if they spark a lively discussion.

Each activity comprises of:

Teacher Notes – these contain:

- Learning Objectives
- Required Prior Knowledge
- Details of any Pre-Activity Set-Ups that must be carried out before the lesson starts
- Step-by-Step Instructions, together with suggested questions, ideal responses, and opportunities for class interaction.
- Ideas for Further Work to build upon the knowledge gained during the demonstration.

Viewing the Demonstrations

Obviously it is crucial that all students can see the demonstration. For that reason it is recommended that you use either an Interactive Whiteboard, or a digital projector hooked up to a computer which has the Autograph software installed upon it. All the activities have opportunities for class participation and interaction, and whilst an Interactive Whiteboard naturally lends itself better to this (especially using Autograph's impressive on-screen keyboard), working with a projector will also be fine.

The Role of the Teacher

The teacher is far more prominent and visible than in the Student Investigations. It is crucially important that they have tried the activities out themselves prior to the lesson. Again, this will enable the teacher to feel more confident delivering the activity and enable them to decide how far into the activity to go depending on the needs and ability of the group.

Saving the Activities

Again, whilst it is not strictly necessary to save the activities once the demonstration is over, it might be wise to do so. These activities can be quickly called upon to illustrate a specific concept, or to answer a previously unforeseen problem, and once again they would make excellent revision aids.

Whiteboard Mode

All of the *Teacher Demonstrations* have been designed in **Whiteboard Mode**, whereas all the *Student Investigations* in the book *Autograph Activities: Student Investigations* take place in non-Whiteboard Mode. The major difference between the two is the way multiple objects are selected, with Whiteboard Mode being far more user-friendly in this regard when Autograph is used with an Interactive Whiteboard, a graphics tablet or a digital projector . You are encouraged to read the teacher notes accompanying the *Autograph Tutorial*, and then to have a go at the *Autograph Additional Teacher Tutorial* specifically designed to address this issue.

The On-Screen Keyboard

Autograph has a handy in-built facility for displaying an **on-screen keyboard**, meaning that any text, equations, or use of the Shift and Ctrl buttons can be entered with the mouse/pen on the Interactive Whiteboard or graphics tablet. Details about making the most of this valuable tool can also be found in the *Autograph Additional Teacher Tutorial*. As some teachers still prefer to use the traditional keyboard in unison with their interactive whiteboard, no specific reference has been made to the on-screen keyboard during the set of *Teacher Demonstrations*. However, whenever there is mention of using the normal keyboard, just be aware that the on-screen keyboard can be used to the exact same effect.

Teacher Notes

Overview

The aim of this tutorial is to enable both teachers and students to become familiar with how Autograph works. The tutorial gives an overview of some of Autograph's major 2D graphing features†. It is very much hands-on, challenging and interactive, allowing the user to really get to grips with how the software works and what it can do. This should increase the user's confidence with the software, hopefully allowing teachers to come up with activities of their own, and students to use the software to aid them with their studies.

Please Note: A series of 10 interactive student investigations are available in the book *Autograph Activities: Student Investigations*.

Note: There is a separate *Autograph Additional Teacher Tutorial*, specifically designed to help teachers make the most of Autograph on an **Interactive Whiteboard** or a **Graphic's Tablet**. The tutorial should only take around ten minutes to complete, and it highlights the two of Autograph's best features:

1. Whiteboard Mode
   All the Teacher Demonstrations in this book are written in Whiteboard Mode, whereas all of the Student Investigations, and this tutorial itself are written in non-Whiteboard Mode. There are a few subtle differences to make Autograph smoother and easier to operate with an interactive whiteboard or a graphics tablet. Therefore, it might be a good idea to have a quick run through this additional tutorial before embarking upon any of the Teacher Demonstrations.

2. The On Screen Keyboard
   Autograph comes complete with its own onscreen keyboard, which means you don't have to be tied to the keyboard attached to your classroom's computer. The onscreen keyboard makes Autograph fully functional using the pen of an interactive whiteboard, or frees you up to walk around the classroom using a graphics tablet. The onscreen keyboard is even more useful as it can be used with any other application, not just Autograph, and the maths symbols in the Extra panel will also work in other applications. Only a couple of characters are exclusive to Autograph font.

Timing

The tutorial is designed to last between forty-five minutes to an hour, based on someone who has never used the software before. This will of course differ depending on the user's past experience and overall computer competency. Once again, the most important elements are covered earlier on in the tutorial, so if not all students reach the end, they will still be in a strong position to tackle the rest of the activities in the book.

Location

For students, this tutorial should ideally take place in a computer suite with one student allocated to each workstation. Paired work is also acceptable, but it is important that each student has the same hands-on experience. If your school has access to Autograph at home, either because your school has an Extended Site Licence, or your students have individual Student Licences, then this tutorial might make a nice homework assignment.

Required Pre Knowledge

Whilst the tutorial touches on concepts such as differentiation, integration and radians, specific knowledge of these is not required. This tutorial is designed for a student about to embark upon a post-16 maths course, and hopefully the little taster they will get of these concepts will whet their appetite for what is to come later.

Pre-Activity Set-Up

None required.

† As this textbook is aimed at Post-16 students, I have avoided including elements such as the transformations of shapes and basic data handling work to the tutorial. In addition, the 3D work is covered in the activities themselves, and hence is also omitted from the tutorial. If these are areas which you are interested in learning more about, or if you want to review any of the features covered in this tutorial, then I recommend accessing the Autograph Video Tutorials contained within the Help menu of Autograph, or visiting the Autograph Website (www.autograph-math.com).

STUDENT WORKSHEET NAME:...

ACTIVITY 1

Autograph Buttons covered:

- Enter equations
- Edit axes
- Zooming in and out
- Dragging
- Select Mode
- Text boxes
- Point Mode
- Undo

Other Autograph Functions covered:

- Solve f(x) = 0
- Solve f′(x) = 0
- Solve f(x) = g(x)
- Change the line/curve appearance
- Adjust the degree of accuracy

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are *not* in **Whiteboard Mode**.

Enter the equation: $y = x^2 + 4x - 3$
Note: To enter x^2, either use the little ² button, press "**alt 2**" together, or type in "**xx**".

Predict

What will this curve look like? Where will it cross the axes? Which direction will it slope?

Click OK.

As you will see, not all of the curve fits on the page, so we need to edit our axes. Edit the axes as follows:

x:	Minimum: −6	Maximum: 4	Numbers: 1	Pips: 0.2
y:	Minimum: −8	Maximum: 6	Numbers: 1	Pips: 0.5

Remove all of the green ticks underneath Auto.
Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Note: Before you attempt to re-scale the axes yourself, often pressing the **Default Scales** button will do a good job of sorting the scales out for you.

Now we have our curve on the page, let's see what Autograph can help us find out about it.

1. WHERE DOES THE CURVE CROSS THE X-AXIS?

Use the **Zoom In** function to take a closer look at the first crossing point which is close to −5.

Notice how the scale automatically adjusts the closer in you get.

Use the **Drag** function to move across the screen to find the other point where the curve crosses the x-axis.

Zoom back out so we are looking at the original graph again.

Note: Pressing **Undo** several times is often a quicker way of getting back to your original view.

Now, there is a way to find where the curve crosses the x-axis much more accurately:

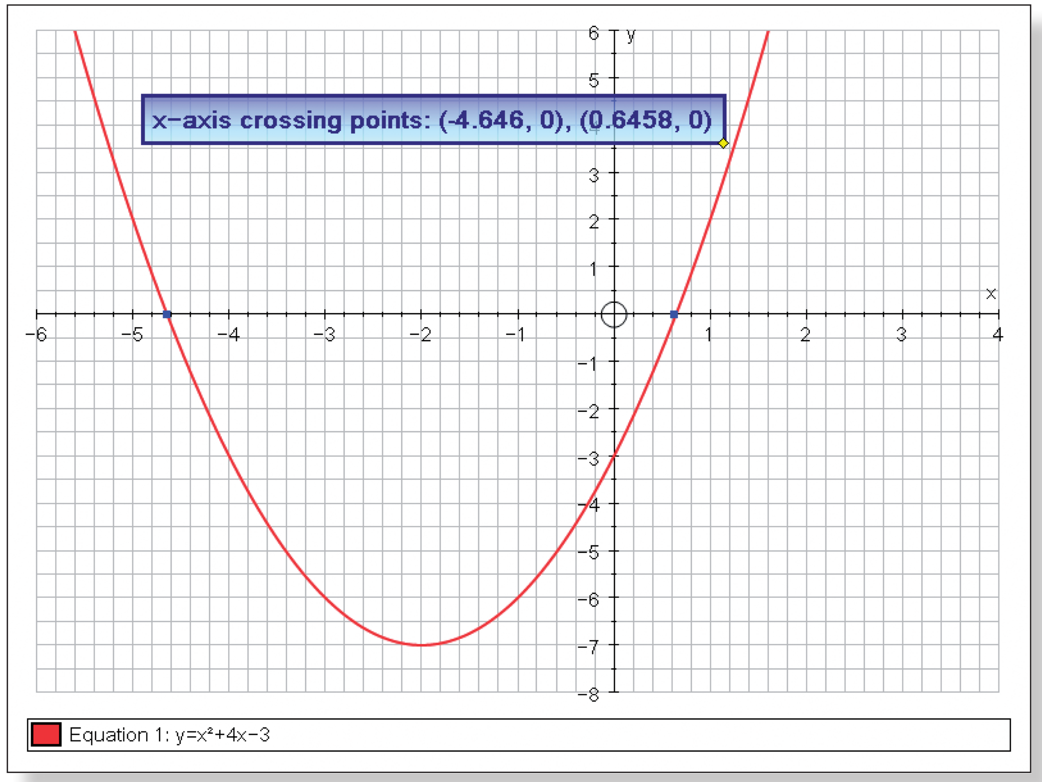
Make sure you are in **Select Mode**.
Left-click on the curve (it should turn black).
Right-click to bring up a menu.
Select **Solve f(x) = 0**.

This will mark on the two points where the curve crosses the x axis, and give you their values to four significant figures in the **Status Bar** at the bottom of the page.

Left-click to select one of the crossing points (it should turn black).
Note: Because these two points are related, they will both be selected.

Click on **Text Box**.
In the text field, delete "Equation solver", and instead write "x-axis crossing points".
Click **OK**.
This displays the results in a moveable text box on the page.

Your page should look something like this:



2. Where does the curve cross another line?

Enter another equation: y = −x − 4
Still on the enter Equation screen, click on Draw Options:

- Change the colour of the line to purple.
- Choose a *dashed* line style.
- Set the line thickness to 3 pts.

Predict

What will this line look like? Where will it cross the axes? Which direction will it slope?

Click OK twice.

You can see that it crosses our curve at two points. Again, we can zoom in to take a closer look at the points of intersection, but we can also use a similar technique to find out their co-ordinates more accurately:

After zooming in, press Undo until you return to the original view of the graph.

Ensure you are in Select Mode.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black).
Note: The use of the **Shift** button to select *more than one object* is very important!

Right-click, select Solve f(x) = g(x), and once again the co-ordinates of the crossing points are marked, and the results are displayed in the status bar.

If you want to improve the accuracy of the calculations:
Go to Page > Edit Settings.
Adjust the level of accuracy to 8 significant figures.

Another quite nice way to achieve all this is to have a moveable co-ordinate on the curve.

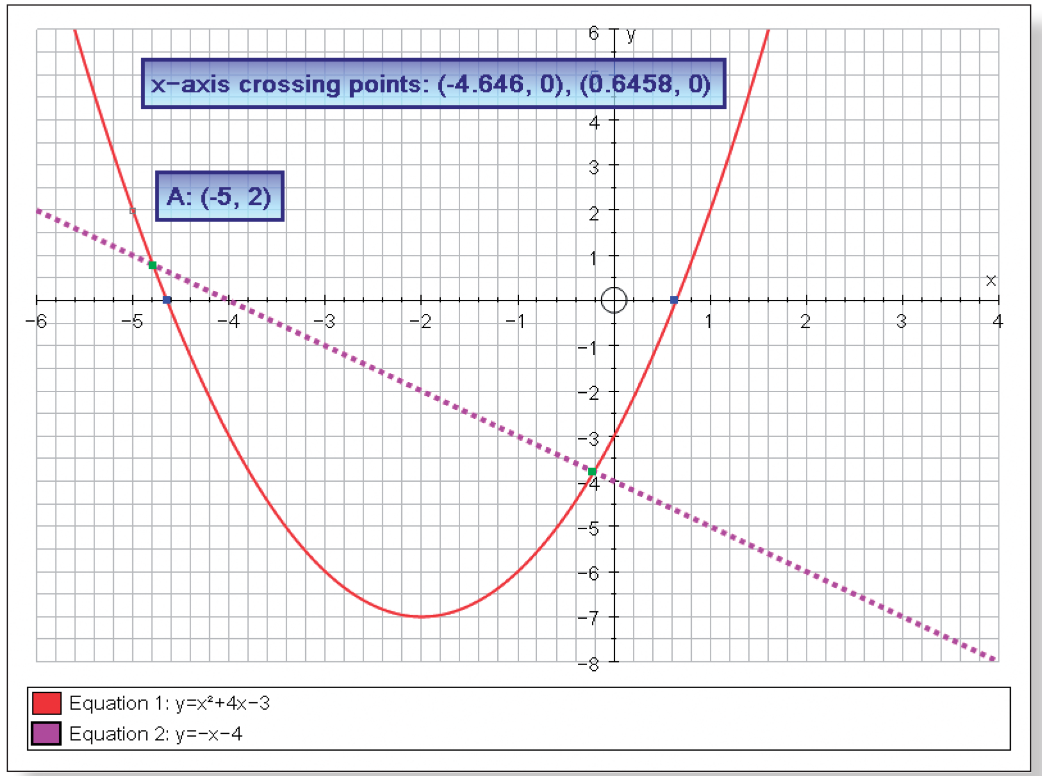
Select Point Mode and place a point somewhere on the left-hand side of the curve.
Note: When the cursor hovers over a section of the curve, it should turn from a cross into a black arrow.

Click on Text Box.
In the text field, delete “Point”, and instead write “A”.
Click OK.
This will keep a record of the current co-ordinates of point A.

Ensure you are in Select Mode and point A is selected (it should have a square around it).
Use the **left** and **right** arrow buttons on the keyboard to move the point along the curve.
Use the **up** and **down** arrow buttons on the keyboard to switch between the curve and the line.
Place the point somewhere on the curve.

Left-click twice on the point, and type in the x-value −5.

Your screen should look something like this:



With the point still selected, if you **right-click** you also have the following options:

- **Move to next f(x) = 0** which will move you to the next crossing of the x axis
- **Move to next f'(x) = 0** which will move you to the next stationary point on the curve.

Activity 2

Autograph Buttons covered:

- Slow Plot
- Re-plot, Pause, Fast Forwarding
- Animate Object
- Gradient Function
- Area Under a Curve
- Enter Co-ordinates

Other Autograph Functions covered:

- Table of Values
- Results Box
- Delete Objects
- Move to Next Intersection

Open up a **New 2D Graph Page**

Edit the axes as follows:

x:	Minimum: −4	Maximum: 4	Numbers: 1	Pips: 0.5
y:	Minimum: −10	Maximum: 15	Numbers: 2	Pips: 1

Remove all of the green ticks underneath Auto

Click on **Slow Plot**

Enter the equation: $y = x^3 + 2x^2 - 5x - 3$

Note: To enter x^3, either use the little 3 button, press "**alt 3**" together, or type in "**xxx**"

Predict

What will this curve look like? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Autograph will begin to plot the graph from left to right.

Pressing **Pause Plotting** at anytime stops and starts the plotting and enables you to focus on the key features of the graph.
Note: Pressing the **spacebar** on your keyboard does the same.

Pressing **Fast-Forward Plotting** immediately speeds to the end of the plotting.

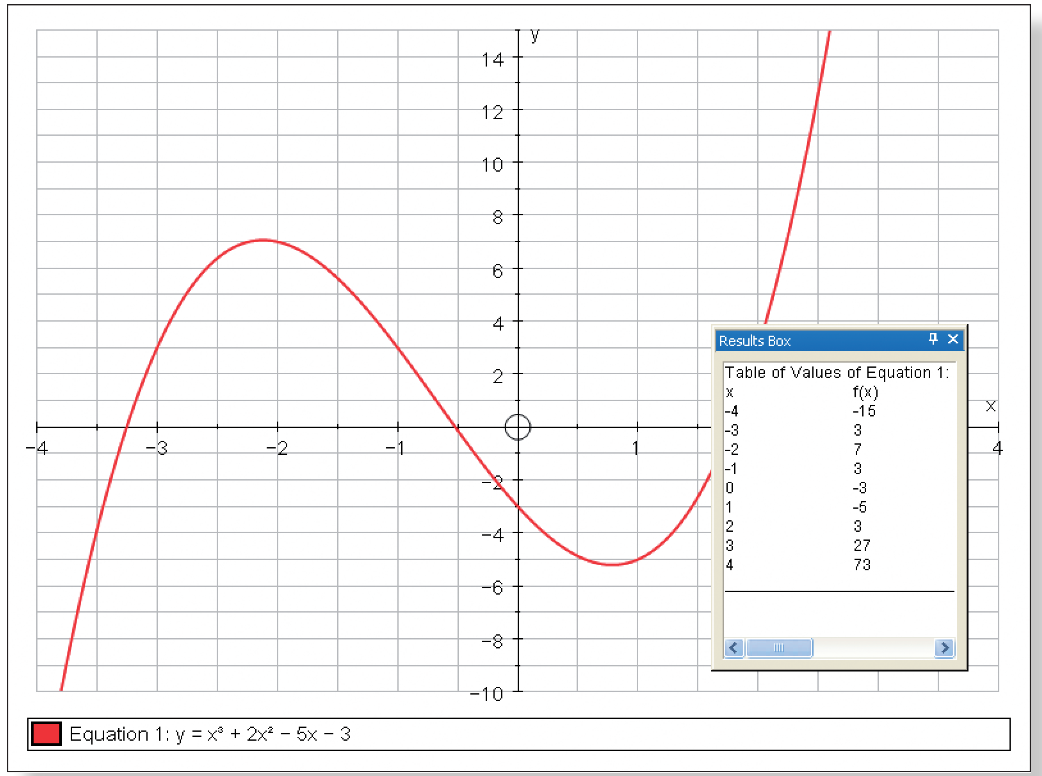
When the curve has finished plotting, click on the **Replot** button.
You now have the option to adjust the plot settings, which means you can adjust both the x-step and the specific section of the graph you want to plot.

Left-click to select the curve (it should turn black).
Right-click and select **Table of Values** from the menu.
Change the **x-step** to 1 and leave the **x-min** and the **x-max** as they are.

This will now give you the corresponding y values for each integer x value on your graph in a **Results Box** at the side of the page.

Note: To view the Results Box at any stage, just click **Results Box**.
The Results Box can also be copied and pasted into another application should you wish.

Your screen should look something like this:



Predict

*What would a **tangent** to this curve look like? What direction would it slope?*

Select **Point Mode** and place a point somewhere on the curve
Right-click and select **Tangent** from the menu.

Left-click to select the tangent (it should turn black).

Select **Text Box** and click **OK**.
The equation of the tangent should now be displayed.

Left-click to select the point on the curve (it should have a square around it).
Use the **left** and **right** arrow keys on the keyboard to move the point along the curve. Notice how the equation of the tangent adjusts.

There is a better way of moving points along a curve then using the left and right buttons.

Make sure the point is selected.

Select the **Animate Object** button, which brings up the **Adjust Position** box.
You now have the option of selecting the exact starting location for the point, and the size of the step which it increases in.
Select a position of −2, and a step of 0.1.
Use the **left-right** buttons to move your point along the curve.

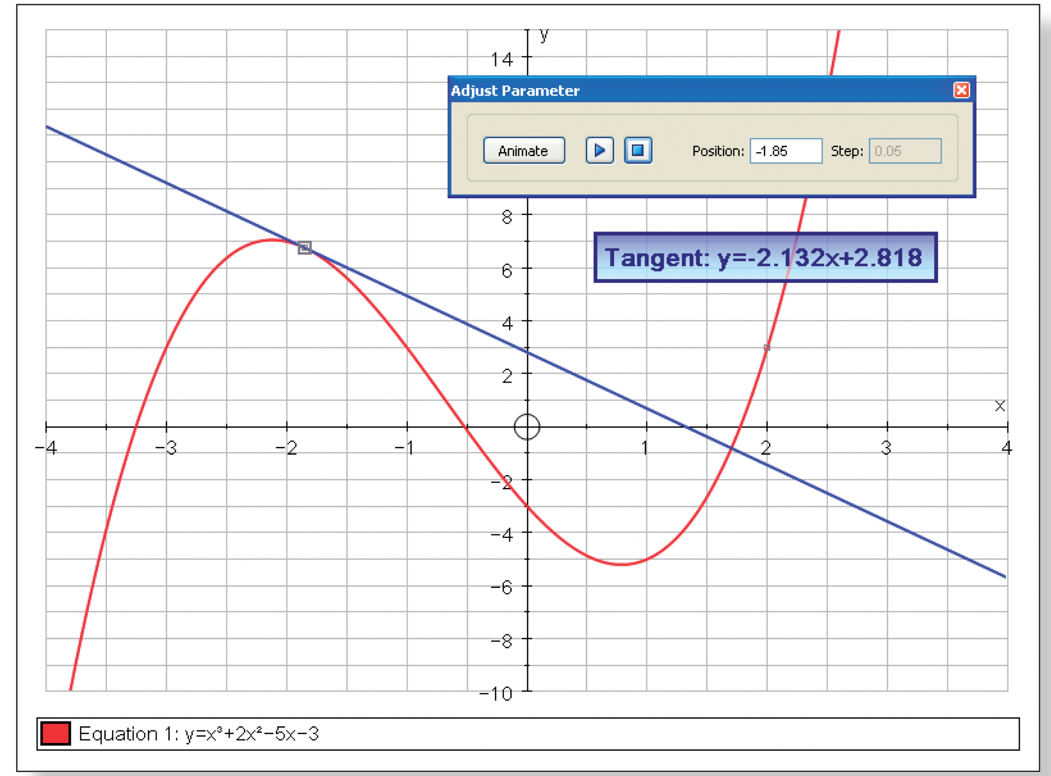
Still in the **Adjust Position** box, click on **Animate**.
Here you can set up Autograph to adjust the step automatically.

Try these settings on **Automatic - Repeat**:

- Animation speed as far to the left as it will go
- Start: −3 Finish: 2 Step: 0.05

Click the **Play** button.

Your screen should look something like this:



When you are ready:

Close the **Adjust Position** box by clicking the red cross in the corner.

Left-click to select the tangent (it should turn black).
Either **right-click** and select **Delete Objects** from the menu, or just press **Delete** on your keyboard.
Note: Don't worry about the warning. This is just to remind you that the Text Box is linked to the tangent, and so will be deleted as well.

You should now be only left with the curve.

Predict

What would the Gradient Function of this curve look like?

Make sure you are still in **Slow Plot** mode (there should be a small blue square around the button).

Left-click to select the curve (it should turn black).

Click on **Gradient Function**.
This will automatically calculate the gradient of the tangent along the curve,

and plot these values as it goes.

The gradient function is set-up to pause at important points, such as each maximum and minimum point, and each point of inflexion.

Press **Pause Plotting** (or the spacebar) to resume the plot.

Predict

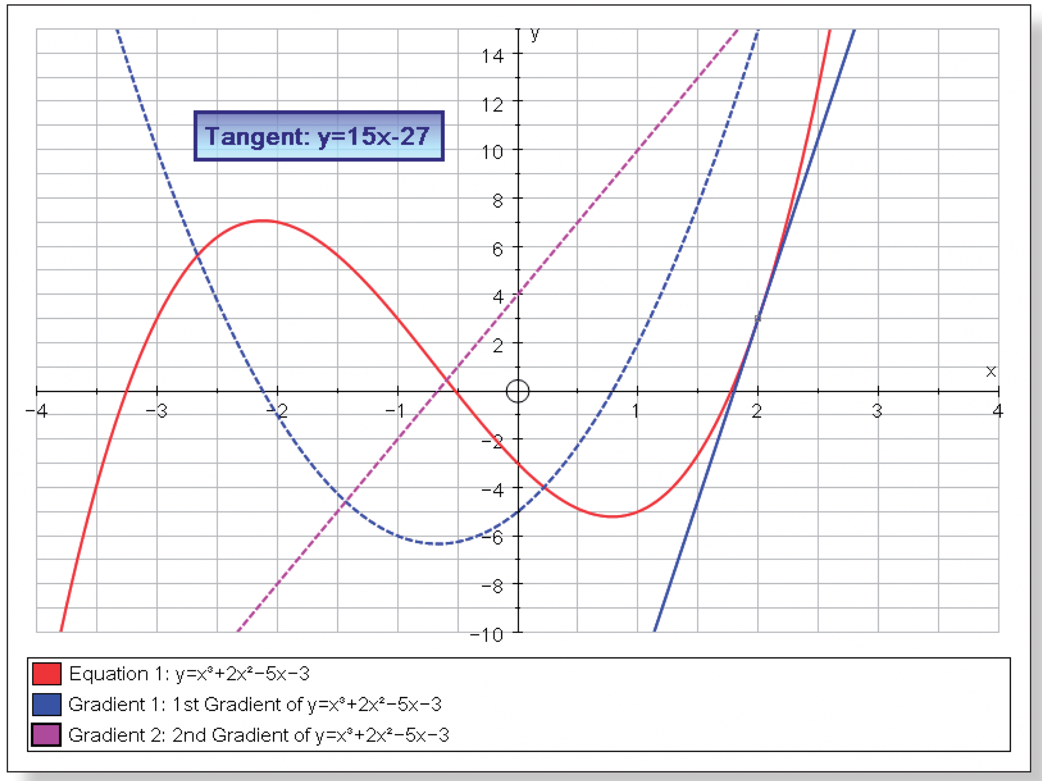
What would the gradient function of the gradient function look like?

Left-click to select this new curve – the gradient function (it should turn black).

Again, click on **Gradient Function**.
This will automatically plot the gradient function of the gradient function, otherwise known as the second gradient function of the original graph.

Note: These lines and curves can then be analysed like any other, whether it be finding points of intersection with the other curves, crossing points with the x-axis, or anything else.

Your screen should look something like this:



Open up a **New 2D Graph Page**.

Edit the axes as follows:
x: Minimum: −4 Maximum: 4
y: Minimum: −6 Maximum: 6
Leave all of the green ticks underneath Auto.

Make sure **Slow Plot** is turned off.

Enter the equation: $y = 5 - x^2$

Predict

What will this curve look like? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Left-click on the curve (it should turn black).

Add a point on the curve with an x value of 0.

Add another point on the curve with an x value of 1.

Left-click on an unoccupied part of the graph area to de-select everything.

Hold down the **Shift** button to select more than one object:
Left-click on both these points (they should both have a square around them).

Right-click and select **Find Area** from the menu.
Use the following settings:
- Trapezium Rule
- Divisions: 5

Predict

What will this look like?

Click **OK**.

Left-click on the marked area under the curve.

Select **Text Box** from the menu and click **OK**.
This will display a numeric estimation for the area under the curve.

Left-click on one of the points.
Use the **left** and **right** arrow buttons on the keyboard to move the point along the curve and observe how the area changes.

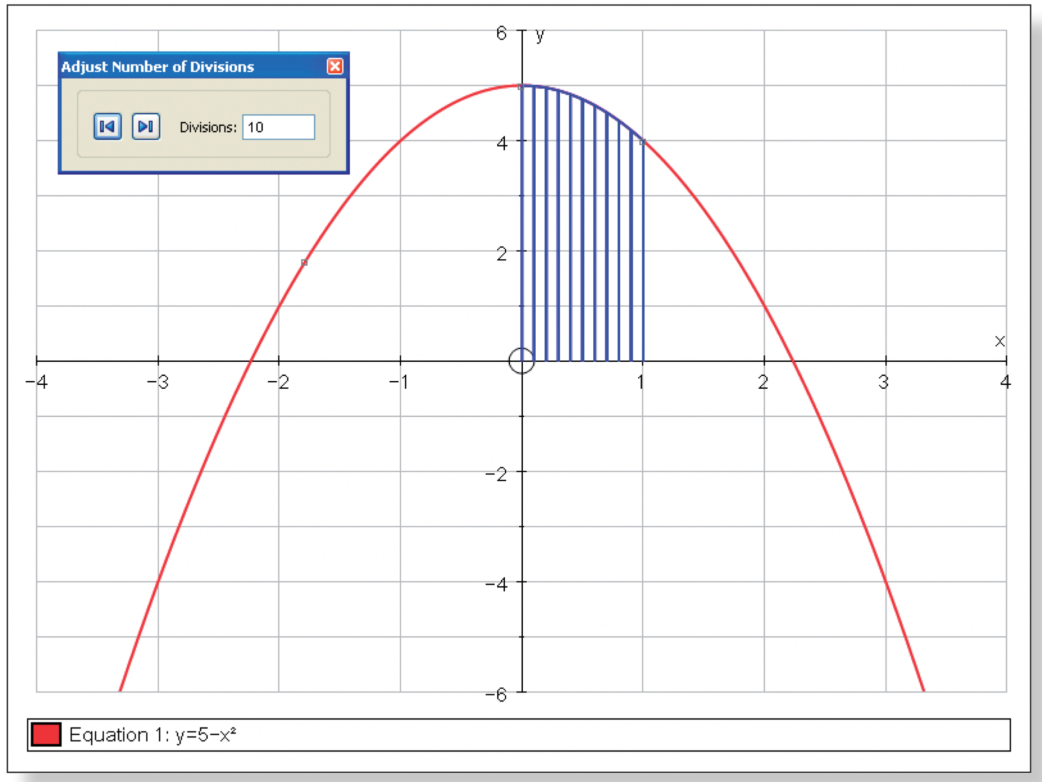
Predict

What effect will increasing the number of divisions have on the estimate for area?

Left-click to select the area under the curve.

Click on **Animate Object**.
You can now adjust the number of divisions and observe the effect it has on the estimate for area.

Your screen should look something like this:



Left-click to select the area under the curve, and **Delete**.

Enter the equation: y = −x

Predict *What will this line look like?*

Click **OK**.

Place a point on the left hand section of the curve which is below the line.

Predict *What do you think the feature Move to Next Intersection will do?*

Make sure you are in **Select Mode**.

Use the **Shift** button to select the point *and* the line.
Right-click and select **Move to next Intersection** from the menu.

Place a point on the middle section of the line which is below the curve.

Make sure you are in **Select Mode**.

Use the **Shift** button to select the new point and the curve.

Right-click and select **Move to Next Intersection** from the menu.

Make sure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to de-select everything.

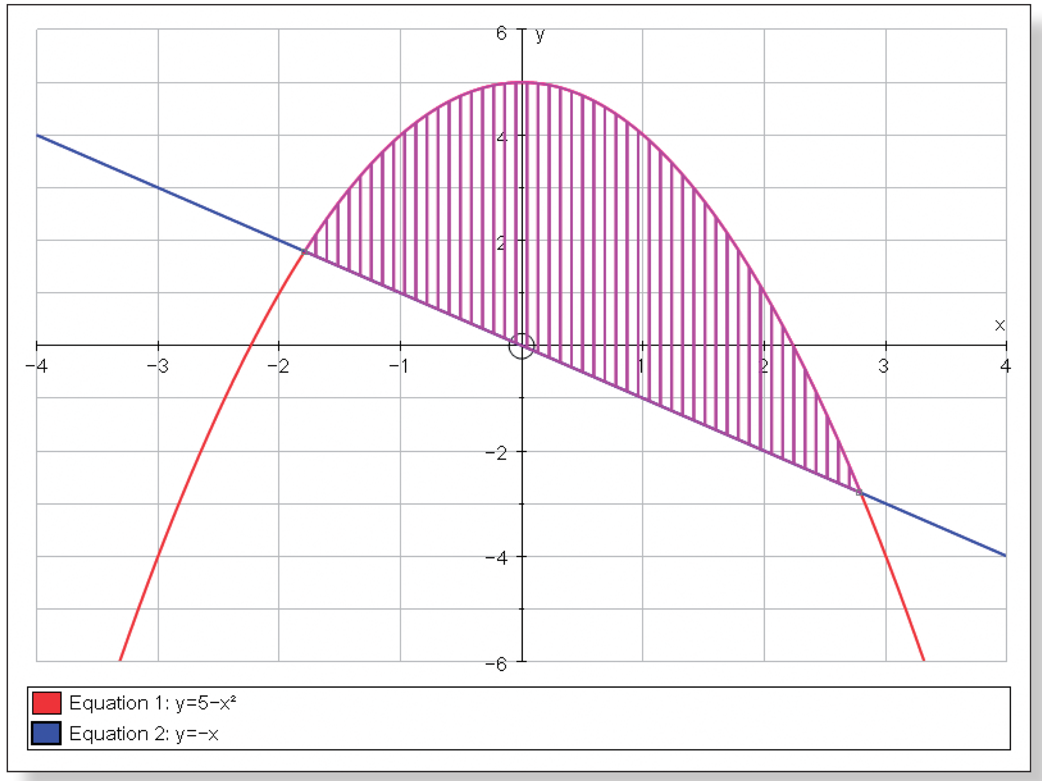
Use the **Shift** button to select both points.

Right-click and choose **Find Area**.
Use the following settings:

- Trapezium Rule
- Divisions: 50

The area between the two curves should now be marked.
You can now use the **Animate Object** function to experiment with different divisions as before.

Your page should look something like this:



Activity 3

Autograph Buttons covered:

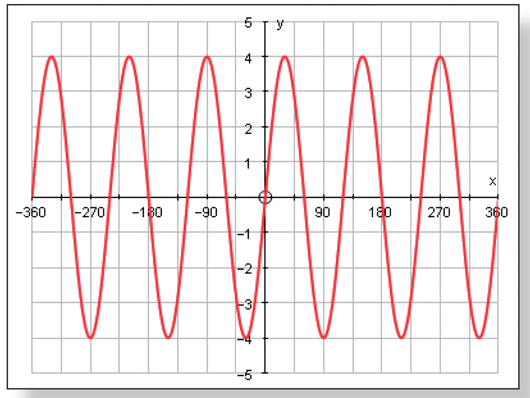
- Enter equations (complicated ones!)
- Default Scale
- Degrees and Radians
- Zoom In Box
- Reset Axes

Often in mathematics we encounter some pretty nasty looking equations. Autograph can handle all of these, but it important you know how to enter them.

We are going to go through a number of equations and go through step-by-step how to enter each of them. Hopefully what you end up with on your screen is pretty similar to the graphs on the right.

Equation 1:

$y = 4\sin(3x)$



Open up a **New 2D Graph Page**.

Ensure you are in **Degree** mode.

Type the following and then press **OK**:

$y = 4\sin(3x)$

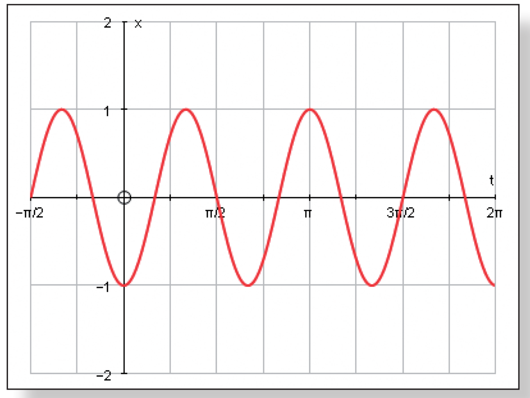
Click on **Default Scales** to improve the graph.

Edit the axes as follows:

x: Minimum: −360 Maximum: 360
y: Minimum: −5 Maximum: 5

Equation 2:

$x = \cos(3t + \pi)$



Open up a **New 2D Graph Page**.

Click on the small black arrow next to this button.

Change the axes to x and t.

Ensure you are in **Radian** mode.

Type the following and then press **OK**:

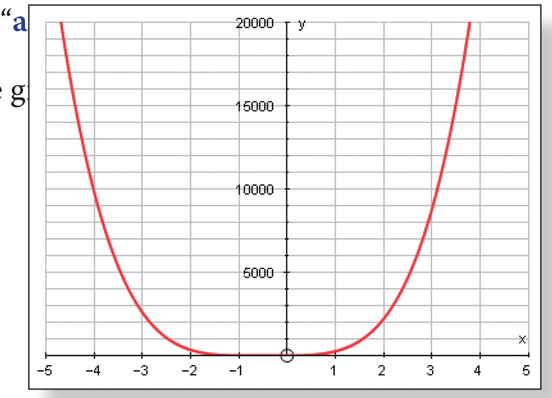
$x = \cos(3t + \pi)$

Note: Either use the π button or press “a

Click on **Default Scales** to improve the g

Equation 3:

$y = \dfrac{(5x+3)^3}{2x}$



Open up a **New 2D Graph Page**.

Type the following and then press **OK**:

y = (5x+3)³/2x

Note: For the cubed, use either the little 3 button or press “**alt 3**” together.

Edit the axes as follows:

x: Minimum: −5 Maximum: 5
y: Minimum: 0 Maximum: 20000

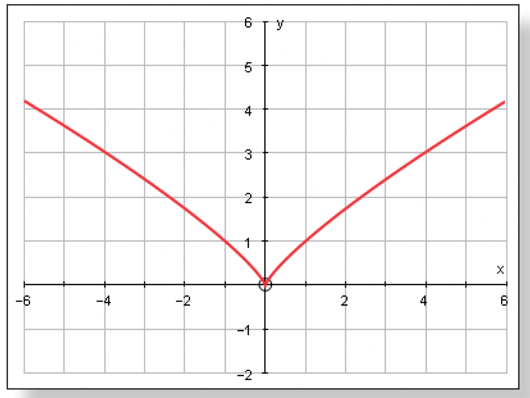
Use the **Zoom In Box** function to take a closer look at the graph around the origin.
Notice how the scale automatically adjusts the closer in you get.

Use the **Drag** function to move across the screen.

Press **Undo** several times is to get back to your original view of the graph.

Equation 4:

$y = x^{\frac{4}{5}}$



Open up a **New 2D Graph Page**.

Click on **Slow Plot** mode.

Type the following and then press **OK**:

y = x^(4/5)

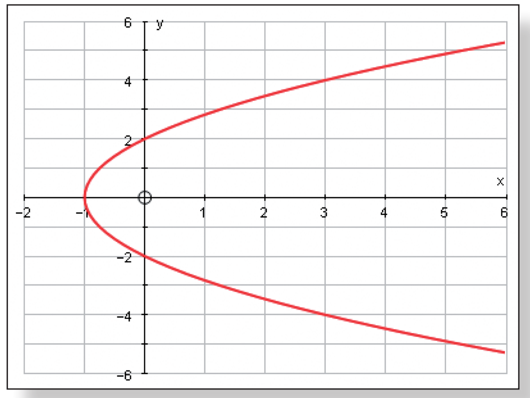
Note: To access the power/hat button press **Shift** and the number which the symbol is above together (usually 6).

Edit the axes as follows:

x: Minimum: −6 Maximum: 6
y: Minimum: −2 Maximum: 2

Equation 5: Parametric

$x = t^2 - 1,\ y = 2t$



Open up a **New 2D Graph Page**.

Ensure **Slow Plot** mode is on.

Type the following and then press **OK**:

$x = t^2 - 1,\ y = 2t$

Edit the axes as follows:

x: Minimum: −2 Maximum: 6
y: Minimum: −6 Maximum: 6

Equation 6:

$y = |x| \pm \sqrt{4 - x^2}$

A nice romantic one to finish...

Predict

Can you guess what this graph will look like?

Open up a New 2D Graph Page.

Ensure Slow Plot mode is on.

Type the following and then press OK:

$y=|x| \pm \sqrt{(4 - x^2)}$

Note: To access the modulus signs, the plus-minus, and the square root, simply press the corresponding buttons in the equation editor.

Note: For more examples of the different types of equations that can be entered into Autograph, see the relevant section in the Help menu.

Activity 4

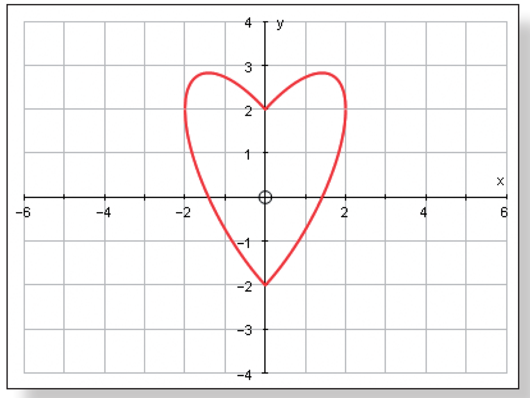
Autograph Buttons covered:

- Equal aspect
- Line segment
- Gradient
- Perpendicular Bisector

Other Autograph Functions covered:

- Graph paper theme
- Hiding the Key
- Circles and Tangents
- Marquee Select
- Use of Shift and Ctrl

Open up a New 2D Graph Page.



Click on Equal Aspect Mode.
This alters the x scale so that the axes are square.

Click on Edit Axes
Go to Appearance
Open the drop-down menu underneath Themes
Select Graph Paper
Click OK

On the top toolbar, click on Axes, and then Show Key to hide the key at the bottom of the screen.

Place a co-ordinate at (1, 3).

Place another co-ordinate at (4, 1).

Predict

What will a line segment between these two points look like? What direction will it slope? Will it cross the axes?

Make sure you are in Select Mode.

Use the Shift button to select both points.

Right-click and select Line Segment from the menu.

Predict

What do you think the Gradient function will do?

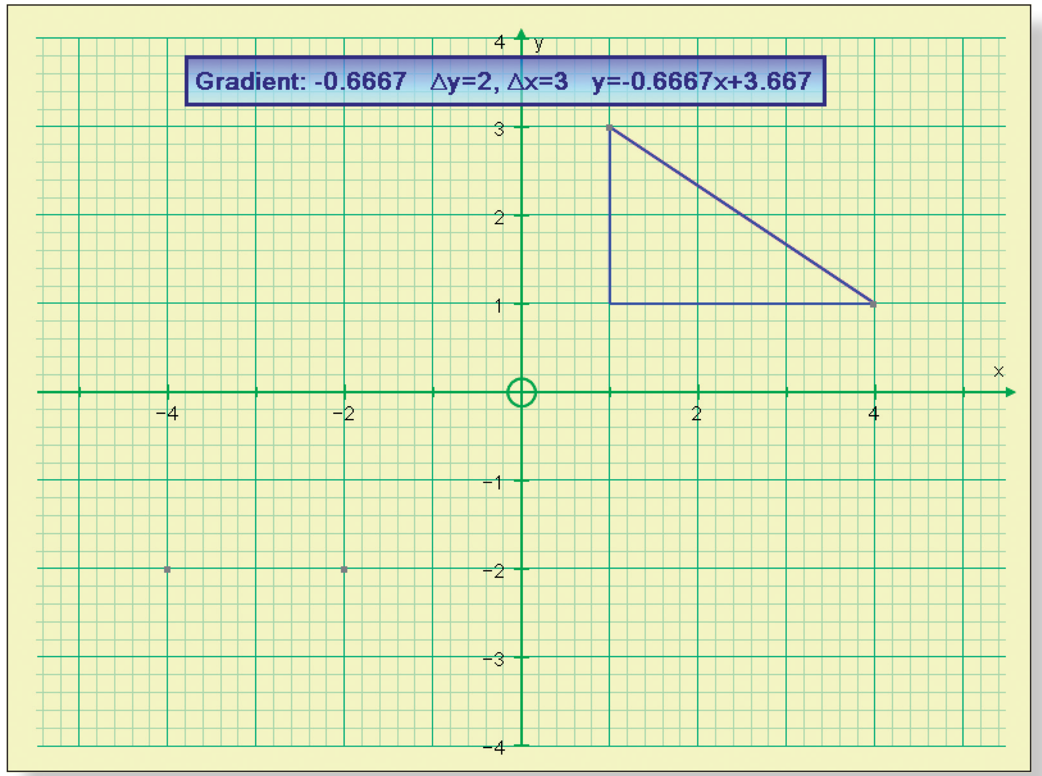
Again, use the Shift button to select both points.

Right-click and select Gradient from the menu.

Left-click to select the gradient triangle.

Select Text Box from the menu and click OK.
This displays the gradient of the line, as well as the equation.

Your screen should look something like this:



You can now select a point and move it using the **left-right** arrows on the key-board to see the effect on the gradient and the equation of the line.

Note: The default in **Advanced Mode** is that the left-right arrows move points 1/10th of a grid spacing. If you hold **Shift** down whilst pressing the arrows, the point moves a whole "grid spacing". If you hold down **Ctrl** whilst pressing the arrows, the point moves by 1/100th of a grid spacing.

Predict *What do you think the Perpendicular Bisector function will do?*

Again, use the **Shift** button to select both points.

Right-click and select **Perpendicular Bisector** from the menu.

Left-click to select the perpendicular bisector.

Select **Text Box** from the menu and click **OK**.
This displays the equation of the line.

Place a co-ordinate at (–4, –2).

Place another co-ordinate at (–2, –2).

Drag a rectangle that surrounds both points.
This is known as **Marquee Selection**, and is a useful way of selecting multiple objects in a given area.

Predict *What do you think the Circle (2pts) function will do?*

Right-click and select **Circle (2pts)** from the menu.

Predict *What will a tangent to this circle look like? Where will it cross the axes?*

Place a point somewhere on the circle.

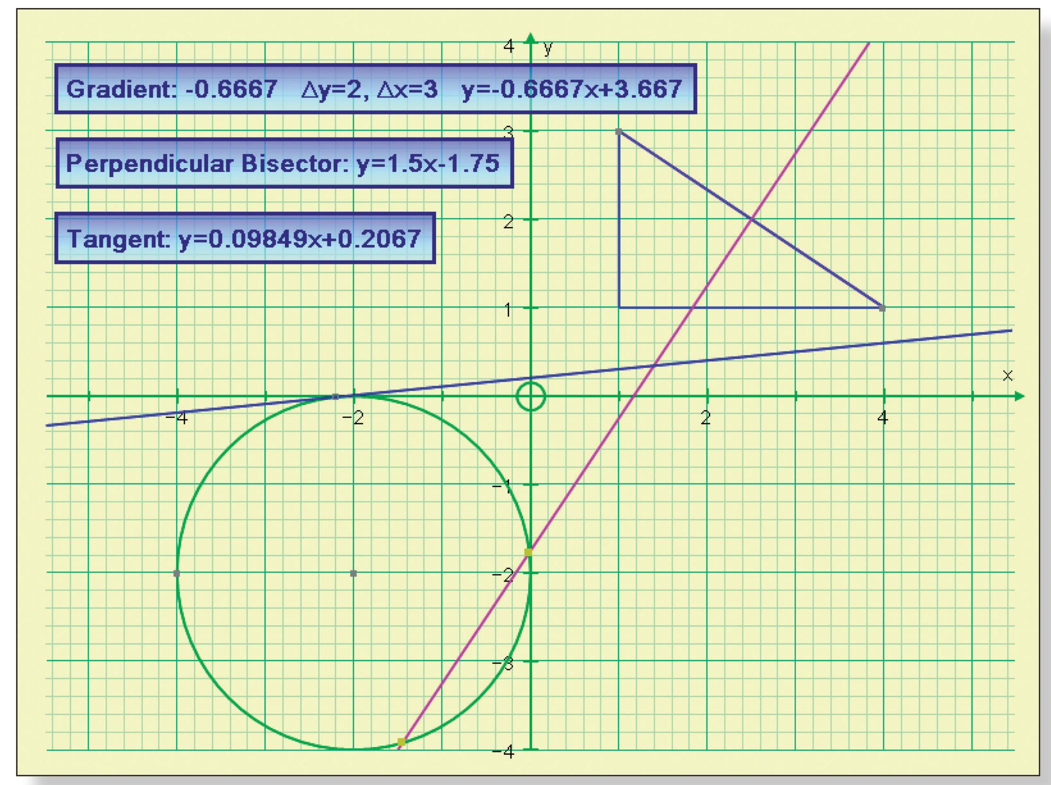
Left-click so just this new point is selected.
Right-click and select **Tangent** from the menu.

Left-click so just the tangent is selected.

Select **Text Box** from the menu and click **OK**.
This displays the equation of the tangent.

You can now use the arrow buttons on the keyboard to move the tangent around the circle.

Your screen should look something like this:



Note: With these various shapes and lines on the page, it is also possible to find the points of intersection in exactly the same way as described in Activity 1.

Activity 5

Autograph Buttons covered:

Constant Controller

Open up a New 2D Graph Page.

Edit the axes as follows:

x: Minimum: −6 Maximum: 6

y: Minimum: −10 Maximum: 10

Leave the green ticks as they are to allow Autograph to select the best grid spacings.

Enter the equation: $y = ax^2 + bx + c$

Still on the Enter Equation screen, click on Edit Constants, and set the values as follows:

$a = 1 \quad b = 4 \quad c = 4$

Predict

What will this curve look like? Where will it cross the axes? Which direction will it slope?

Click OK twice.

Left-click to select the curve (it should turn black).

Select Text Box.

Tick the box next to Show Detailed Object Text and click OK.

The equation of the curve and the current values of the constants should now be displayed in a moveable Text Box.

Click on Constant Controller.

You can now change the values of any of the three constants and observe the effect it has on the shape and position of the curve.

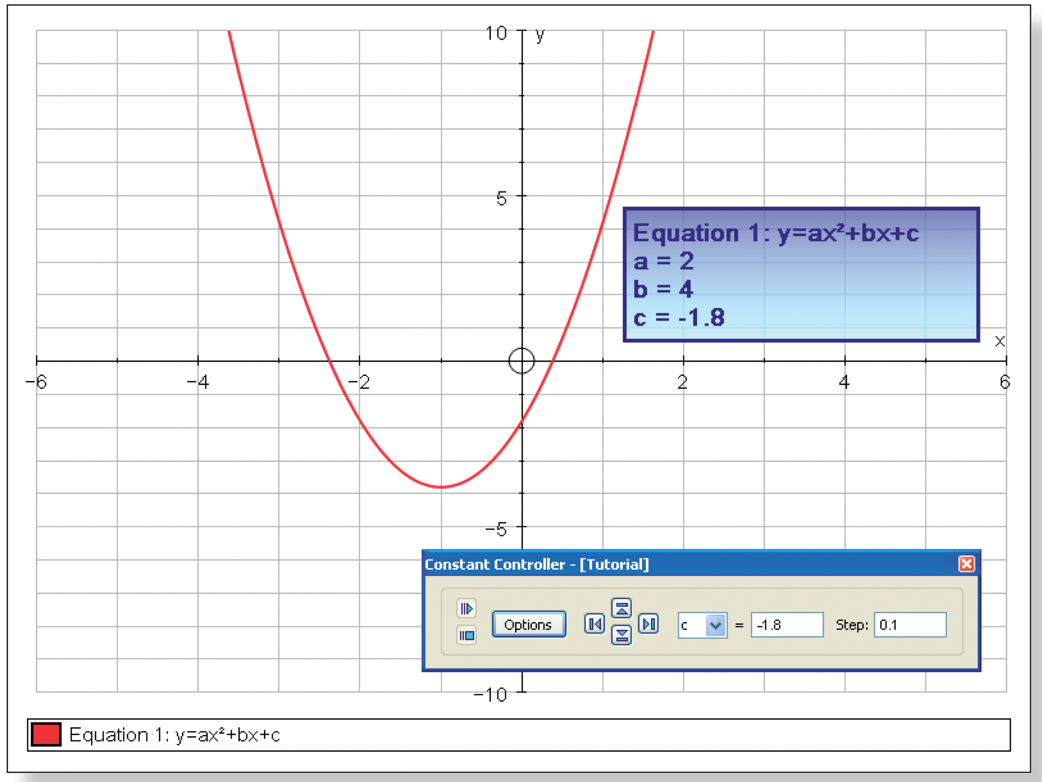
The drop-down menu is used to switch between constants.

The **up-down** buttons alter the value of the constant.

The **left-right** buttons alter the size of the step.

Experiment by adjusting the values of each constant, trying to predict what effect it will have on the shape and position of the graph.

Your page should look something like this:



Set the value of each constant to 1.

Select constant c.

Click on Options on the Constant Controller.

Select Family Plot.

Adjust the parameters as follows:

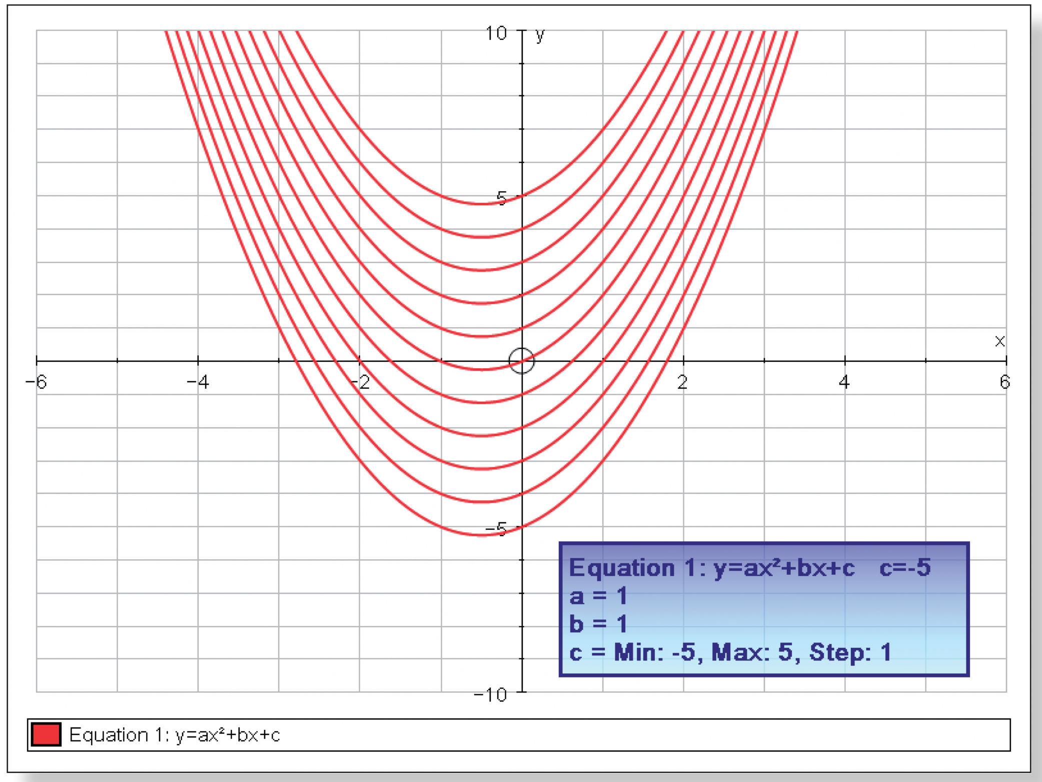
Start: −5 Finish: 5 Step: 1

Predict

What will this look like?

Click OK.

Your screen should look something like this:



Click on Options again.
This time select Animation.
Keep the Animation Speed and the Start and Finish the same, but adjust the Step to 0.1.
Click OK and press the Play button.
This automatically adjusts the value of c for you so you can observe its effects on the shape and position of the curve.

Open up a New 2D Graph Page.

Edit the axes as follows: x from −10 to 10 and y from −40 to 40.

Click on Function Definitions:

- Define f(x) to be $10\sqrt{x}$

Note: to type the √ sign, press "alt r" together.

- Define g(x) to be $x^2 - 4$

Ensure Slow Plot mode is on.

Enter the equation: y = f(x)

Predict

What will this look like?

Click OK.
Enter another equation: y = g(x)

Predict

What will this look like?

Click OK.
Enter the equation: y = f(x) + g(x)

Predict

What will this look like?

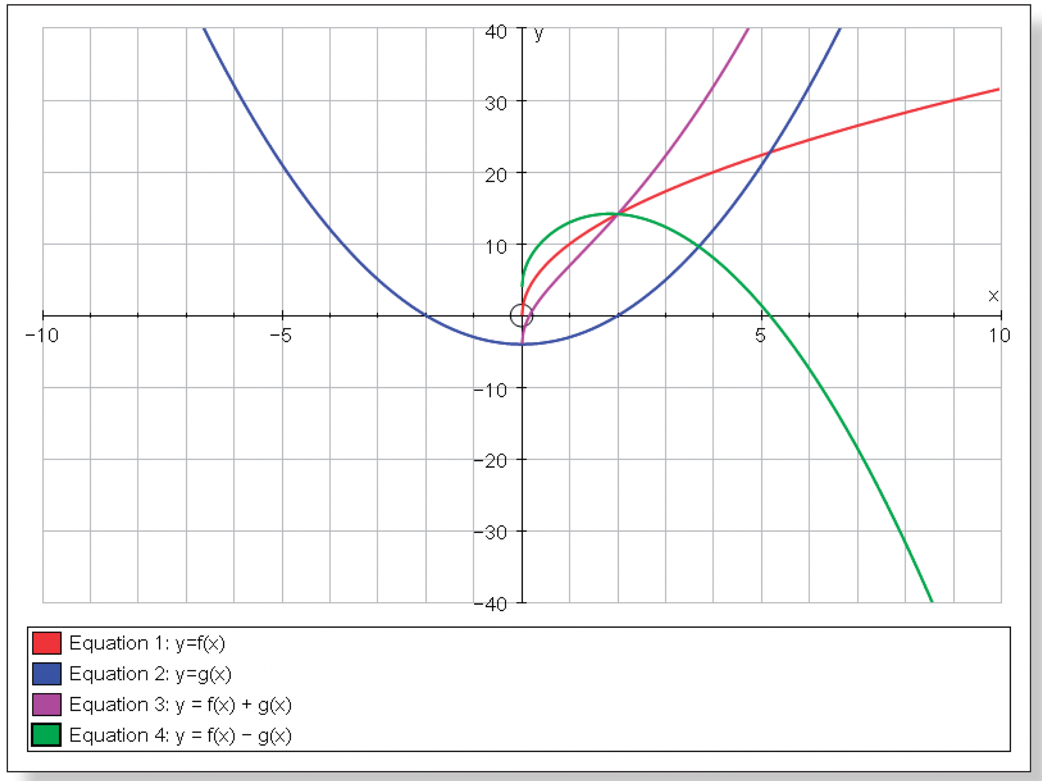
Click OK.
Enter another equation: y = f(x) − g(x)

Predict

What will this look like?

Click OK.

Your screen should look something like this:



Click on Manage List.

Select Equation 3: y = f(x) + g(x)
Delete it.

Do the same with Equation 4: y = f(x) − g(x)

Click OK.

You should now be left with the original two curves.

Carefully enter the equation: y = f(g(x))

What will this look like?

Click OK.

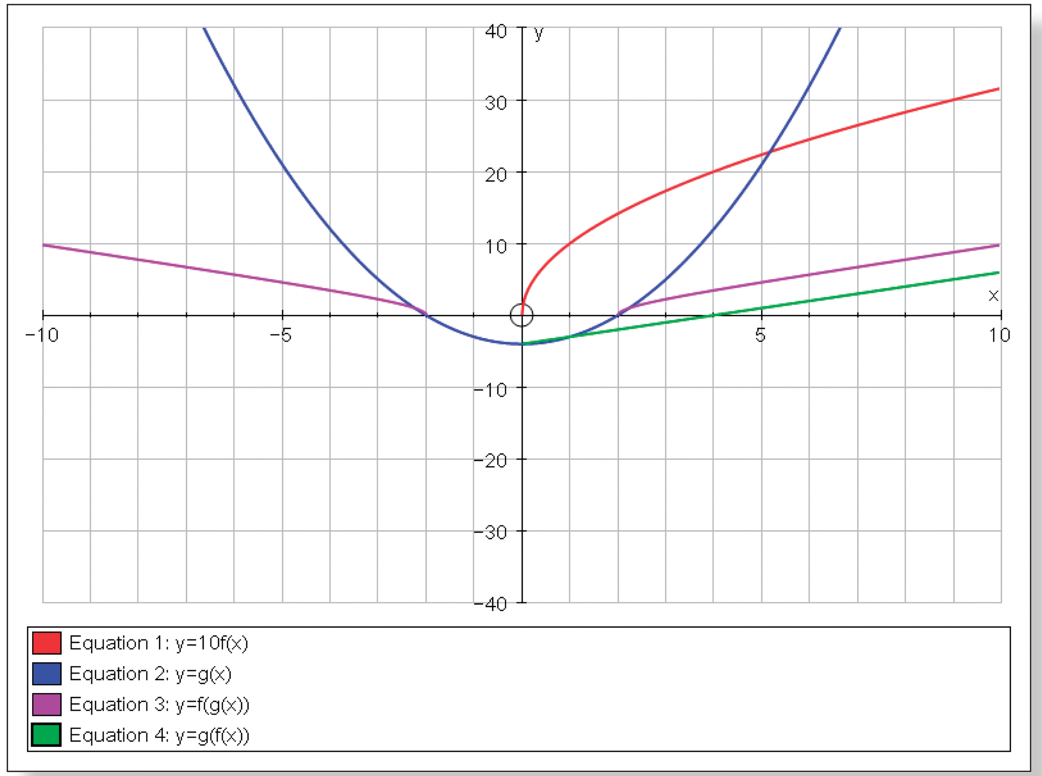
Carefully enter another equation: y = g(f(x))

Predict

What will this look like?

Click OK.

Your screen should look something like this:



Use the zoom and Drag functions to have a closer look at the intersections of these curves.

Whiteboard Mode and On-Screen Keyboard

Note: This additional tutorial is designed to help you get the most out of Autograph on the Interactive Whiteboard (IWB) for classroom demonstrations, and hence ideally it should take place on an IWB.

New Autograph Buttons covered:

Whiteboard Mode

Keyboard

Open up Autograph in Advanced Mode and ensure you have a blank 2D Graph Page.

Select Whiteboard Mode.

Edit the axes as follows:

x: Minimum: −12 Maximum: 12 Numbers: 1 Pips: 1

y: Minimum: −6 Maximum: 6 Numbers: 1 Pips: 1

Remove all of the green ticks underneath Auto.

Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Select Equal Aspect Mode.
This will automatically adjust the x-axis so the axes are square.

In the View menu on the top toolbar, select Keyboard.
The on-screen keyboard should now appear.
Click on the Text button to display a greater range of the keyboard buttons.
Drag the on-screen keyboard to a convenient place on the screen.

Place a point at (3, 3).

The point should still be selected (it should have a square around it), but if not simply left-click to select it again.

Click on Text Box.
Use the on-screen keyboard to change the word “Point” to “A” and click OK.
Drag the Text Box to a convenient space on the screen.

Place a point at (−2, −4).

Label the point “B”.

Place a point at (−4, −3).

Label the point “C”.

Note: At this stage, all three points are selected (they all have little squares around them). If you want to work on just one (or two) points, you must first de-select everything.

Very Important: De-Selecting Everything
Whilst in **Select Mode**, **left-click** on an unoccupied part of the graph area. This de-selects everything and is extremely useful in all Autograph activities in **Whiteboard Mode**.
If you don't de-select everything before moving onto another task, you might find that Autograph thinks you still want to work with a certain point or line, and this could lead to problems!
Note: Pressing the **Esc** button on the keyboard performs the same function.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select Point B (it should have a square around it).

Click on the **Text** button again to show a more limited range of the keyboard buttons.

We are now going to move Point B using the **arrow buttons** on the on-screen keyboard.

The default in Advanced Mode is that the left-right arrows move points $1/10^{th}$ of a gird spacing.
If you hold **Shift** down whilst pressing the arrows, the point moves a whole "grid spacing".
If you hold down **Ctrl** whilst pressing the arrows, the point moves by $1/100^{th}$ of a grid spacing.
Note: It is important when using the on-screen keyboard to remember that the **Shift**, **Ctrl**, **Alt** and **Caps Lock** buttons remain pressed down until you click on them again.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select point A and then point C (they should both have squares around them)
Note: In **Whiteboard Mode**, it is enough to simply click on objects in turn to select them.

Right-click and choose **Line Segment** from the menu.

Click on the **Data** and **Text** buttons to show the full range of the on-screen keyboard buttons.

Use the on-screen keyboard to enter the equation: $y = x^2$ and click **OK**.
Note: Hitting the **Enter/Return** button on the keyboard is often a quicker way of opening the **Add Equation** screen.
Note: To enter x^2, either use the little ² button, press "**alt 2**" together, or type in "**xx**".

At the top of the screen go to **Axes** > **Show Key**.

This should make the key at the bottom of the screen disappear and is extremely useful when you don't wish your students to see the equations of lines and curves on the screen.

Note: This can also be done by **right-clicking** on the **Key** towards the bottom of the screen where it says "Equation 1: $y = x^2$", and from the menu **left-click** on **Show Key**.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select the line segment and the curve (they should both turn black).
Right-click and choose Solve f(x)=g(x) from the menu.

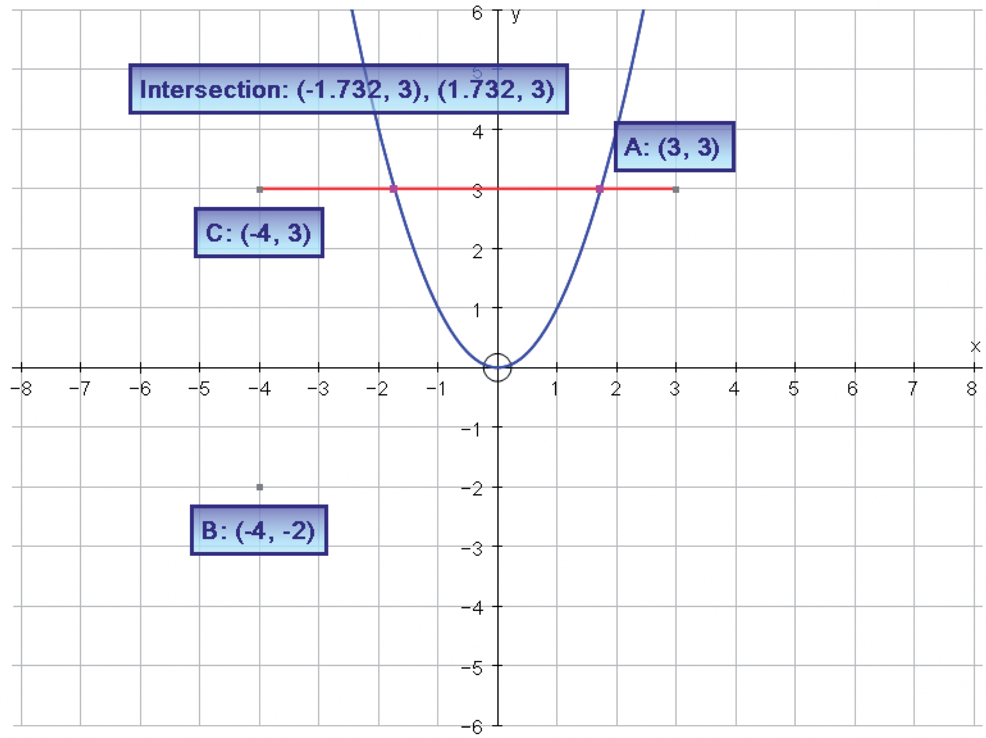
The points of intersection of the curve and the line segment should now be marked on the graph.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click on one of the points of intersection (they should both turn black).

Click on **Text Box**.
Use the on-screen keyboard to change the words "Equation Solve" to "Intersection" and click **OK**.
Drag the **Text Box** to a convenient space on the screen.

Your screen should look something like this:



Note: The **onscreen keyboard** is extremely useful as it can be used with any other application, not just Autograph, and the maths symbols in the Extra panel will also work in other applications. Only a couple of characters are exclusive to Autograph font.

The more you use Autograph, the more little short-cuts you find to make life even easier for yourself. Here is a collection of some of my favourite handy hints and tips.

Keyboard Shortcuts

The following keyboard shortcuts are particularly useful when entering equations, writing in Text Boxes, or defining functions:

π	**Alt-p**	σ	**Alt-s**	x^2	**Alt-2** or **xx**
√	**Alt-r**	μ	**Alt-m**	x^n	**Alt-n**
| |	**Alt-l** (modulus)	α	**Alt-a**	x^x	**Alt-x**
θ	**Alt-t**	β	**Alt-b**	½	**Alt-h**

Spacebar pauses and restarts during **Slow Plot** mode.

Enter takes you straight from the graph area to the **Enter Equation** screen.

The Use of Undo

Undo is not only useful for correcting mistakes, and returning to the original view of a graph following several zooms and drags, it can also be used as a means of revealing a pre-prepared set of objects. Simply prepare the page how you want, delete the objects you don't want the class to see, and then whenever you are ready for them to see them, just hit Undo!

Moving Points

If you select a point and use the left-right arrows on the keyboard to move it, then by default Autograph will move the point 1/10th of a grid space. But if you want to move it more (or less) then try this:

If you hold **Shift** down whilst pressing the arrows, the point moves a whole grid spacing.

If you hold down **Ctrl** whilst pressing the arrows, the point moves by 1/100th of a grid spacing.

Manage Equation List

If you have two or more lines drawn on top of each other, and you want to be sure you have the selected the equation you want, just click on **Manage Equation List**, and you will be able to access all equations for editing or deleting from here.

Note: Double **left-clicking** on the “top” equation in the graph area should also automatically select the one “below” it, but the above method is fail-safe!

The Marque Select Tool

When in Select Mode, you can drag a rectangular shape across the screen, and any objects contained within this rectangle will automatically become selected.

Selecting All Objects

If you wish to select the vast majority of objects on a screen, it can often be quicker to select all the objects at once, and then de-select a few of those.

In the Edit menu, go to Select All.

Now just left-click on any objects you don’t want to select to de-select them!

Note: If you are NOT in Whiteboard Mode, you must hold down Shift whilst de-selecting objects.

Viewing in 3D

When working in 3D, the Drag function is very handy for having a look around your shapes. But this function is even better when combined with a few buttons:

+ **Ctrl** Zooms in and out

+ **Shift** Shifts the camera left and right

And remember, you can restore the original view of your 3D page with one click of a button (**x-y-z Orientation**).

Putting Hide Objects onto the Toolbar

Hide Object and **Reveal Object** are two extremely useful functions in Autograph, and if you follow these steps you can have these options just one click away on your toolbar:

Enter the equation y = x

Left-click to select the line (it should turn black).

Right-click in the black space to the right of the top toolbar and a small menu should appear.

From this menu, select **Customise...**

With the **Customise** window still open, click on the **Object** menu from the top toolbar, and **Hide Object** should be one of the options.

You should now be able to drag **Hide Object** to the top toolbar, where it will remain as an option.

You can add **Reveal Object** to the toolbar (or indeed any function) in exactly the same way.

The Autograph Keyboard

Not only is the keyboard incredibly useful to use when running Autograph on an interactive whiteboard or via a graphics tablet, it can also be used in other applications such as Word, Excel or even when writing an email. So, your days of searching for that elusive theta or fractional notation may be over!

The Trapezium Rule

Learning Objectives

- To be able to understand how using both rectangles and trapeziums can give an approximation to the area under a curve.
- To be able to understand why the Trapezium Rule gives a more accurate approximation of the area under a curve.
- To be able to understand under which circumstances the Trapezium Rule gives and over-estimate of the area under a curve, and when it gives an under-estimate.

Required Pre-Knowledge

- To be able to calculate the areas of rectangles and trapeziums.
- To be aware of the concept of integration.

Pre-Activity Set-Up

Open up Autograph in **Advanced Mode** and ensure you have a blank 2D graph page.

Select **Whiteboard Mode**.

Enter the equation: $y = (1 + x^3)^{-1}$
Note: To enter the power, either use the little button, or hold down "**alt**" and press "**1**".

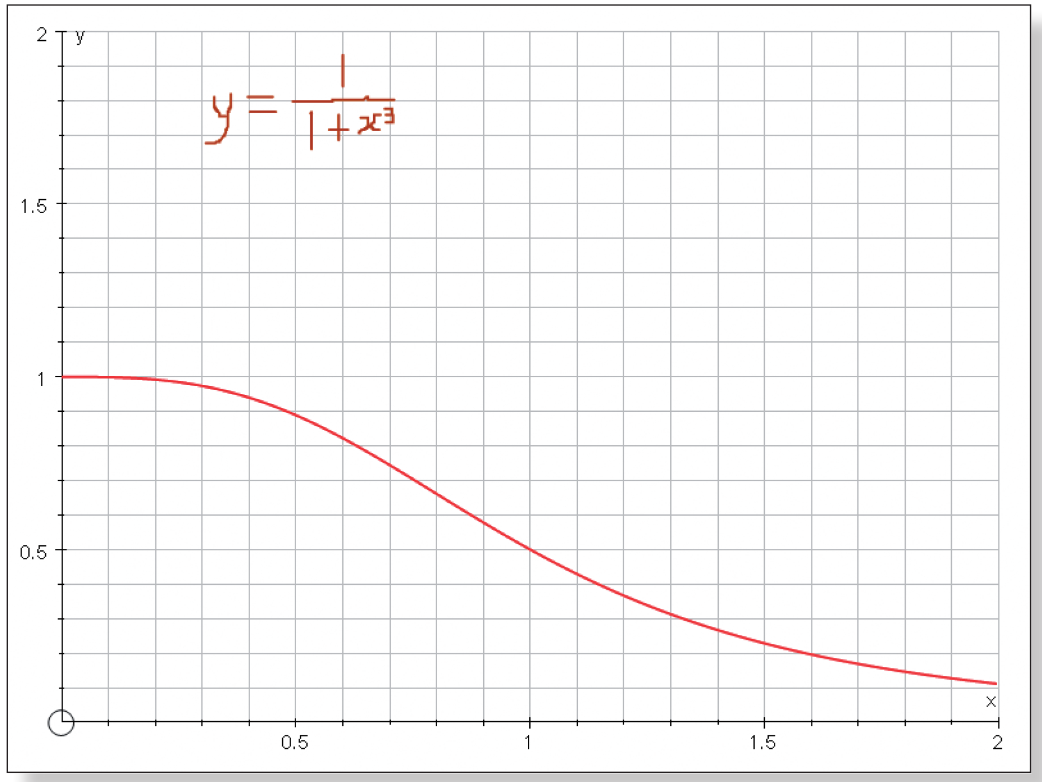
Edit the axes as follows:

x:	Minimum: 0	Maximum: 2	Numbers: 0.5	Pips: 0.1
y:	Minimum: 0	Maximum: 2	Numbers: 0.5	Pips: 0.1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Using the **Scribble Tool**, write the equation of the graph in the form $y = \frac{1}{1 + x^3}$ somewhere above the graph using the **Scribble Tool**.
Note: If you do not have an interactive whiteboard, then either display this equation elsewhere, or simply inform the class that this is the equation of the line.

On the top toolbar, click on **Axes**, and then **Show Key** to hide the key at the bottom of the screen.
Your screen should now look something like this:



Step-by-Step Instructions

Activity 1: The Problem with Integration

Teacher: How would you find the area under this curve between x = 0 and x = 1?
Or: How would you calculate: $\int_0^1 \frac{1}{1+x^3}\,dx$?

Ideal Response: Integrate!

Teacher: Okay then, off you go!

Allow the students a few minutes to try and integrate the expression, before announcing that for the time being expressions like this are too complicated to integrate.

Teacher: Can anybody think of another way of finding out the area under the curve?

Prompt: Maybe not the exact area, but at least a pretty good estimation? Could we perhaps split up the area using a nice simple shape?

Ideal Response: Split up the area under the curve using rectangles!

Activity 2: The Rectangle Rule

At this point you can invite a student to come to the front to demonstrate exactly how rectangles would help get an approximation of the area.

Use the **Erase** tool to rub out any mistakes.
If you want to get rid of all scribbles, click on **Edit** > **Select all Scribbles**, and

press **delete** on the keyboard (or **right-click** on the graph itself and select **Delete Objects** from the menu).

When ready, follow these steps:

Left-click on the curve with your mouse (it should turn black).
Right-click and a menu should appear.
Select **Find Area**.
Select the following:
Rectangle (-), Start Point: 0, End Point: 1, Divisions 5.
Click **OK**.

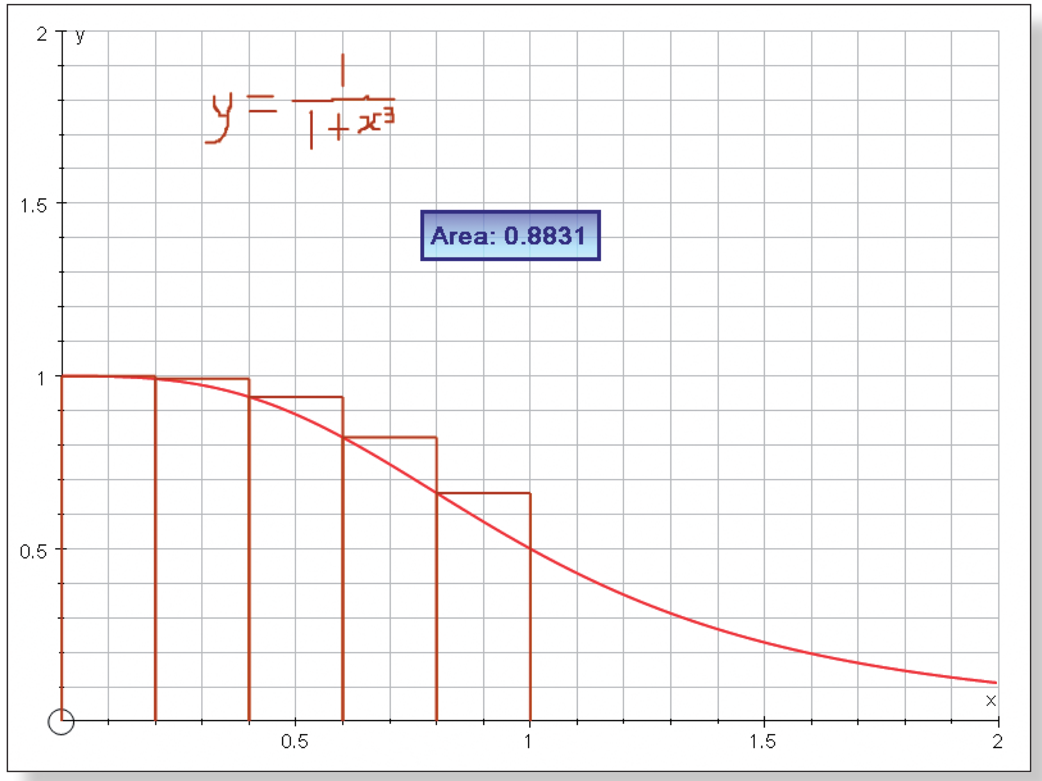
Left-click on an unoccupied part of the graph area to de-select everything.

Left-click on any of the rectangles to select them.

Choose **Text Box** from the top of the screen, and click **OK**.

The approximation to the area under the curve as given by the 5 rectangles should now appear.

Your screen should look something like this:



You can now ask any of the following questions:

Teacher: Is this an over-estimate or an under-estimate of the actual area under the curve?

Ideal Response: Over-estimate.

At this point you could **left-click** on the curve again, **right-click** to call up the menu, select **Find Area**, and this time choose **Rectangle (+)** to demonstrate an under-estimate:

Teacher: Still using rectangles, how could we improve the accuracy of our approximation?

Ideal Response: Use more rectangles.

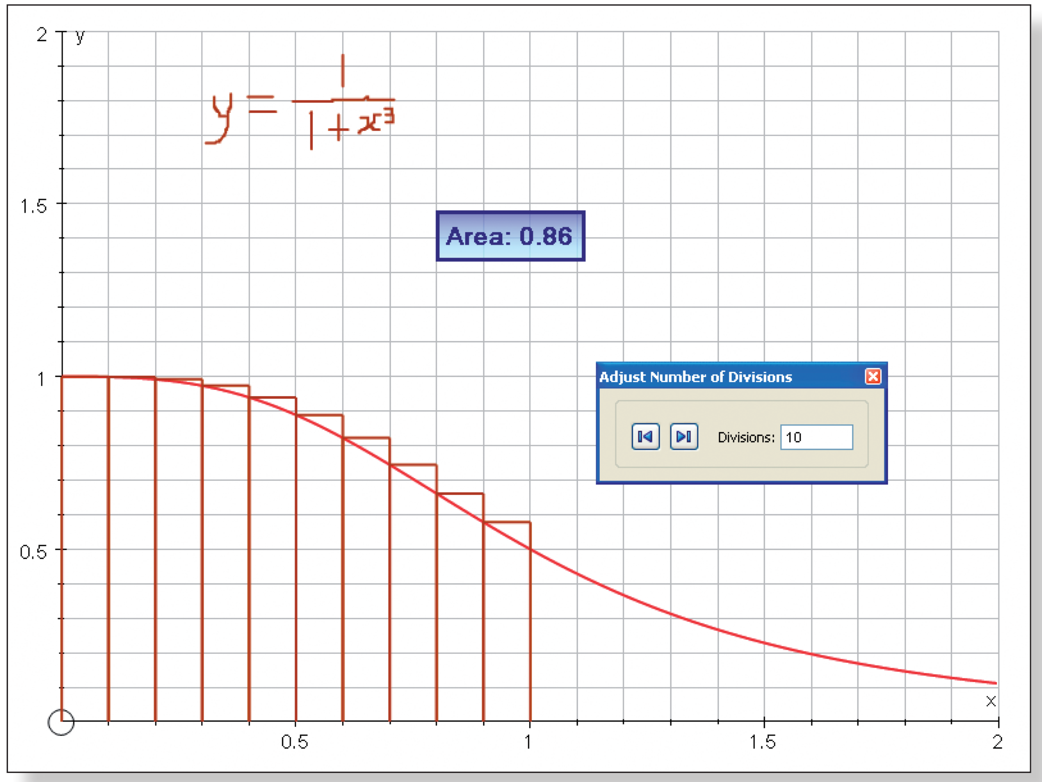
Left-click on an unoccupied section of the graph area to de-select everything.

Left-click the most recent set of rectangles (the under-estimate), **right-click** and select **Delete Object**.
Note: Pressing **delete** on the keyboard has the same effect once the object is selected.

Left-click on the remaining set of rectangles.

Right-click to call up the menu, and select **Animate Object**.

You now have the ability to dynamically adjust the number of rectangles and encourage the class to observe how it improves the estimation of the area:



Teacher: Can anybody think of a better shape to use when estimating the area?

Prompt: What is it about the shape of the rectangle that causes it to be inaccurate?

Ideal Response: Use trapeziums!

Activity 3: The Trapezium Rule

Delete the existing rectangles as described above.
Note: don't worry about the on-screen warning, this is just to inform you that the Text Box will also be deleted as it is linked to the area.

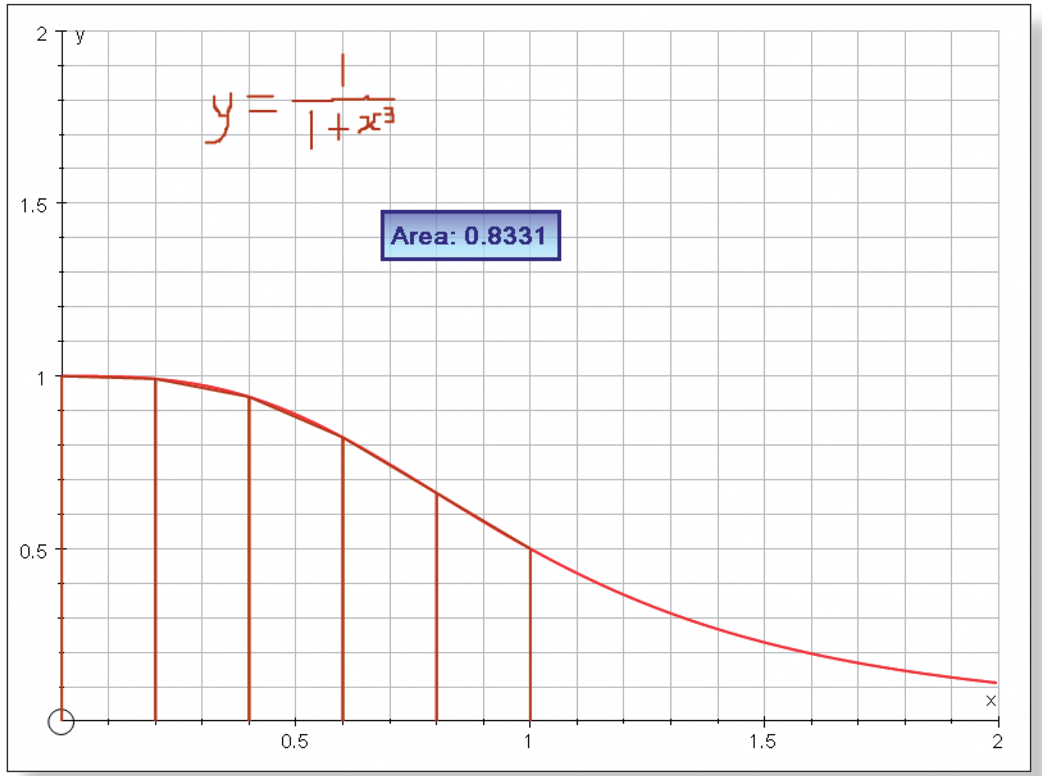
Repeat the steps above to find the area under the curve, but this time select the following: Trapezium Rule, Start Point: 0, End Point: 1, Divisions: 5.

Click OK.

Again, left-click to select the trapeziums.

Select Text Box and click OK.

Your screen should now look something like this:



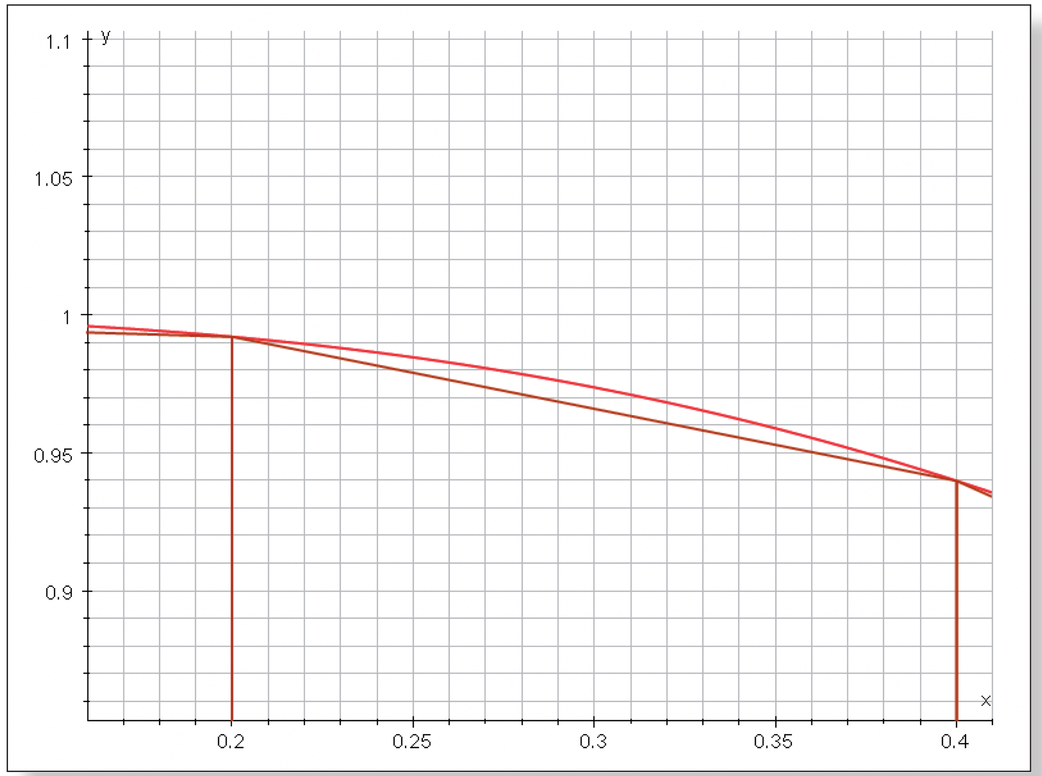
Teacher: We can see that this will give is a more accurate approximation. But is this an over-estimate of the area, or an under-estimate?

Ideal Response: Under-estimate.

Use the Zoom In function to get a closer look at the top of some of those Trapeziums to highlight that it is in fact still an under-estimate of the area. Notice how the scale automatically adjusts the closer you get.

Use the Drag mode to move around to different points on the curve.

Your screen should look something like this:



Use the Zoom Out function to the original view of the graph.

Note: Pressing the Undo button several times is often a quicker way of doing this.

Again at this point you can use the Animation function to look at the effect of increasing the number of divisions has on improving the accuracy of the approximation.

Teacher: Will the Trapezium Rule always give us an under-estimation of the area?
Or:

Teacher: Can anyone think of (or come up and draw) an example of a curve which the Trapezium Rule would give us an over-estimation of the area underneath it?

If the students do want to sketch a curve:

Open up a blank 2D Graph Page.

Encourage the students to use the Scribble Tool to sketch their graphs.

Use the Erase tool to rub out any mistakes.

When you are ready:

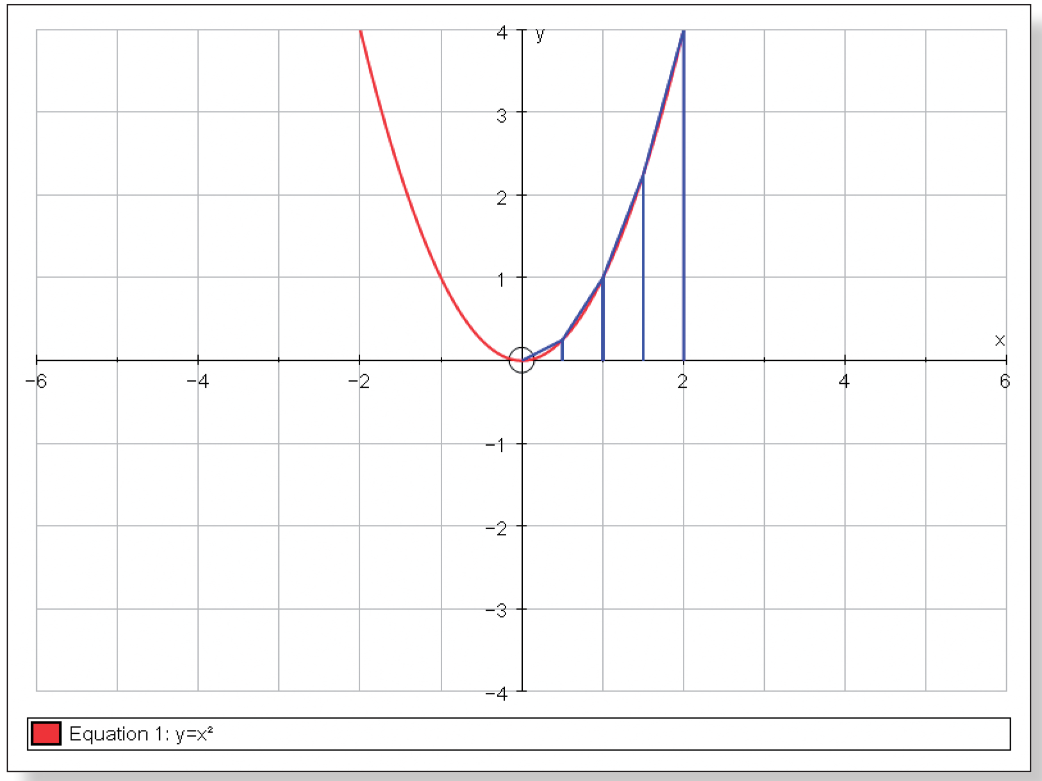
Open up a blank 2D Graph Page.

Enter the equation: $y = x^2$

Left-click to select the curve (it should turn black).

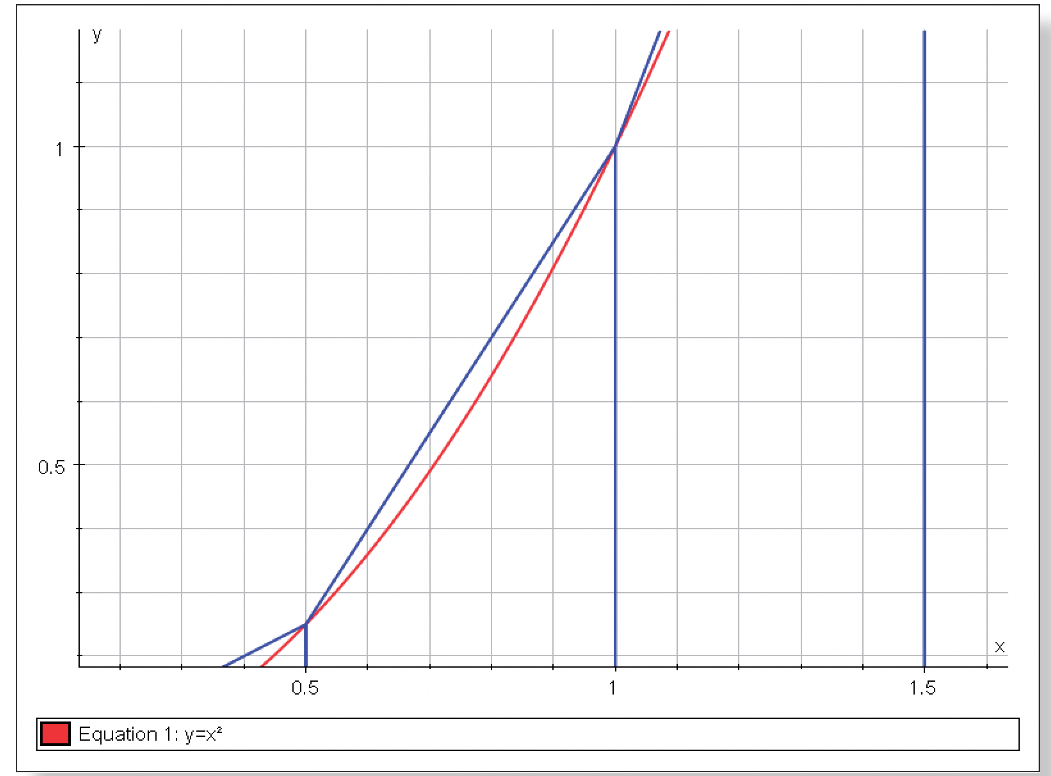
Right-click to get the menu, and select **Find Area**.
Choose the following settings.
Trapezium Rule, Start Point: 0, End Point: 2, Divisions: 4.

Your screen should look something like this:



Again, use the **Zoom In** and **Drag** function to get a closer look to illustrate that we now have an over-estimation of the area.

Your screen should look something like this:



Teacher: Can anyone explain why the **Trapezium Rule** in the first example gave us an under-estimate of the true area, and yet in the second example it gave us an over-estimate?

Prompt: Think about the slope of the graphs, and not just whether they are positive and negative. Think about how the graphs "bend". Can anybody remember the technical term for the type of "bendiness"?

Ideal Response: It depends on the **Concavity** of the function!

Teacher: So, it looks like using Trapeziums can give us a more accurate approximation of the area underneath a curve whenever integration lets us down. Now all we need to know is how to calculate the area of all these trapeziums, knowing both the height and the base. If only we had a nice formula…

Note: A simple Autograph diagram like the ones in this demonstration might be ideal to talk though the **Trapezium Rule Formula**.

Ideas for Further Work

- The students should now be in a position to learn the formula for the **Trapezium Rule**, and to practise using it to estimate the area underneath a curve.
- They should also be well prepared for exam-style questions which often like to ask about over and under estimations.
- Further investigation into the **concavity** of functions an the effect on the **Trapezium Rule** using Autograph could be conducted in a similar way as

outlined in this demonstration

- A possible extension activity would be to look at the **Trapezium Rule** as being halfway between the two rectangle rules. Take an always increasing function and find the area between two points using all three methods. The **Trapezium Rule** always gives an answer exactly halfway between the rectangle rule. You can look closely at one trapezium and create a triangle near the top, rotate it to show how the trapezium is halfway between the two rectangle approximations.

Learning Objectives

To be able to understand some of the common problems and difficulties surrounding integration, such as:

- Not accounting for negative areas
- Improper Integrals
- Unbounded Functions
- Unbounded Intervals

Required Pre-Knowledge

- To be able to sketch curves involving both positive and negative powers of x.
- To be able to calculate simple, definite and indefinite integrals such as:

$$\int 3x^2\,dx \qquad \int_2^4 x^2 - 3x + 4\,dx \qquad \int_1^4 \frac{3}{x^3}\,dx$$

Note: Knowledge of infinite series would be helpful, but is not essential. Indeed, this topic is a nice way of introducing the idea of an infinite series converging to a finite amount.

Pre-activity Set-up

None required.

Step-by-Step Instructions

Activity 1: Negative Areas

Teacher: Just to warm you up, can you work out the value of this integral:

$$\int_0^3 x^2 - 4x + 3\,dx$$

Prompt: Pupils may need reminding of the techniques of integration learnt in previous lessons.

Ideal Response: $9 - 18 + 9 = 0$

Teacher: Does that sound right to you? Why would the area under this curve be equal to zero?

Prompt: What type of curve is it? Think of the shape. Where does it cross the x-axis?

Ideal Response: Some section of the curve lies above the x-axis, which gives us a positive area, but some section of the curve lies below the x-axis, which gives us a negative area. These two areas must be cancelling each other out to leave us with zero.

Teacher: Can anybody come and draw a sketch of the curve and show us why the answer is coming out at zero?

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

Edit the axes as follows:

x: Minimum: −2 Maximum: 6 Numbers: 1 Pips: 1

y: Minimum: −2 Maximum: 4 Numbers: 1 Pips: 1

Remove all of the green ticks underneath **Auto**.

Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Using the **Scribble Tool**, encourage the students to attempt to sketch the curve on the grid.

Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press **Delete** on the keyboard (or **Right-Click** on the graph area itself and select **Delete Objects** from the menu).

When the correct curve is drawn and the students have identified that some parts of the curve lie above the line and some parts lie below the line, explain that we are now going to let the computer draw the curve accurately:

Go to **Edit** in the top toolbar, then **Select All Scribbles**, and then press **Delete** on the keyboard.

Click on **Slow Plot** mode.

Enter the equation: $y = x^2 - 4x + 3$

Note: To enter x^2, either use the little 2 button or press "**xx**".

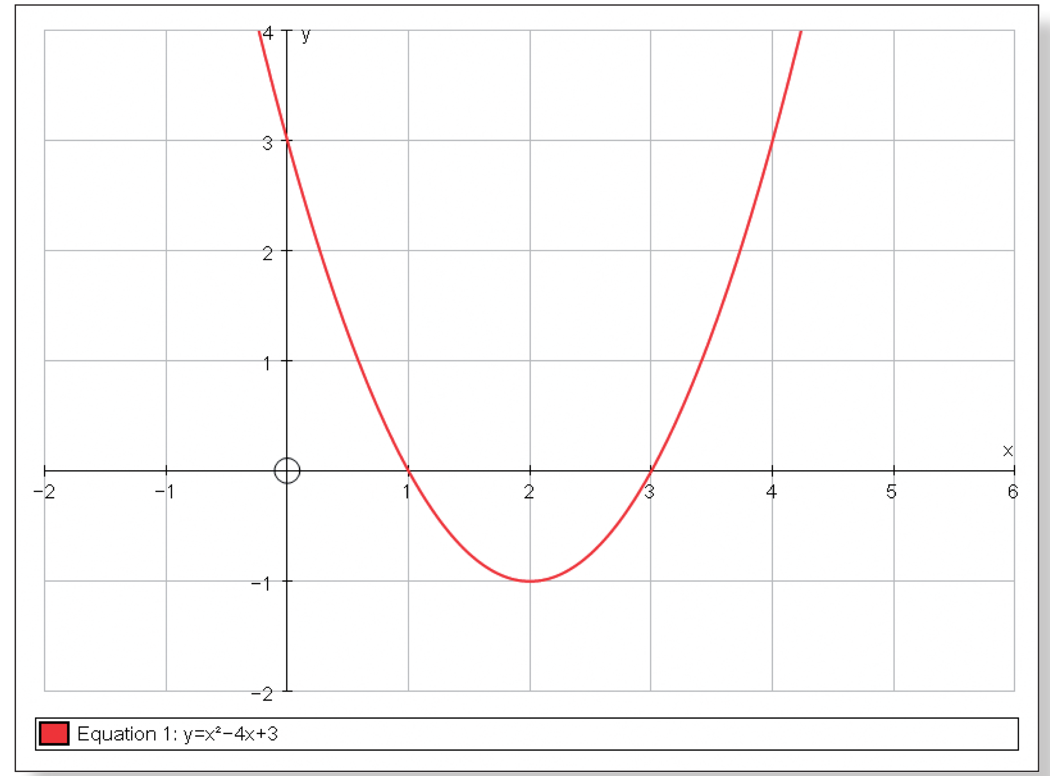
Click **OK**.

The curve should begin to appear on the screen.

Press **Pause Plotting** both to stop the process, or to resume it to focus on the key features of the graph.

Note: Pressing the **Spacebar** on the keyboard has the same effect!

Your screen should look like this:



Teacher: So, how are we going to work out the area between the curve and the x-axis between $x = 0$ and $x = 3$?

Ideal Response: Do two separate integrals.

Teacher: Can anybody tell me what those two separate integrals are?

Ideal Response: $\int_0^1 x^2 - 4x + 3\,dx$ and $\int_1^3 x^2 - 4x + 3\,dx$

Teacher: Okay, see if you can work out the answer.

Ideal Response: The first integral gives us an answer of $\frac{4}{3}$. The second integral gives us an answer of $-\frac{4}{3}$. So, ignoring the minus sign, the actual area of the integral is equal to $\frac{8}{3} = 2\frac{2}{3}$.

Teacher: Let's just quickly check that using Autograph.

Left-click to select the curve (it should turn black).

Add a point onto the curve with an x value of 0.

Add two more points, with x values of 1 and 3.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select the first two points: (0, 3) and (1, 0).

Note: They should both have little squares around them to show they are both selected.

Right-click and choose **Find Area** from the menu.

Select Simpson's Rule.
Alter the number of divisions to 500.
Click OK.

Note: As the curve is a parabola, Simpson's Rule would of course give the exact area under the curve if only one division was used. However, 500 divisions gives us a clearer visual representation of the area on the screen, and is important for what comes later...

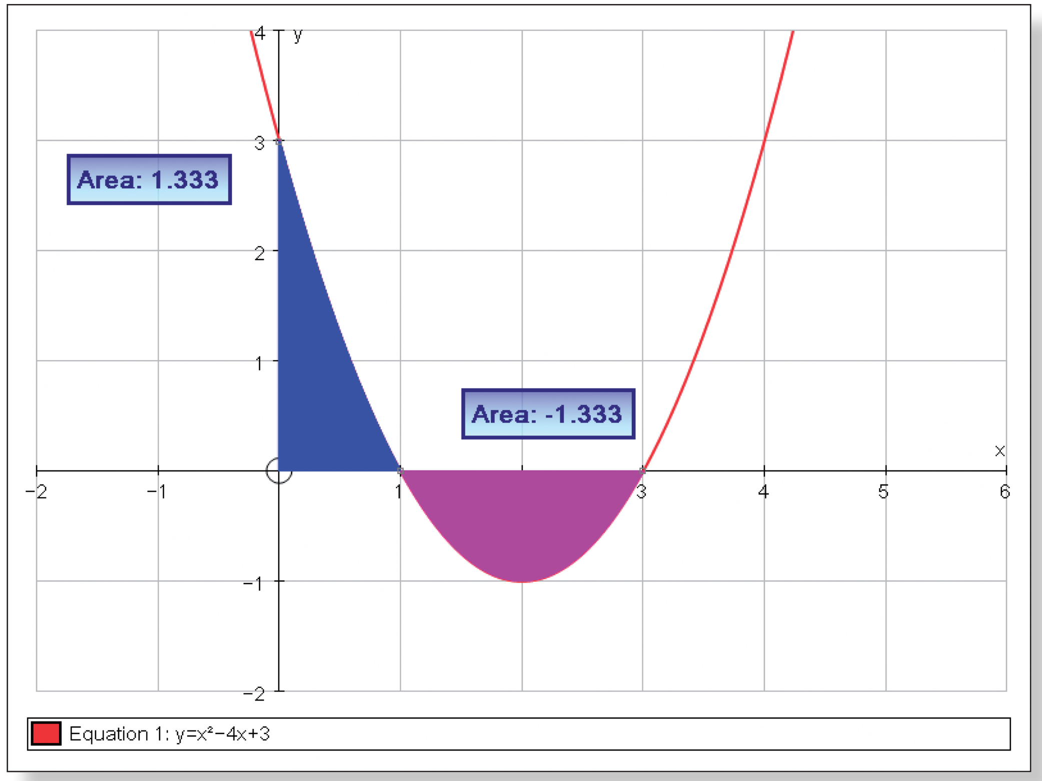
Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select the coloured in area underneath the curve (it should turn black).

Choose Text Box from the menu.
Click OK.
The value of the area under the curve (1.333) should now be displayed.
Note: To adjust the accuracy, go to Page > Edit Settings > Number of significant figures.

Repeat these steps but using the points (1, 0) and (3, 0).

When complete, your page should look something like this:



Teacher: I hope this example highlights how important it is to always do a quick sketch of any function before you try to integrate it. That way you can never get tripped up with negative areas!

At this point you could give the pupils some similar questions involving negative areas to practise.

For Example:

$$\int_{-1}^{2} 4x^3 \, dx \qquad \int_{0}^{2} x(x-1)(x-2) \, dx$$

Activity 2: Improper Integrals

(a) Unbounded Functions

Teacher: Have a go at trying to evaluate this integral:

$$\int_{0}^{2} \frac{1}{x^2} \, dx$$

Give the students a few minutes to attempt this question. It is crucial that they discover the difficulties for themselves, and then they are more open and receptive to solutions!

Prompt: What particular value of x is causing the problem, and why?

Ideal Response: The zero!

Teacher: As we said before, the key to sorting out any integral is to try and do a sketch first. Can anybody come up and sketch this function for positive value of x?

Prompt: What points do we know definitely lie on the curve? Think about what happens to y when x is really big. How about when x is really small?

Open up a New 2D Graph Page.

Edit the axes as follows:

x:	Minimum: 0	Maximum: 6	Numbers: 1	Pips: 1
y:	Minimum: 0	Maximum: 6	Numbers: 1	Pips: 1

Remove all of the green ticks underneath Auto.

Allow the students to attempt to sketch the curve on the grid using the Scribble Tool.

Use the Erase Tool to clear away any mistakes.

When the correct curve is drawn explain that we are now going to let the computer draw the curve more accurately:

Go to Edit in the top toolbar, then Select All Scribbles, and then press Delete on the keyboard.

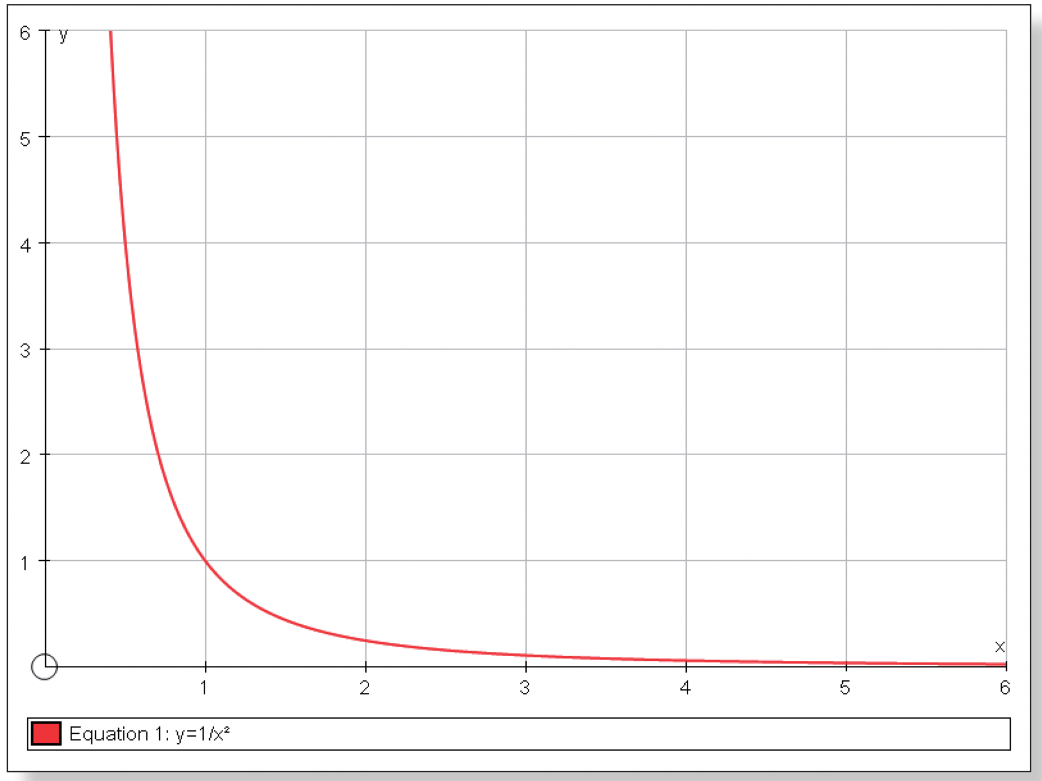
Again, ensure Slow Plot mode is turned on.

Enter the equation: $y = 1/x^2$
Note: you could also enter this equation as: y = x^(−2)

The curve should begin to appear on the screen.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process, or to resume it to focus on the key features of the graph.

Your screen should look something like this:



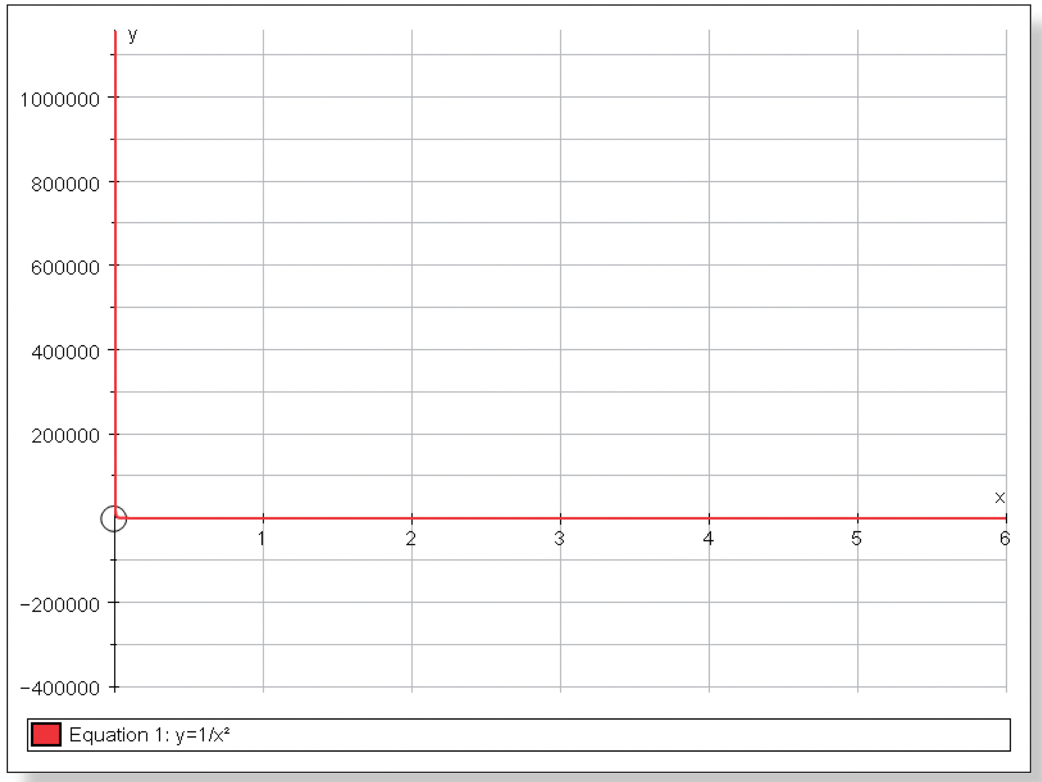
Teacher: Can anybody explain what is happening as we approach x = 0?

Ideal Response: The curve shoots off towards infinity!

Teacher: Yes! Let’s take a closer look at that…

Use the **Zoom Out y** function to have a look at larger values of y.
Notice how the scale automatically adjusts the more you zoom out.

Your screen should look something like this:



Press the **Undo** button several times to return to the original view of the graph.

Teacher: As we can see, the problem with this integral is that the function shoots off towards infinity as we get closer to x = 0. The technical way of saying this is that the function is unbounded, and we call integrals like this Improper Integrals.

Teacher: Can anybody predict what will happen to the area under the curve as we approach the value of x = 0?

Ideal Response: The area shoots off towards infinity as well!

Teacher: Well, let’s have a look…

Left-click to select the curve (it should turn black).

Add a point onto the curve with an x value of 2.

Add another point with an x value of 1.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click to select the two points.

Right-click and choose **Find Area** from the menu.
Select **Simpson’s Rule**.
Divisions: 500.
Click **OK**.

Left-click on an unoccupied part of the graph area to *de-select everything*.

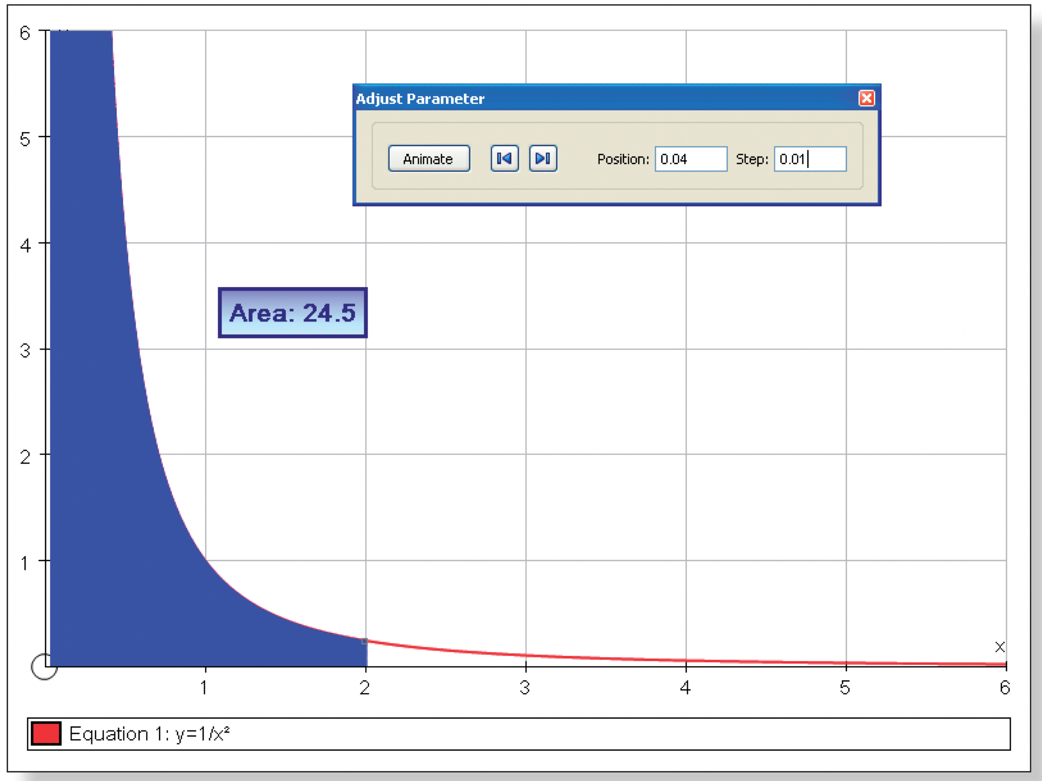
Left-click to select the coloured in area under the curve (it should turn black).

Choose **Text Box** from the menu and click **OK**.
The value of the area under the curve (0.5) should now be displayed.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click to select the first point on the curve (1, 1).

Select **Animate**.
Now, move the point closer to the value of 0 using the left arrow.
When the position reaches 0.1, adjust the step to 0.01 and continue.
Your screen should look something like this:



At this point you can either begin to teach the students how to deal with improper integrals with unbounded functions, or explain that you will tackle this matter later on.

(b) Unbounded Intervals

Teacher: Well, if that integral caused us problems, what on earth are we going to do with this one?

$$\int_2^{\infty} 1/x^2\,dx$$

Prompt: Refer them to the graph on the screen. What is happening to the curve as x gets bigger and bigger? Would the area under the curve between x = 5 and x = 6 be a lot? How about between x = 10 and x = 100?

Teacher: I wonder, will the curve ever touch or cross the x axis?

Ideal Response: No!

Teacher: Why not?

Prompt: Think about the function itself. Try putting some really big numbers into it. What do you notice?

Ideal Response: The function can never be negative because of the squared term, so it must never cross the x-axis. It just keeps getting closer and closer to it.

Teacher: Sounds good. Let's have a look…

Use the hand and zoom functions to investigate what is happening to the curve as x is getting bigger. Point out the scale on the y axis so the students are aware that the curve is getting so close to the x axis, but still is not touching.

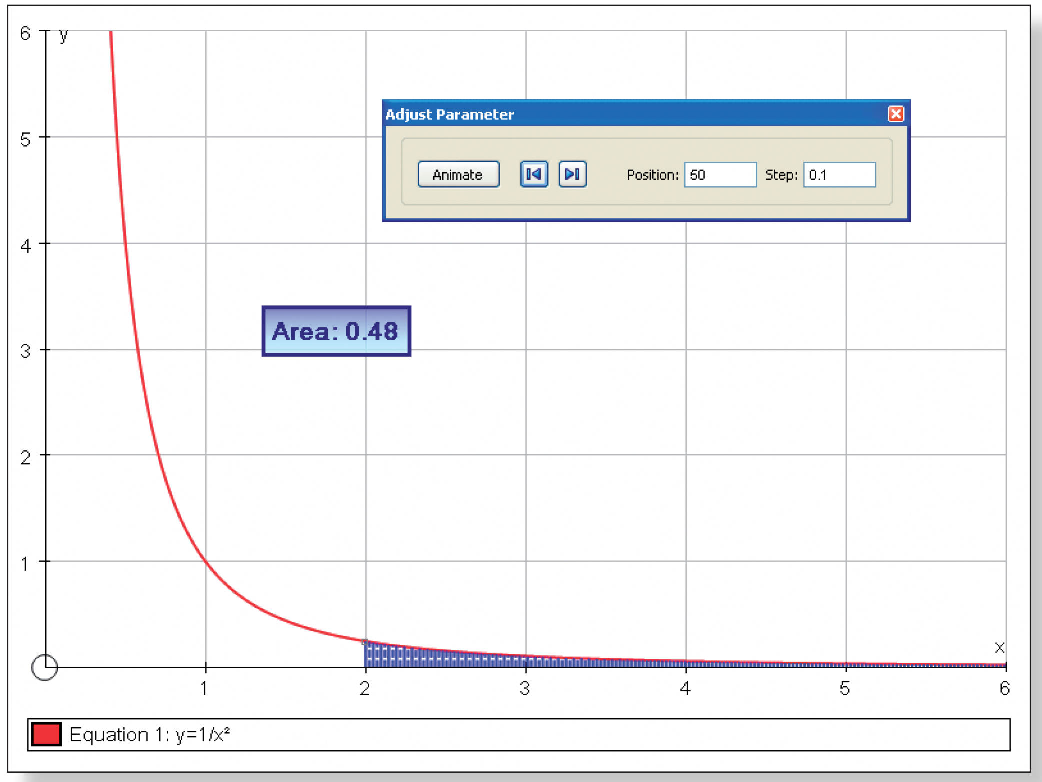
Return to the original view of the graph.

Note: Pressing the **Undo** button several times is often a quicker way of doing this.

Teacher: Believe it or not, even though one of the limits of our integral is infinite, we can still work out the exact value of the area under the curve. Watch what happens to the value of the area as we increase the upper limit.

Left-click twice on the area under the curve.
Adjust the number of divisions to 1,000.
Click **OK**.

Still using the **Animate** function, change the position of the first point to 2, and the step to 0.1.
Now keep pressing the right button so the upper limit increases.
Your screen should now look like this:



Continue doing this, perhaps adjusting the step, first to 1, and then to 10. Adjust the accuracy (**Page** > **Edit Settings** > **Number of significant figures**) to make the point even clearer.

Emphasise the fact that even though the size of the steps is increasing, the change in the area is actually falling.

Teacher: What value does the area seem to be converging to/tending towards?

Ideal Response: 0.5

Note: As **Simpson's Rule** is still only a numerical approximation, and as a maximum of 1,000 divisions are permitted, be careful not to go above x values of 1,000 as it appears that the area actually goes above 0.5!

Teacher: So, even though one of the limits of our integral did not have a finite value, we were still able to calculate the size of the area underneath the curve. The technical way of saying this is that the interval of the integral was **unbounded**, and this is the second type of **Improper Integral** which you could encounter.

At this point, you might like to repeat the above investigation but with the curve:

$$\int \frac{1}{\sqrt{x}}\,dx$$

This curve works the other way around – integrals approaching $x = 0$ can be evaluated, whereas those approaching infinity cannot.

Ideas for Further Work

- The students should now have a good understanding of the concepts of negative area and the two types of improper integrals. This should make it easier for them to learn and understand the techniques involved in solving questions like this.
- In terms of the two types of Improper Integrals, the students should now be in a good position to grasp the concept of limits, and hopefully enjoy success in what is a notoriously difficult and misunderstood concept.
- Investigations into infinite series follows on nicely from this topic.

LEARNING OBJECTIVES

- To be introduced to the concept of volume of revolution.
- To understand through a dynamic visual demonstration why the formula for finding the volume of a solid formed by rotating a function around the x-axis is: $V = \int_a^b \pi y^2 \, dx$
- To consolidate understanding of the concept of limits, sums and the skills of integration.

REQUIRED PRE-KNOWLEDGE

- To be able to integrate functions involving positive powers of x.
- To be able to use the answer to find the area bounded by the curve and the x-axis between two given limits.
- To be aware of the formula for finding the volume of a cylinder.
- To be comfortable with Sigma Σ notation.

PRE-ACTIVITY SET-UP

Open up Autograph in **Advanced Mode**.

Open up a **New 3D Graph Page**.

Select **Whiteboard Mode**.

Change the colour of the background to **Medium**.

Change the position of the key to the right-hand side of the screen.

Enter the equation: $y = x^2 + 2$
Note: To enter x^2, either use the little 2 button, or type "**xx**", or press "**alt 2**" together.
Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D Equation**.

Click **OK**.

Click on the little arrow next to **x-y-z Orientation**.

Select **x-y Orientation**.

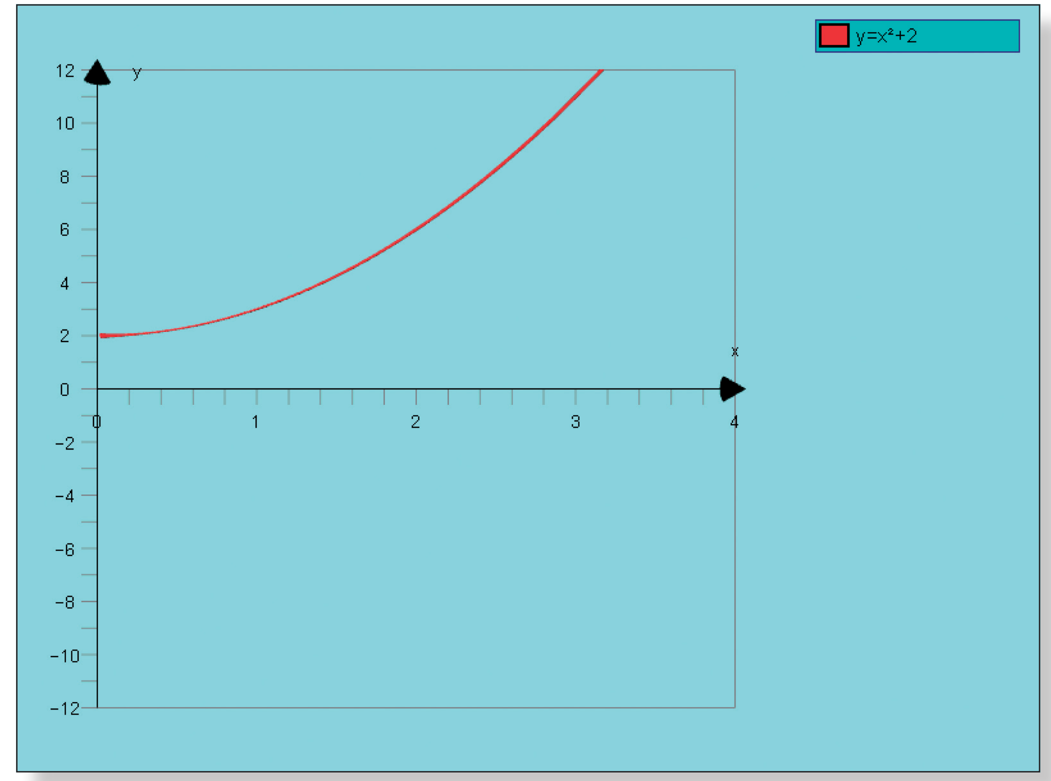
Edit the axes as follows:

x:	Minimum: 0	Maximum: 4	Numbers: 1	Pips: 0.2
y:	Minimum: −12	Maximum: 12	Numbers: 2	Pips: 1
z:	Minimum: −12	Maximum: 12	Numbers: 2	Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Still on the **Edit Axes** screen, click on the **Options** tab.
Under **Axes**, remove the tick next to **Always Outside**.
Click **OK**.

Your screen should now look something like this:



STEP-BY-STEP INSTRUCTIONS

ACTIVITY 1: WARMING UP

Teacher: Okay, on the screen I have drawn a portion of the graph of $y = x^2 + 2$. Just to warm you up, can you work out the area bounded by the curve and the x-axis between $x = 1$ and $x = 3$.

Prompt: What does "bounded by the curve and the x-axis" mean? How do we work out the area underneath a curve? How could we write what the question is asking us using notation? Does this help:

$$A = \int_1^3 x^2 + 2 \, dx$$

Ideal Response: We need to integrate:

$$A = \int_1^3 x^2 + 2\,dx$$
$$= \left[\frac{x^3}{3} + 2x\right]_1^3$$
$$= 12\frac{2}{3}$$

Teacher: Sounds good. Let's use Autograph to check the answer…

Left-click on the curve (it should turn black).

Enter a co-ordinate with an x value of 1 (leave z unchanged).

Enter another co-ordinate, this time with an x value of 3 (leave z unchanged).

Two points should now be marked on the curve.

Left-click on an unoccupied part of the graph area to *de-select everything*.

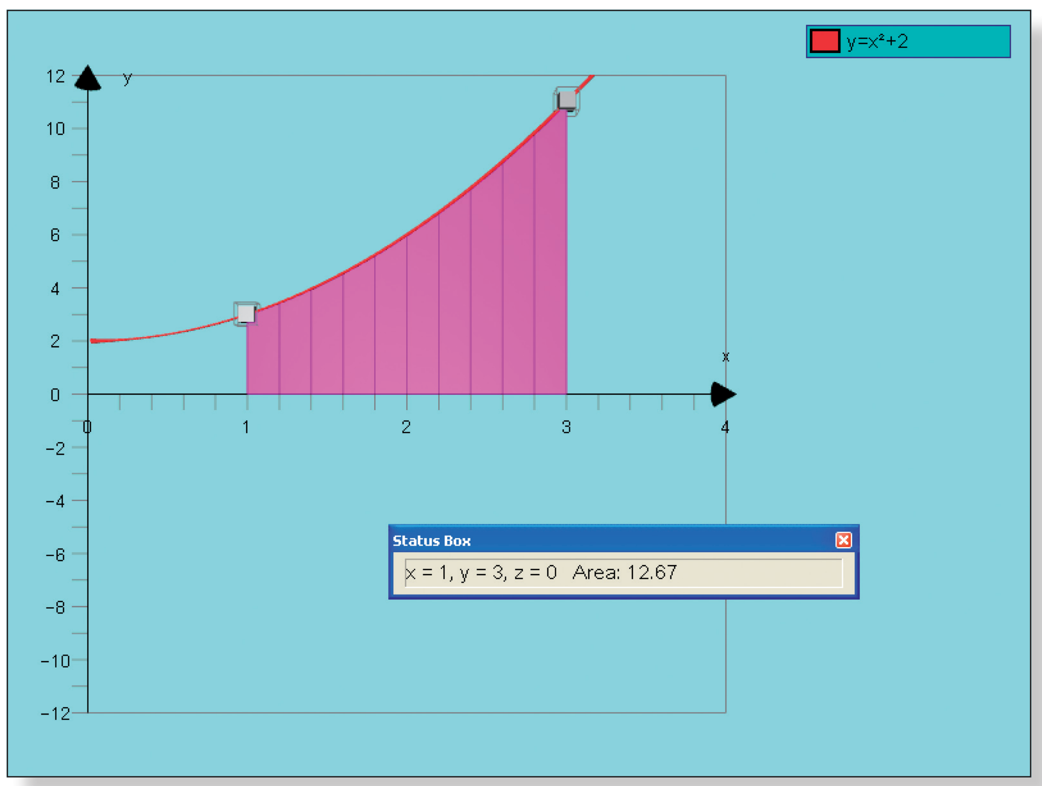
Left-click on both points (they should both have little squares around them).

Right-click and select **Find Area** from the menu.

Select **Simpsons Rule**, leave the number of divisions at 5, and click **OK**.

In the **View** menu on the top toolbar, select **View Status Box**, and move it to a convenient position on the page.

Your screen should look something like this:



Activity 2: Picturing the Third Dimension

Close the **Status Box** by clicking on the red cross in the corner.

Teacher: Okay, now the tricky part. Say that instead of just finding the area underneath the curve, we actually wanted to find the volume of the solid formed when rotating this curve around the x-axis. Now, to start with, does anybody have any idea what that solid might look like?

Prompt: Try to picture in your mind rotating the curve all the way around the x-axis. What sort of shape do we end up with?

Give the students time to discuss this and share their ideas before proceeding.

Teacher: Let's use Autograph to have a look…

Click on **Slow Plot** mode.

Left-click on an unoccupied part of the graph area to *de-select everything*.

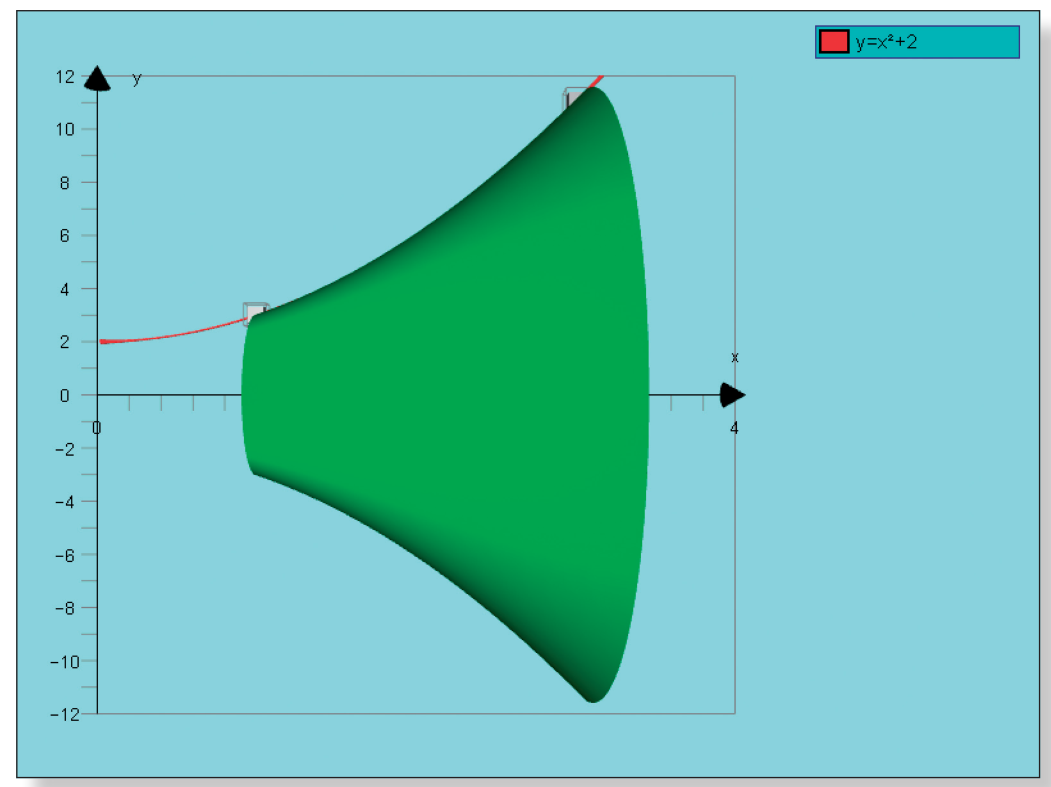
Left-click on the area underneath the curve (it should turn white).

Right-click and select **Find Volume** from the menu, and click **OK**.

The volume of revolution should now begin to plot.

If your students wish to see this again, simply press **Undo** and repeat the steps above.

Your screen should look something like this:



Use the **Drag** tool to have a good look around the graph to emphasise its shape.

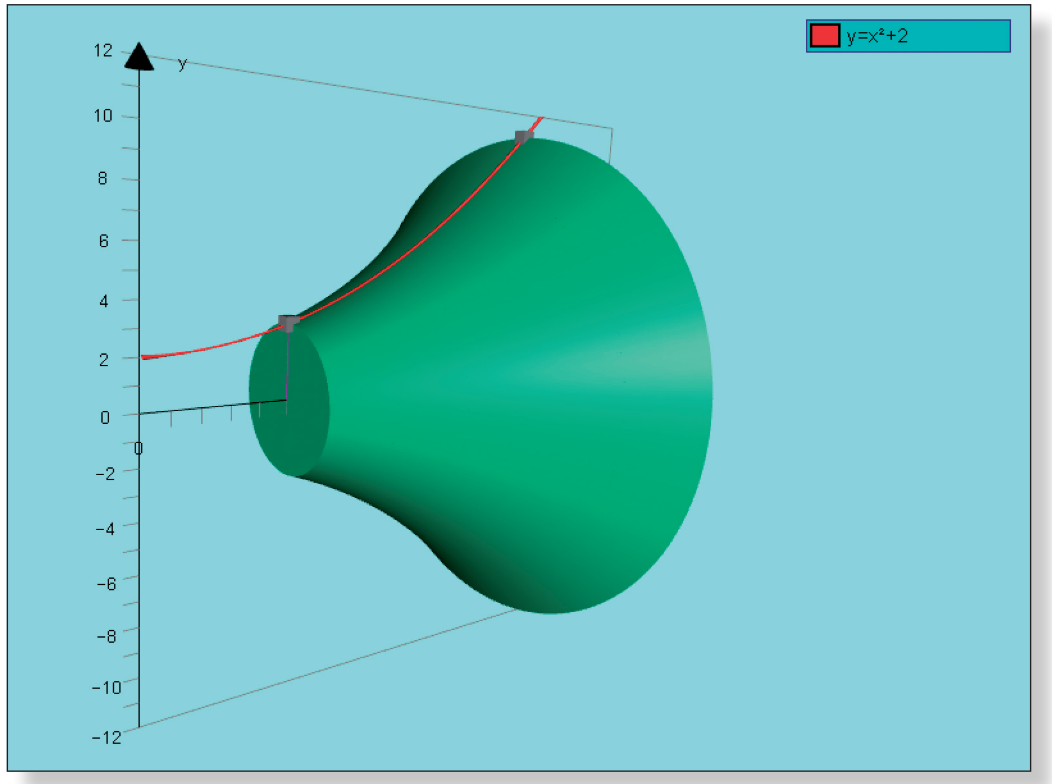
Note: The following functions may also prove useful:

+ **Ctrl** Zooms in and out.

+ **Shift** Moves the entire graph.

Restores the original view of the graph (**x-y Orinetation**).

After having a look around, your screen might look something like this:



Activity 3: Splitting Up the Shape

Select **x-y Orientation** to return to the original view of the graph.

Make sure you are in **Select Mode**.

Teacher: Okay, so having seen what the solid looks like, we now need to come up with a formula that allows us to work out its volume, and the way we are going to do this is very similar to how we came up with our method of differentiating functions in the first place…

Note: At this point it is possible to encourage a discussion about how the students might go about finding the volume, but it is unlikely they will be able to come up with the correct answer on their own. This does not matter too much as the real power of this demonstration comes from the students visualising and understanding this process that follows, and not necessarily coming up with the process themselves.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the lower half of the green volume (it should turn grey).

Right-click and select **Delete** from the menu, or simply press **Delete** on the keyboard.

You should now be left with the area under the curve marked on the graph.

Teacher: Okay, we are going to begin by looking at a smaller section of the curve to see if that helps us…

Left-click on the curve itself (it should turn grey).

Enter a co-ordinate with an x value of 1+h (leave z unchanged).

Teacher: Now, because the value of h is automatically set to 1, our new point has an x value of 2. Now once again we can mark the area underneath this curve between these two points…

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on both the point at x = 1, and the point at x = 2 (they should both have little squares around them).

Right-click and select **Find Area** from the menu.

Select **Simpsons Rule**, leave the number of divisions at 5, and click **OK**.

Teacher: And once again I can use to Autograph to show me the solid that is formed when rotating this portion of the curve around the x-axis…

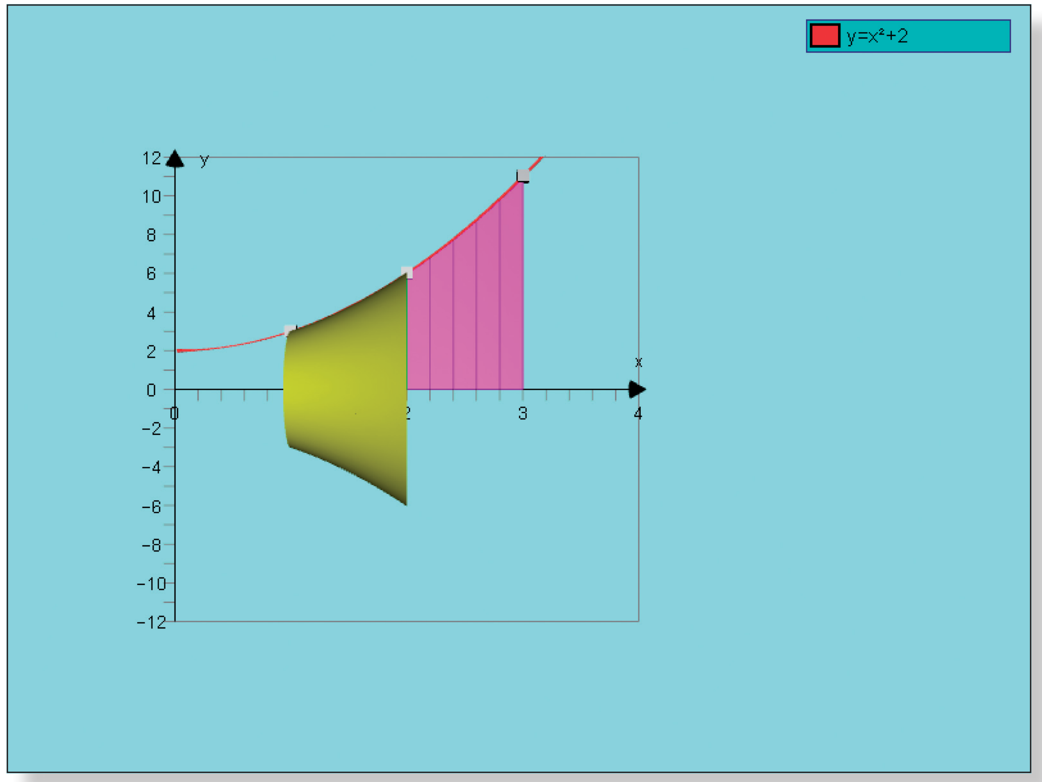
Make sure **Slow Plot** mode is still on.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the area underneath the portion of curve (it should turn white).

Right-click and select **Find Volume** from the menu, and click **OK**.

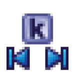
Your screen should look something like this:



Teacher: Now, in a moment I am going to reduce the value of h. As the value of h gets smaller and smaller, what shape will the yellow solid start to resemble?

Prompt: Try to picture in your mind the gap between the x co-ordinates reducing. Think what effect that will have on the solid. What shape will this look like when the gap is really, really small?

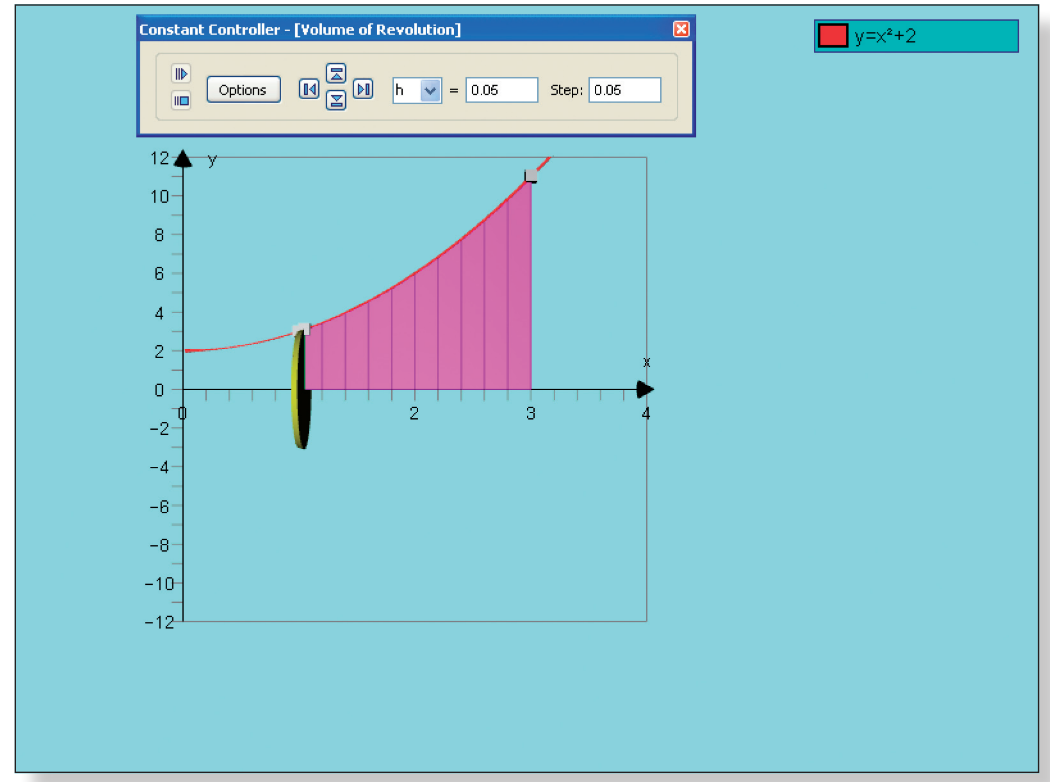
Ideal Response: A cylinder!



Click on the **Constant Controller**.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the value of the step.

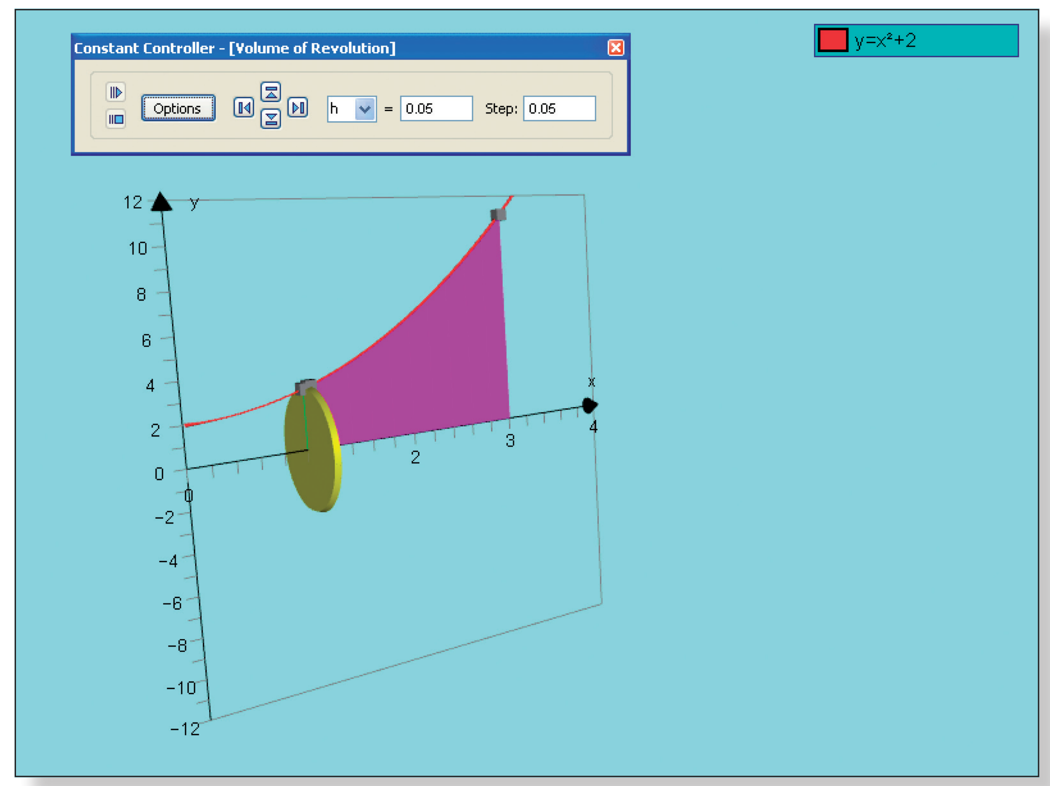
Reduce the value of h, reducing the value of the step as the point approaches x = 1.

Your screen should look something like this:



Use the **Drag** tool to have a good look around the graph to emphasise its shape.

Your screen should look something like this:



Teacher: Now comes the all important question: What is the volume of this cylinder?

Prompt: What is the formula for working out the volume of the cylinder? On our graph, what is the radius? What is the height?

Ideal Response: The formula for working out the volume of a cylinder is: $V = \pi r^2 h$. On our diagram, the radius of the cylinder is the distance from the x-axis to the circumference, which is given by the y value. The height of the cylinder is the distance along the x-axis, which is given by h. And so the formula for working out the volume of the cylinder is $V = \pi y^2 h$.

Teacher: Now this is not a *perfect cylinder*. As you can see from the graph, the curve slopes upwards, and so the radius of the left-hand face would be smaller than that of the right-hand face. But as the value of h gets smaller, and smaller and smaller, this difference reduces, and the solid formed becomes more and more like a cylinder.

Further reduce the value of h to emphasise this crucial point.

+ **Ctrl** Zoom in to have a closer look!

Give the students a few moments to digest this before moving on.

Teacher: Now what is also crucially important, is that we could have placed this cylinder *anywhere* on this curve, and the formula to work out it' volume would have been exactly the same…

Adjust the value of h to 1.5.

Make sure you are in **Select Mode**.

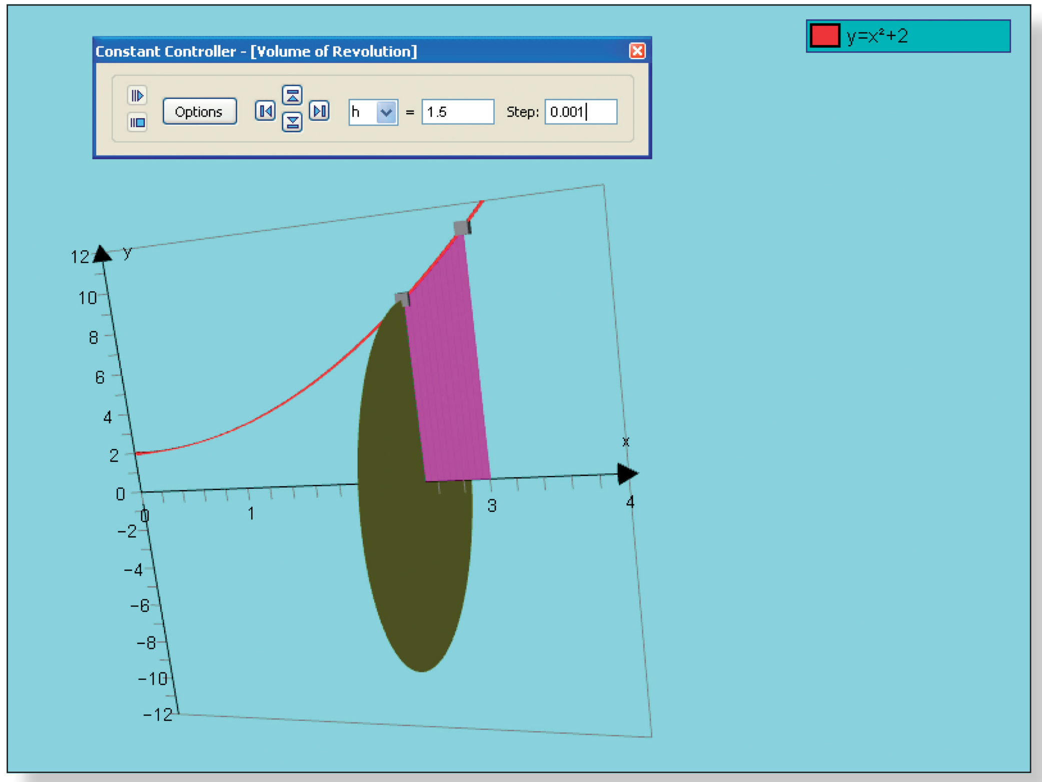
Left-click on the point at $x = 1$ (it should have a little square around it).

Use the **left-right arrows** on the keyboard to adjust the position of the point until it is at $x = 2.5$.

Use the **Drag** tool to have a good look around the graph to emphasise its shape.

Point out that the radius of the cylinder is still given by y, and the height by h.

Your screen should look something like this:



Teacher: Now, can anybody see how this helps us work out the volume of the funny-shaped solid that we started with?

Prompt: Think about the fact that the formula for working out the volume of the cylinder remains the same wherever we are on the curve.

Ideal Response: The volume of the funny-shaped solid was just made up of loads and loads of cylinders. If we can work out the volume of those and add them all together, then we have the volume of the funny-shaped solid!

Activity 4: Deriving the Formula

Teacher: Excellent. Now, this bit can be a little tricky to understand, so give it your full attention… Instead of calling the tiny distance along the x-axis h, we are going to call it dx. This is the same dx that we know and love from our work on differentiation and integration. What does it mean?

Ideal Response: A small change in x.

Teacher: Good. And remember, just like with differentiation, the change is so small you can hardly see it, which is exactly what we need in order to make our series of solids resemble cylinders. So, bearing that in mind, what is the volume of each cylinder?

Ideal Response: $V = \pi y^2\, dx$

Teacher: Good. And would you agree that the area of the funny-shaped solid approximately equal to the sum of the volume of all these cylinders, which can be written as: $V \approx \sum_{x=1}^{x=3} \pi y^2\, dx$?

Prompt: You may need to explain that the equation is only an approximation (hence no = sign) because of the sloping nature of the curve as discussed earlier.

Teacher: Good. Now, the smaller the size of dx, the more accurate our approximation of the volume. The exact size of the volume is given by the limit as $dx \rightarrow 0$, and we are left with this…

$$V = \lim_{dx \rightarrow 0} \sum_{x=1}^{x=3} \pi y^2 \, dx = \int_1^3 \pi y^2 \, dx$$

Give the students time to digest this.

Teacher: In other words, to find the volume of the solid formed when a function has been rotated around the x-axis, you must square the function, multiply by π, and then integrate as normal. Do you reckon you could use this formula to work out the volume of the funny-shaped solid?…

Prompt: It might be worth holding back on pointing out that the calculation is made easier by placing π before the integral sign. Hopefully the students will discover this for themselves once they have had more experience with these types of questions.

Ideal Response:

$$V = \int_1^3 \pi y^2 \, dx$$

$$= \pi \int_1^3 x^4 + 4x^2 + 4 \, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_1^3$$

$$= 91\frac{1}{15}\pi$$

Teacher: Sounds good. Let's use Autograph to check the answer…

Make sure you are in **Select Mode**.

Adjust the value of h to 2.

Select **x-y Orientation** to return to the original view of the graph.

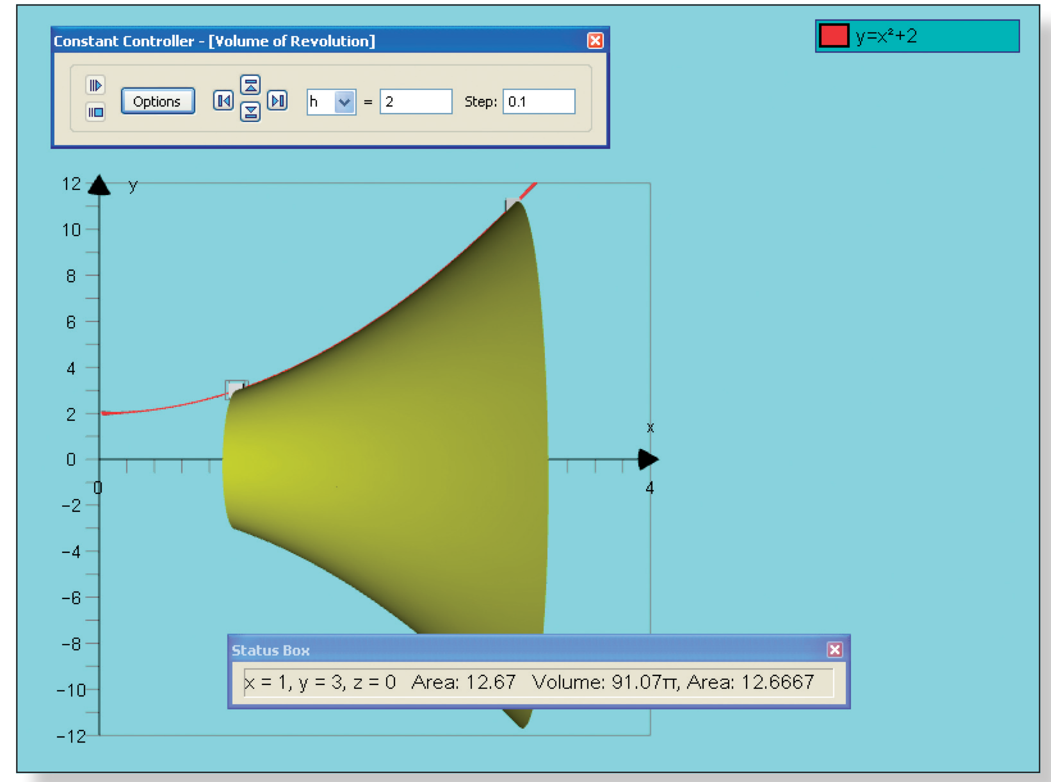
In the **View** menu on the top toolbar, select **View Status Box**, and move it to a convenient position on the page.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point at x = 2.5 (it should have a little square around it).

Use the **left-right arrows** on the keyboard to adjust the position of the point until it is back at x = 1.

The correct volume should be displayed in the **Status Box**, and your screen should look something like this:



Ideas for Further Work

- Further practice the volume of the solid formed by rotating functions around the x-axis.
- Introduce finding the volume of functions rotated around the y-axis. **Note:** This can be clearly demonstrated in Autograph using a similar method as described above.
- Introduce finding the volume of solids formed by rotating the area between two intersecting functions around either axes.
- To consolidate and deepen understanding of volumes of revolution, try Student Investigation 2: Volumes of Revolution.

Learning Objectives

- To appreciate that the gradient of a curve is not constant like a straight line.
- To be able to understand the concept of approximating the gradient at a point using limits, leading onto Differentiation from First Principles.

Required Pre-Knowledge

- To be able to work out the gradient of a straight line using the right-angled triangle method ("change in y divided by change in x").
- To know the shape of the $y = x^2$ graph.
- To be comfortable manipulating simple algebra, such as multiplying brackets and cancelling terms.

Pre-activity Set-up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

At the top of the screen go to **Page** > **Edit Settings**, and adjust the number of significant figures up to 8. This will increase the accuracy of our calculations.

Select **Whiteboard Mode**.

Edit the axes as follows:

x:	Minimum: −6	Maximum: 6	Numbers: 1	Pips: 0.5
y:	Minimum: −4	Maximum: 25	Numbers: 2	Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Enter the equation: $y = 8$

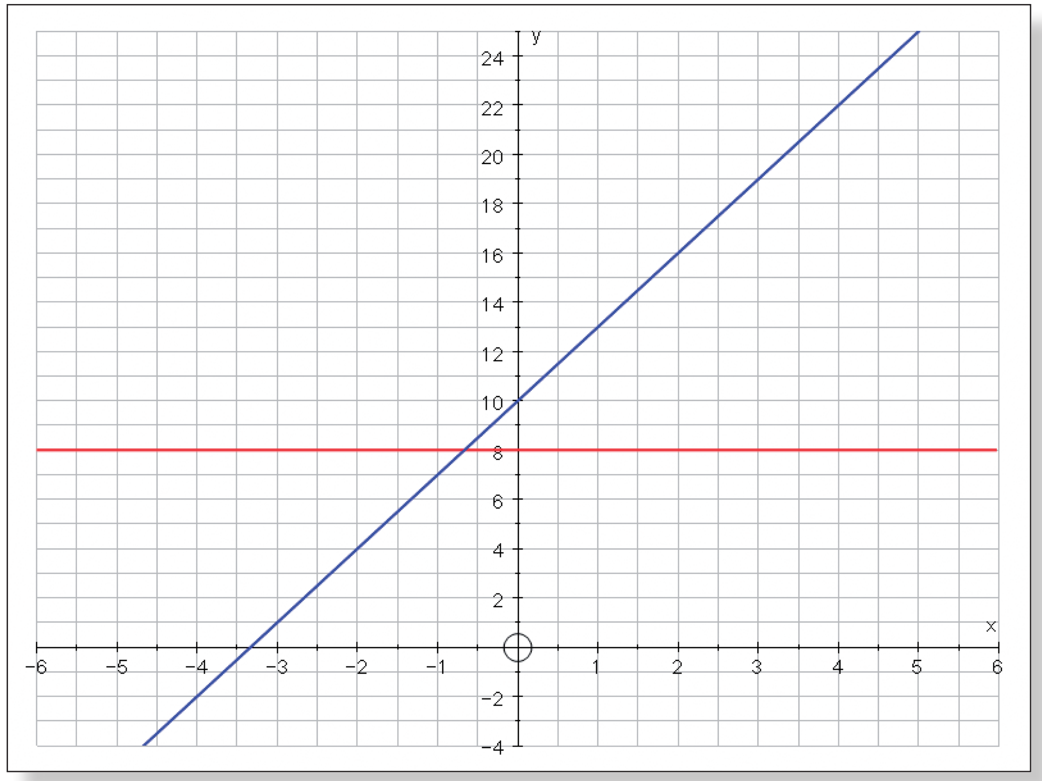
Enter another equation: $y = 3x + 10$

At the top of the screen go to **Axes** > **Show Key**.

This should make the key at the bottom of the screen disappear.

Note: This can also be done by **right-clicking** on the **Key** towards the bottom of the screen where it says "Equation 1: y = 8", and from the menu **left-click** on **Show Key**.

Your screen should look like this:



Step-by-Step Instructions

Activity 1: A Reminder about Gradient

Teacher: Just to warm you up, what is the equation of the *red* line on the screen?

Prompt: Think about what every single co-ordinate on that line has in common.

Ideal Response: $y = 8$

Teacher: And what is the *gradient* of the *red* line?

Ideal Response: 0

Teacher: And is that true for every single point on that line?

Ideal Response: Yes!

Teacher: Okay, so how about the *blue* line? How would we work out the *gradient* of that?

Prompt: What does gradient actually mean? What makes one line steeper than another?

Ideal Response: We need to work out the slope of the line. We need to draw a right-angled triangle / Work out how much the y values change as the x values change.

Teacher: Can somebody come to the board to show us how to do that?

Encourage the students to work out the gradient of the line on the screen using the **Scribble Tool**.

Use the Erase tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > Select all scribbles, and press **Delete** on the keyboard (or **right-click** on the graph itself and select Delete Objects from the menu).

Ideal Response: The gradient of the line is 3.

Teacher: And is the gradient of the *blue* line always 3 no matter where you are on that line? In other words, is it true that the gradient is always constant?

Ideal Response: Yes!

Teacher: Let's just use Autograph to quickly show that…

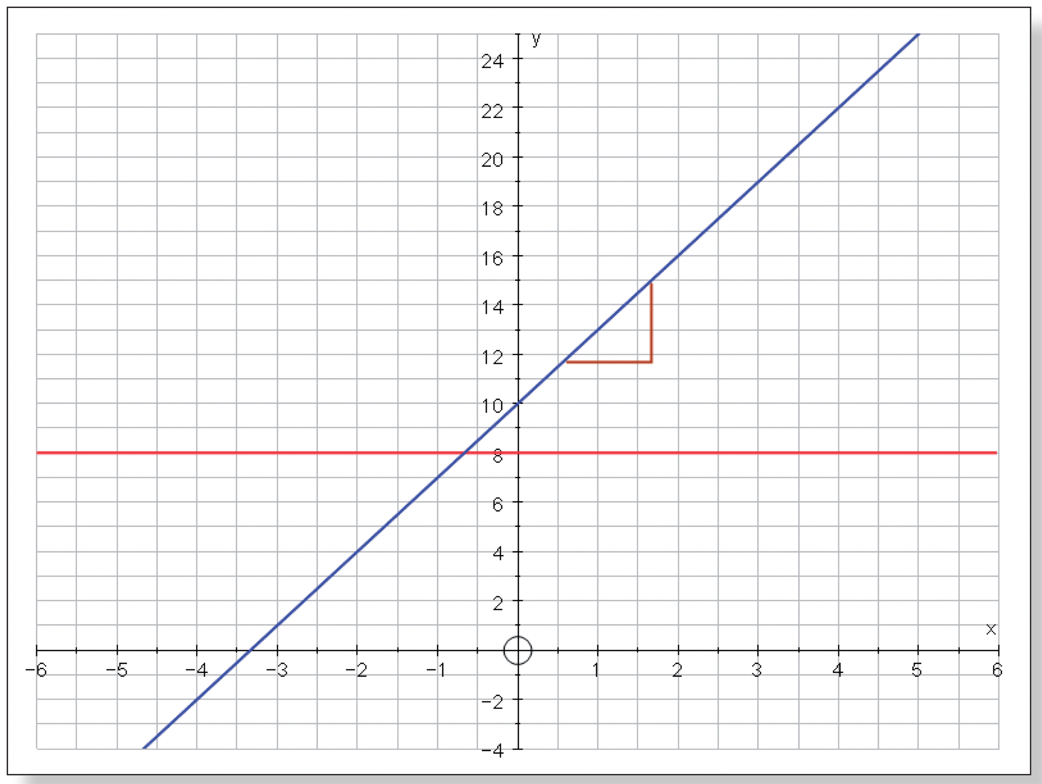
Delete all scribbles as described above.

Draw a "gradient right-angled triangle" with base 1 and height 3.
Note: You must make sure you draw this as a continuous line, so don't take the pen off the screen!

Left-click to select the triangle (it should turn black)

Whilst holding down the left mouse button, you can now move the triangle anywhere along the line, and clearly demonstrate that the gradient is always constant.

Your screen should look something like this:



Teacher: And just before we move on, what is the equation of the *blue* line?

Prompt: Where does the line cross the y-axis?

Ideal Response: $y = 3x + 10$

Activity 2: The Gradient Function of a Quadratic

Go to Edit in the top tool bar, then Select All, and then press **Delete** on the keyboard.

Both lines and all sets of scribbles should now have disappeared, leaving you with the set of axes again.

Teacher: Can somebody quickly describe what the graph of $y = x^2$ looks like?

Click on Slow Plot mode.

Enter the equation: $y = x^2$
Note: To enter x^2, either use the little 2 button or press "xx".
Click OK.

The curve should begin to appear on the screen.

Press Pause Plotting both to stop the process, or to resume it to focus on the key features of the graph.

Note: Pressing the spacebar on the keyboard has the same effect!

Teacher: What do you notice about the gradient of this curve, $y = x^2$?

Prompt: Think about the slope of the line on the left-hand side of the graph, compared to the slope of the line on the right-hand side of the graph.

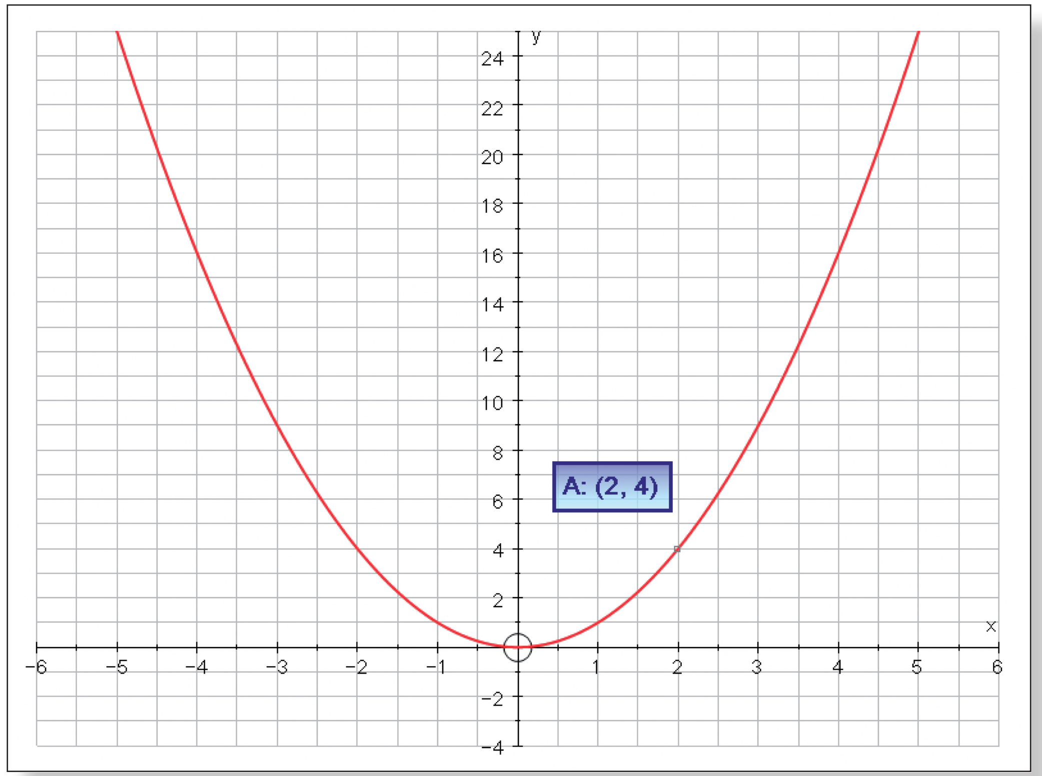
Ideal Response: The gradient is always changing / As you move from the left-hand side of the graph to the right, the gradient goes from negative to positive / The gradient gets steeper from left to right from the origin.

Left-click on the curve (it should turn black).

Enter the point with an x value of 2.
This should display the point (2, 4) on the curve.

Select Text Box.
Change the word "Point" to "A" and click OK.
The point should now be labelled.

Your screen should look something like this:



Teacher: Does anybody have any idea how we might work out, or at least get an approximation, the gradient of the curve at this particular point?

Prompt: Think about how we worked out the gradient of the straight line before.

Note: If, at this point, a student suggests drawing a *tangent* to the curve and working out the gradient of that line, explain that this is an excellent suggestion, and indeed it is ultimately the correct answer, but the only problem is that it is hard to know exactly where to draw the tangent. You could even demonstrate with a ruler or something else with a straight edge that you could quite reasonably argue several different locations for such a tangent. Explain that the method we are going to use actually tells us exactly where to draw that tangent and so takes all the guess work out of it!

Ideal Response: Try drawing the right-angled triangles like before.

Make sure the curve is still selected (it should still be black).

Enter another point, this time with an x value of 3.
This should display the point (3, 9) on the curve.

Select **Text Box**.
Change the word "Point" to "B" and click **OK**.
Both points and their co-ordinates should now be labelled.

Teacher: Okay, if I was to draw a straight line between points A and B, what would be the gradient of the line?

Prompt: Picture the right-angled triangle in your mind / think about how much the y value has changed and how much the x value has changed.

Ideal Response: 5.

Teacher: Good, now to save us having to work out the gradient each time, I am going to let Autograph do it.

Left-click on an unoccupied part of the graph area to *de-select everything*.

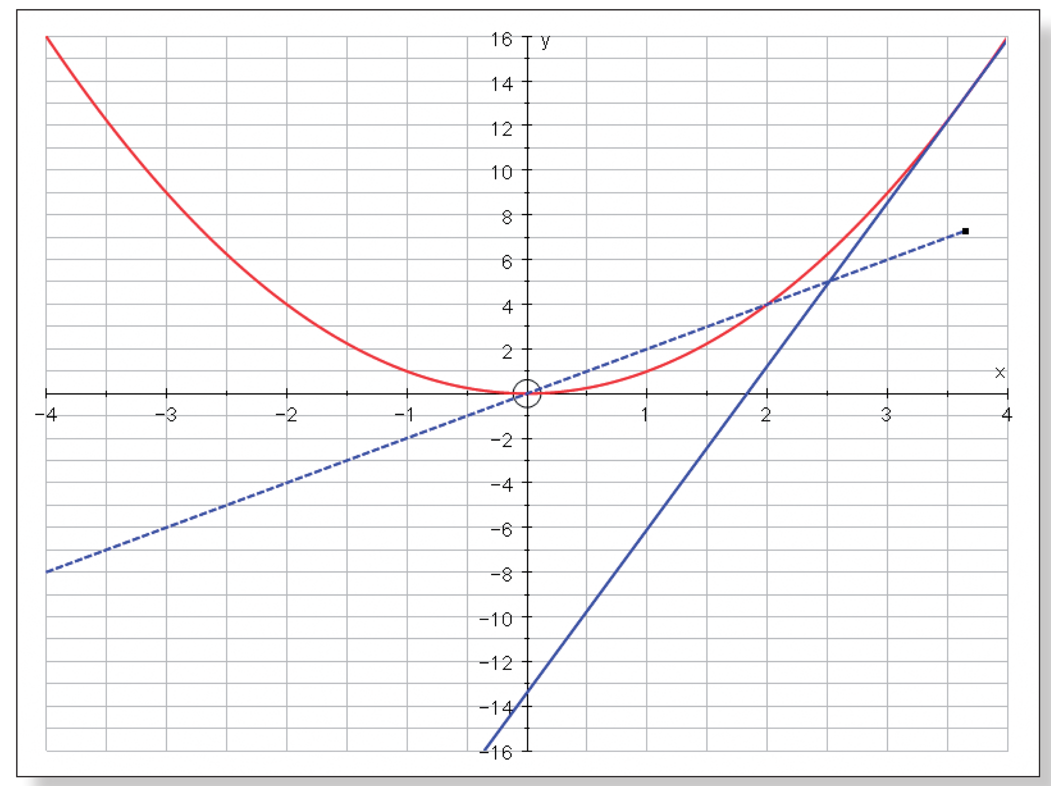
Left-click on points A and B to select them (they should both have squares around them).

Right-click and select **Gradient** from the menu.
This should create a right-angled triangle between the two points.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the right-angle triangle (it should turn black).
From the top menu click on **View**, and then **Status Box**.
A **Status Box** containing important information about the straight line connecting those two points should now be displayed.
Note: The **Status Box** only displays information about objects which are currently selected. If the information ever disappears, just **left-click** to select the object in question again.

Your screen should look something like this:



Teacher: Okay, just to explain what information this **Status Box** is showing:

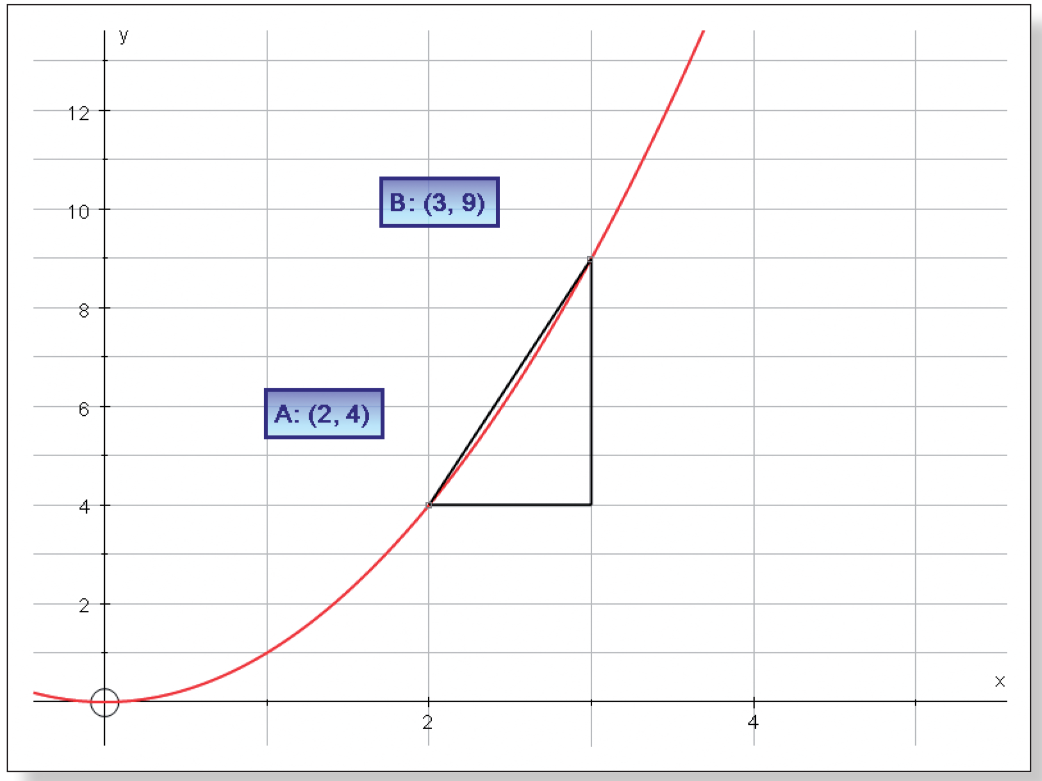
Δy means how much the y value has changed going from Point A to Point B.
Δx means how much the x value has changed going from Point A to Point B.
The first number is the gradient of this line, which is derived from dividing the

change in the y value by the change in the x value.
So, the good news is that Autograph agrees with our calculation for the gradient of the line between A and B.

Teacher: Now, whilst the line connecting A and B is a pretty good approximation of the gradient at the curve at Point A, it is not brilliant. As you can see, if we zoom in a bit closer, there is quite a big gap between the curve and the line.

Use the Drag and Zoom In functions to get a closer look at what is going on.
Notice how the scale automatically adjusts the closer in we get.

Your screen should look something like this:



Use the Drag and Zoom Out functions to return to the original view of the graph.

Note: Pressing the Undo button several times is often a quicker way of doing this.

Teacher: Can anybody think of a way in which we could improve our approximation of the gradient of the curve at Point A?

Prompt: Remember, we want the gradient at Point A. Point B is quite a long way away...

Ideal Response: Move point B closer!

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select Point B (it should have a little square around it).

Select Animation from the top toolbar.

You can now control the position of Point B with the **left-right arrows**.
Set the value of the step to 0.1.
The Status Box keeps a record of the current x and y values of Point B and the value of the gradient.

Note: A nice shortcut is to use the arrow keys on the keyboard. The **left-right** control to position of point B, and the **up-down** control the size of the step

Move Point B closer and closer to Point A.
Note: You may have to adjust to position of the text boxes to stop them getting in the way.

Draw the students' attention to the value of the gradient.
Stop when point B has an x value of 2.1.

Teacher: What do you think will happen to the value of the gradient if I move Point B right on top of Point A?

Prompt: Think about how we work out the value of the gradient. What will the change in y and the change in x be now? What happens when you divide by zero?

Ideal Response: The gradient is undetermined / incalculable / infinite / doesn't make sense.

Teacher: So, how can we keep improving our approximation to the gradient of the curve at Point A without the gradient becoming undefined?

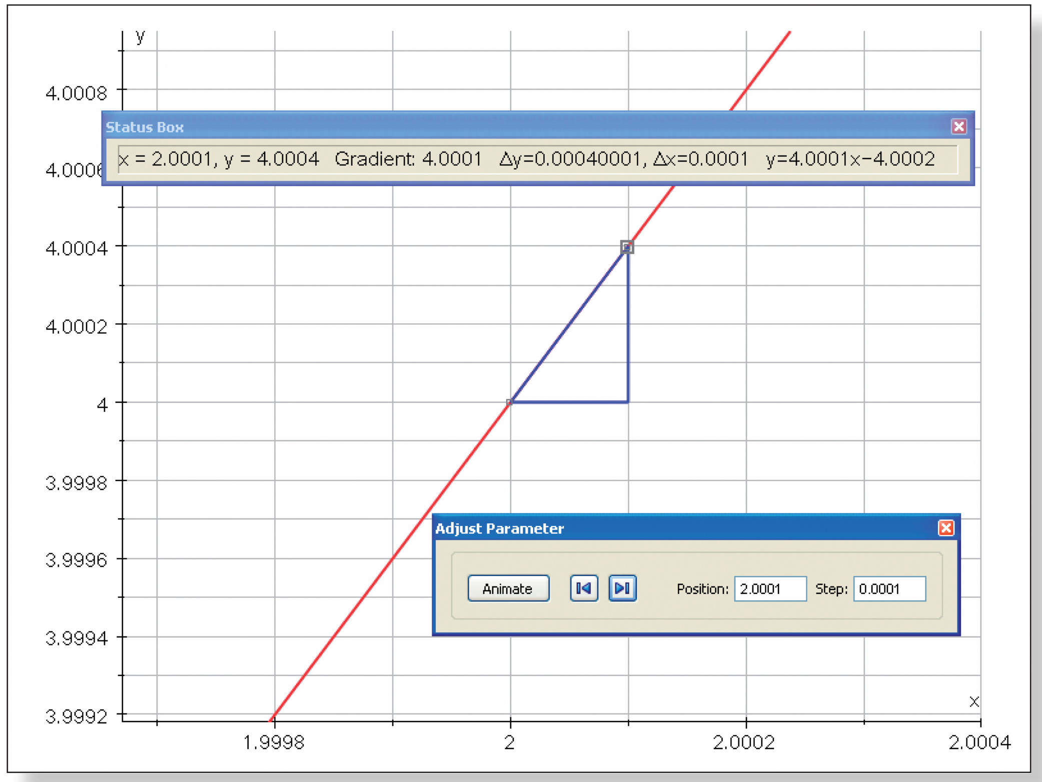
Ideal Response: Make the step size smaller.

Adjust the size of the step to 0.01.

Use the Drag and Zoom In functions to get a closer look at what is going on.
Notice how the scale is adjusting as the numbers we are dealing with get smaller and smaller.
Note: The text boxes containing the co-ordinates of Points A and B will disappear, but the Animation Controller and the information contained in the Status Box will remain visible no matter how much you zoom in.

Continue adjusting the step and zooming in.
Really emphasise the scale on the axes, so the students appreciate just how small these distances really are.

Your screen should look something like this:



Teacher: We are now working out the gradient between two points which are an incredibly small distance apart. Point B is now incredibly close to Point A, but crucially it is not quite at Point A. The closer we get, the better the approximation of the gradient at Point A. Of course, we could keep going closer and closer, but what number does the value of the gradient seem to be approaching?

Ideal Response: 4!

Teacher: Yes, and in fact it is possible to show, by using a technique that follows directly on from what we have been doing here, that the gradient of the curve $y = x^2$ at the point where $x = 2$ is exactly equal to 4. This technique is the foundation upon which many incredibly important areas of maths are built upon. All we need to do is to imagine that the distance between the x values of Point A and Point B is as small a number as we can imagine, and we can call that number h…

Ideas for Further Work

- This demonstration naturally lends itself to be followed up by a demonstration of *Differentiation from First Principles*. In fact, the necessary right-angled triangle is already on the screen, and so you could use the scribble function to label the base of the triangle "h", Point A as (2, 2), Point B as $(2 + h, (2 + h)2)$ and so on.
- Another nice way of doing this demonstration on Autograph is to define $f(x)$ to be whatever you like, and then plot $y = \frac{f(x + h) - f(x)}{h}$. Decreasing the value of h from here has the same effect.

- Alternatively, you could skip this step and just explain that differentiation is a way to reduce the distance between the points to as small as it could be, and then teach the students the rules of differentiation from there.
- Student Investigation S5 in the sister textbook is also a nice follow-up to this introductory work.

Learning Objectives

- To re-enforce the concept of the Gradient Function.
- To understand how the gradient functions of trigonometric functions can be derived by inspection and by plotting the gradient of tangents.
- To interactively find the gradient functions of y = sin(x), y = cos(x) and y = tan(x), and work out their equations.

Required Pre-Knowledge

- To know the shape of the trigonometric functions: y = sin(x), y = cos(x) and y = tan(x).
- To be comfortable working in radians.
- To understand the concept of a tangent.
- To be comfortable with the concept of the gradient function and its role in differentiation.

Pre-Activity Set-Up

Open up Autograph in Advanced Mode and ensure you have a blank 2D Graph Page.

For this activity, you will need three identical 2D Graph Pages, each set up as follows:

Page - 1:

Select Whiteboard Mode.

Ensure you are working in Radians.

Edit the axes as follows:

x:	Minimum: −4π	Maximum: 4π	Numbers: π	Pips: π/3
y:	Minimum: −3	Maximum: 3	Numbers: 1	Pips: 0.5

Note: To enter the π symbol, press "Alt p" at the same time or type "pi".
Remove all of the green ticks underneath Auto.
Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Still in the Edit Axes menu:
Click on the Appearance tab.
Open the drop-down menu underneath Themes.

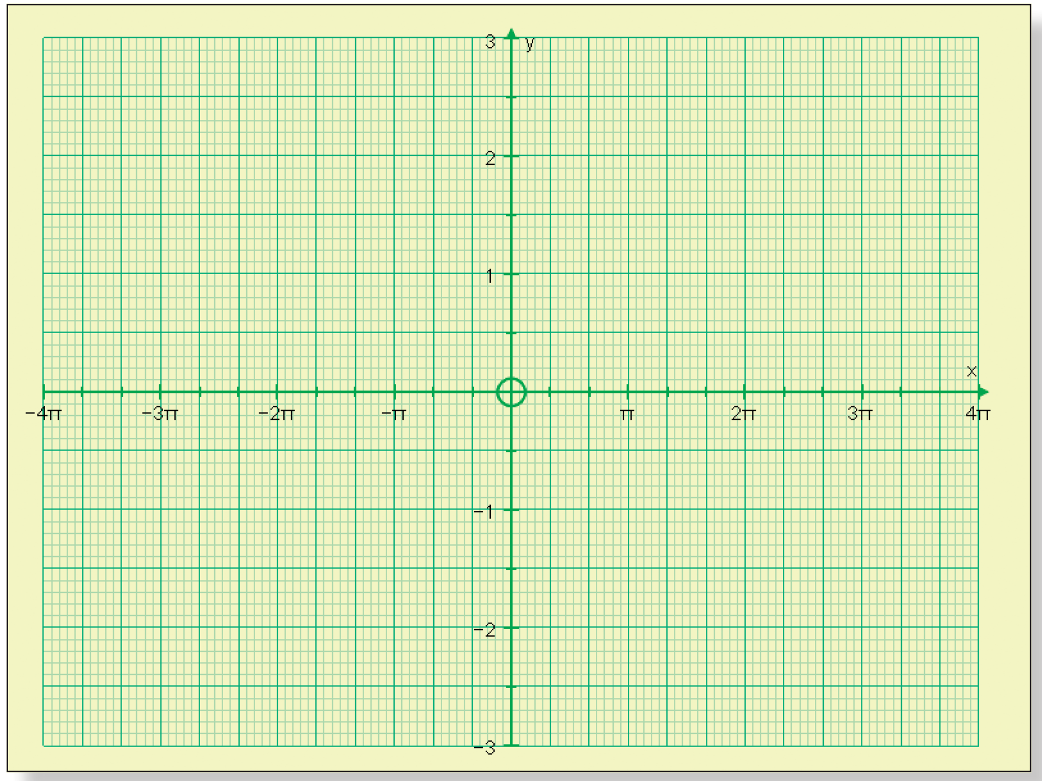
Select Graph Paper.
Click OK.

From the Axes menu on the very top toolbar, select Show Key.

This should make the Key disappear.

Open another two blank 2D Graph Pages (Page - 2 and Page - 3) and repeat the set-up instructions.

Your three pages should each look something like this:



Your three pages are now available as Tabs on the top of the screen and can be accessed any time by simply left-clicking on them.

Step-by-Step Instructions

Activity 1: y = sin(x)

Open Page - 1 by clicking the Tab.

Note: If you wish to re-name this or any other page, simply click on Page > Edit Settings and write in a name of your choice.

Teacher: Okay, to warm you up, can anybody come to the front and do a quick sketch of y = sin(x).

Prompt: Notice we are working in Radians. Does the graph go through the origin? What is the period of the graph? What is the amplitude? Where does it cross the axes?

Encourage students to come to the front to sketch their curves using the **Scribble Tool**.

Use the **Erase** tool to rub out any mistakes

If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press delete on the keyboard (or **Right-click** on the graph area itself and select **Delete Objects** from the menu).

When you are ready:

Click on **Slow Plot** mode.

Enter the equation: $y = \sin(x)$

The curve should begin to be drawn on the screen.

Press **Pause Plotting** both to stop the process and to resume it to focus on the key features of the graph.

Note: The **Spacebar** can also be used to serve this function.

Teacher: Okay, so how would we go about finding the **Gradient Function** of $y = \sin(x)$?

Prompt: What does the gradient function mean? How can drawing lots of tangents help us come up with the **Gradient Function**?

Ideal Response: The **Gradient Function** is just the gradient of the tangent to the curve at each point along the curve. So, if we could draw tangents at lots of different points along the curve and plot their gradients, then we could gradually build up the gradient function of $y = \sin(x)$.

Teacher: Good. Now, before we start, can anybody picture in their minds what kind of shape the gradient function of $y = \sin(x)$ will take?

Give the students a few minutes to think about this and discuss it, before moving on.

Left-click on the curve (it should turn black).

Enter a point on the curve with x co-ordinate 0.

The point (0, 0) should now appear on the curve.

Click on **Text Box**.
Change the word "Point" to "A" and click **OK**.

The co-ordinates of point A should now be displayed.

This will allow us to keep an eye on the x value of our point.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on point A (it should have a square around it).

Right-click and select **Tangent** from the menu.

A tangent to the curve at point A should now be displayed.

Teacher: Looking at the tangent, can anybody have a guess at its equation?

Prompt: What points does it go through? What is its gradient?

Ideal Response: $y = x$.

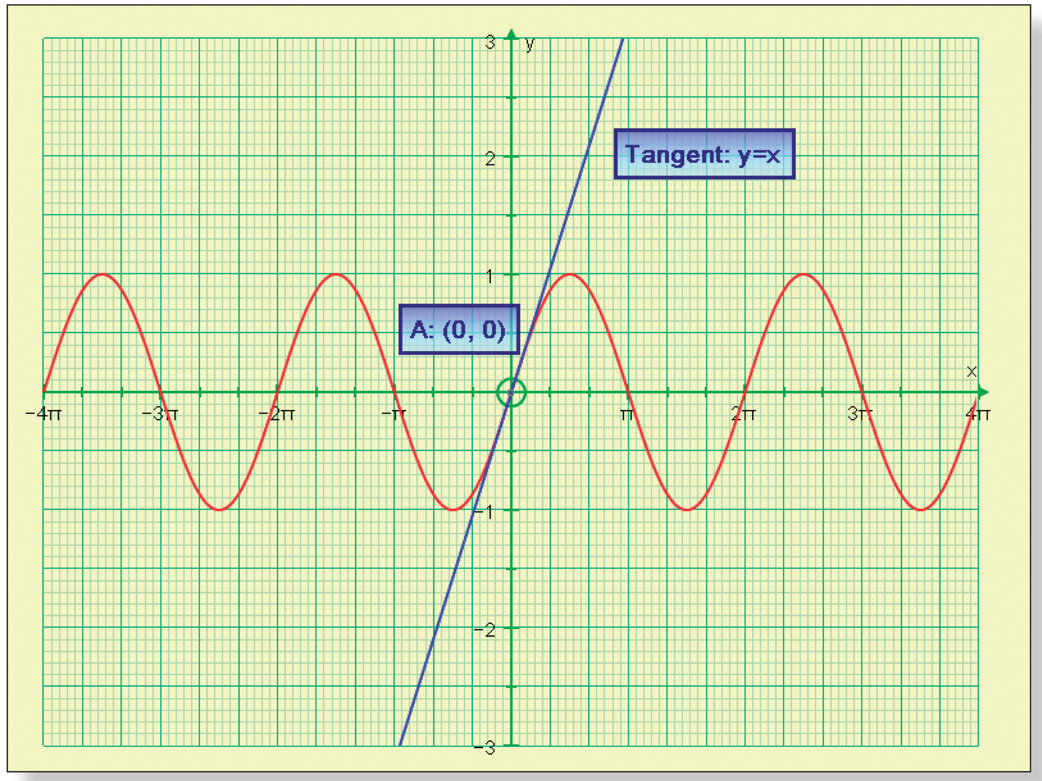
Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the tangent (it should turn black).

Click on **Text Box**.
Click **OK**.

The equation of the tangent should now be displayed.

Your page should look something like this:



Teacher: What is the gradient of $y = \sin(x)$ at the point (0, 0), and how can you tell from the equation of the tangent?

Ideal Response: The gradient is 1, because the co-efficient in front of the x in the equation of the tangent is 1.

Teacher: Good, so when $x = 0$, the gradient is also equal to 1, so I can place a point on my curve at (0, 1).

Use the **Scribble** tool to mark the point on the graph.

Teacher: Okay, so now I need two volunteers to come to the front and help us discover the **Gradient Function** of y = sin(x). One of you will be in charge of moving the position of point A, and the other will be in change of plotting the gradient.

Left-click on an unoccupied part of the graph area to *de-select everything.*

Left-click on point A (it should have a square around it).

Open the **Animation Controller**.

Teacher: Now, just before we start, how many points shall we plot, and which ones in particular shall we plot?

Prompt: Which points are important to ensure we get the shape of our gradient functions?

Ideal Response: Perhaps take the points at regular intervals, maybe every π/3. But we must also make sure we take the gradient at each turning point as this will be important in determining the shape of our gradient function.

Teacher: Excellent. So, off you go…

Step-by-Step Instructions

The first student can use the **left-right** buttons on the **Animation Controller** to move Point A along the curve.

They can adjust the step to whatever they like, remembering that pressing "**alt p**" together, or typing "**pi**" will give them the π symbol.

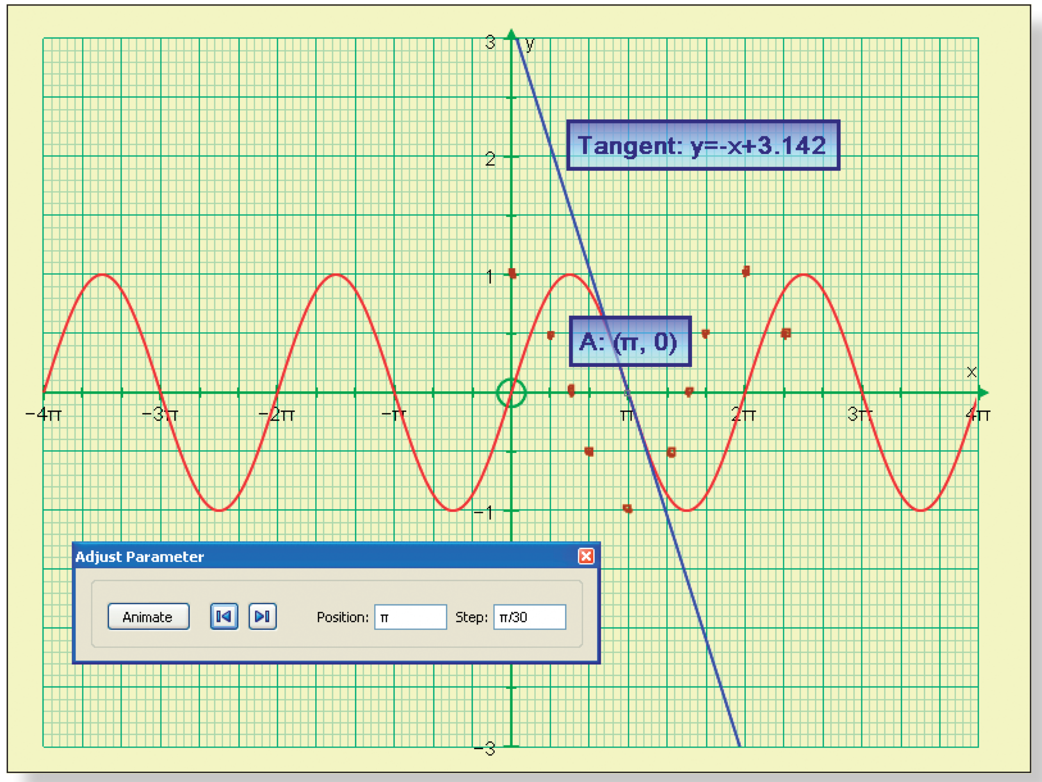
Alternatively, the **left-right** buttons on the keyboard can be used to move point A along the curve, and the **up-down** buttons to adjust the size of the step.

Each time a chosen point is reached, the second student can use the **Scribble** tool to mark the gradient of the tangent (from the equation of the tangent), against the x value of the point (from the co-ordinates of point A).

Use the **Erase** tool to rub out any mistakes.

Note: It may be necessary to adjust the position of the **Text Boxes** if they get in the way.

Your screen should look something like this:



Note: If the students spot that the function repeats, encourage them to use this to quickly finish off their sketch and just use the tangent function to check the gradient at key points.

Teacher: Does anybody recognise the **Gradient Function**?

Prompt: Look at where it crosses the axes, look at the turning points, look at the period and amplitude.

Ideal Response: It's y = cos(x)!

Teacher: Sounds good, but let's just check that…

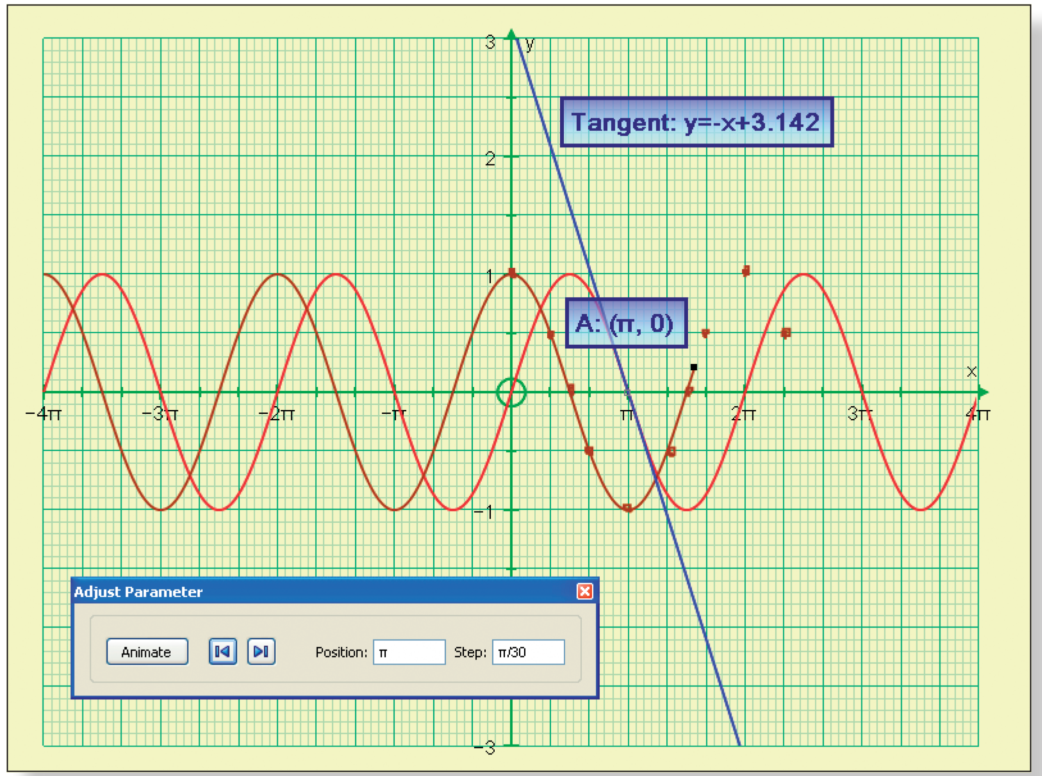
Make sure **Slow Plot** mode is still on.

Enter the equation: y = cos(x)

The curve should begin to be drawn on the screen, hopefully going through the students' points.

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Teacher: So, the Gradient Function of y = sin(x) is y = cos(x). I wonder what the gradient function of y = cos(x) is…

ACTIVITY 2: y = cos(x)

Open Page - 2 by clicking on the Tab.

Teacher: Now, we have just seen what the graph of y = cos(x) looks like, so let me just quickly plot it again on this fresh page.

Make sure Slow Plot mode is still on.

Enter the equation: y = cos(x)

The curve should begin to be drawn on the screen.

Teacher: Now, just before we use the same technique as before, can anybody picture in their minds what the Gradient Function of y = cos(x) would look like and have a go at telling us the equation?

Note: Even if a student does get the equation correct, it is still worth using the tangent function to quickly plot a few points as before to build up the Gradient Function.

Repeat the instructions above, which are briefly re-capped here:

Left-click on the curve (it should turn black).

Enter a point on the curve with x co-ordinate 0, which should give the point (0, 1).

Click on Text Box.
Change the word "Point" to "B" to label the point.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on point B (it should have a square around it).

Right-click and select Tangent from the menu.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the tangent (it should turn black).

Click on Text Box to display the equation of the tangent.

Use the Animation Controller or the left-right arrow buttons on the keyboard to adjust the position of B.

Record the gradients at each point using the Scribble tool.

Note: With the Gradient Function of y = cos(x), it is especially important that the students find the turning points, i.e. the points where the gradient of y = cos(x) is 1 and −1.

Teacher: Does anybody recognise the Gradient Function?

Prompt: Look at where it crosses the axes, look at the turning points, look at the period and amplitude. Is it just y = sin(x), or is it something slightly different?

Ideal Response: It's y = −sin(x)!

Teacher: Again, that sounds good, but let's just check…

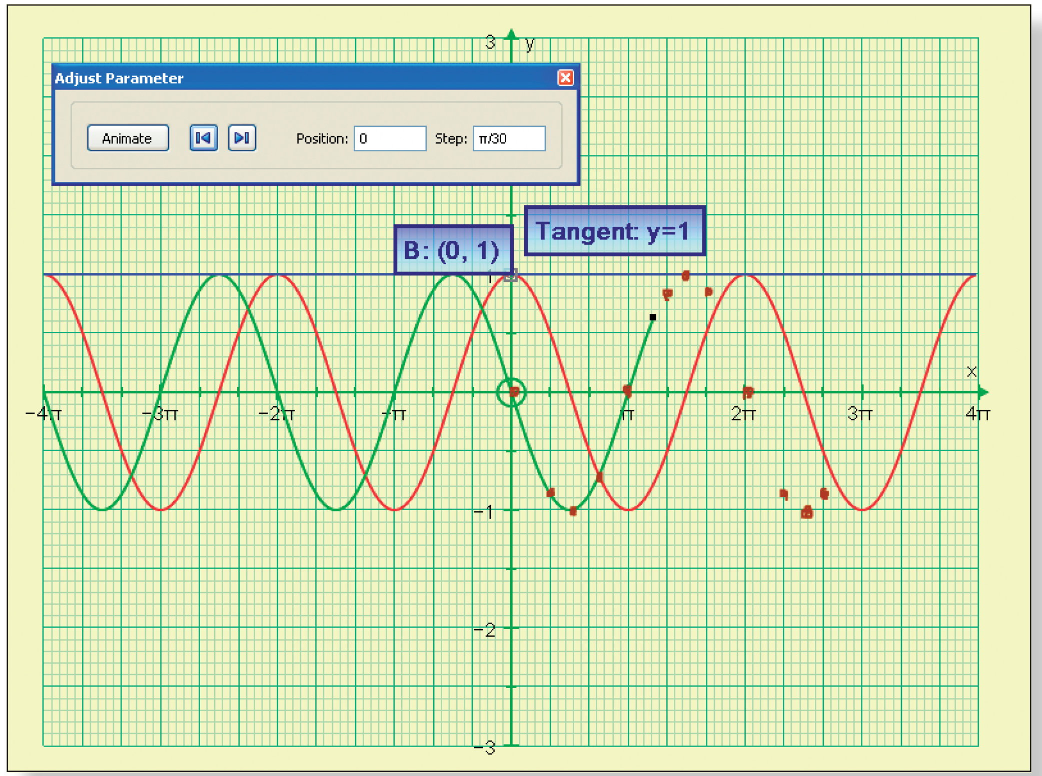
Make sure Slow Plot mode is still on.

Enter the equation: y = −sin(x)

The curve should begin to be drawn on the screen, hopefully going through the student's points.

Press Pause Plotting (or the Spacebar on the keyboard) both to stop the process, and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Teacher: So, the Gradient Function of y = cos(x) is y = −sin(x). This leaves one question: what on earth is the Gradient Function of y = tan(x)?

ACTIVITY 3: y = tan(x)

Open Page - 3 by clicking the Tab.

Teacher: Now, the graph of y = tan(x) is incredibly tricky to draw. Is anybody feeling brave enough to come to the board and give it a go?

Prompt: At what points does the graph cross the x-axis? At what points is the graph undefined / have asymptotes?

Encourage students to come to the front to sketch their curves.

Use the Erase tool to rub out any mistakes.

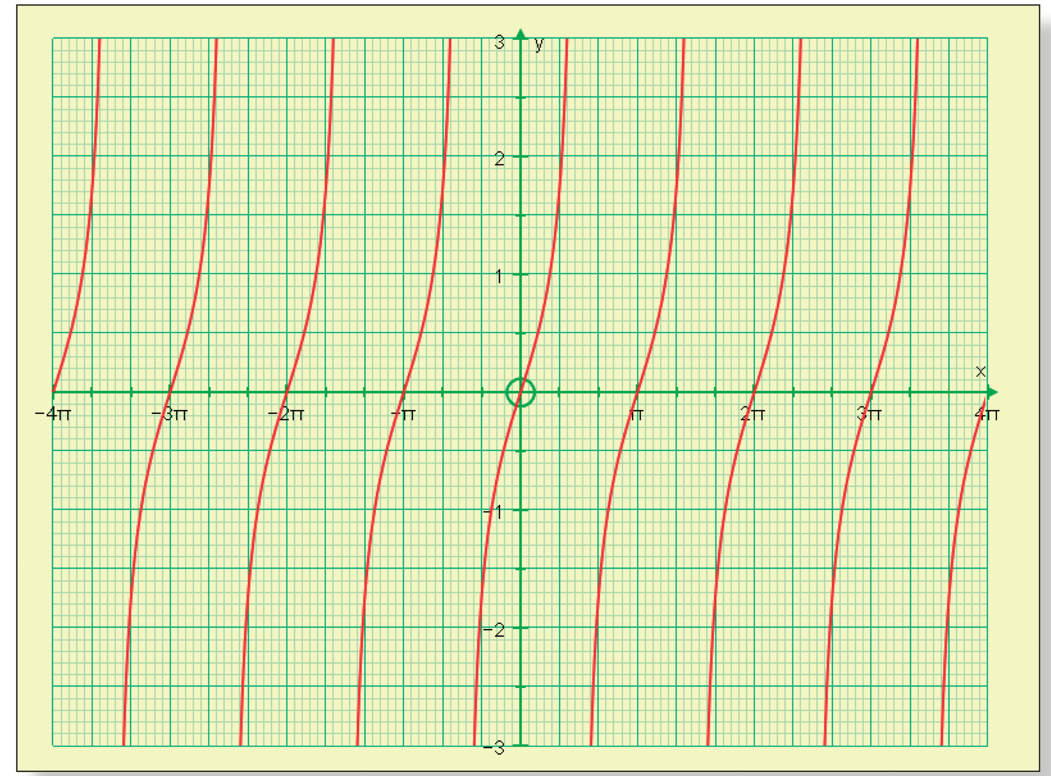
If you want to get rid of all scribbles, click on Edit > Select All Scribbles, and press delete on the keyboard (or Right-click on the graph area and select Delete Objects from the menu).

When you are ready:

Ensure Slow Plot mode is turned on.

Enter the equation: y = tan(x)

Your page should look something like this:



Teacher: Okay, now just to warn you, the Gradient Function is not a function that you will be very familiar with. But, looking at the graph of y = tan(x), can anybody tell me anything about the gradient function?

Prompt: Think about the gradient of y = tan(x). Is it ever negative? What does this mean about the location of the Gradient Function? What is the gradient at the asymptotes? What is the gradient of the graph when it crosses the x-axis? What is happening to the gradient either side of the x-axis?

Ideal Response: The gradient of y = tan(x) is always positive, so the Gradient Function will always lie above the x-axis. The gradient at the asymptotes is infinite / undefined. The gradient of the curve when it crosses the x-axis is equal to 1, and either side of this the gradient is steeper. This means there must be a series of minimum points on the Gradient Function with a y value of 1.

Again, encourage students to come to the front to sketch their curves using the Scribble Tool. Encourage them to mark on the points that they know, and build it up from there.

Use the Erase tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > Select All Scribbles, and press delete on the keyboard (or Right-click on the graph area itself and select Delete Objects from the menu).

Teacher: Okay, let's try using our tangents again.

This proceeds like the method above, an is briefly re-capped here:

Left-click on the curve (it should turn black).

Enter a point on the curve with x co-ordinate 0, which should give the point (0, 0).

Click on **Text Box**.
Change the word "Point" to "C" to label the point.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on point C (it should have a square around it).

Right-click and select **Tangent** from the menu.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the tangent (it should turn black).

Click on **Text Box** to display the equation of the tangent.

Use the **Animation Controller** or the **left-right arrow buttons** on the keyboard to adjust the position of C.

Record the gradients at each point using the **Scribble** tool.

Note: With the graph of $y = \tan(x)$, it is worth paying special attention to the gradient as the curve approaches the asymptotes and as it crosses the x-axis. Also, many of the gradients are too large to plot on the graph, so encourage the students to reduce the size of the step and to concentrate on points around where the curve crosses the x-axis.

Teacher: Well, we can notice the pattern, but the function itself appears a very strange one. Let's use a special function on Autograph to have a better look at it.

Left-click on an unoccupied part of the graph area to *de-select everything*.

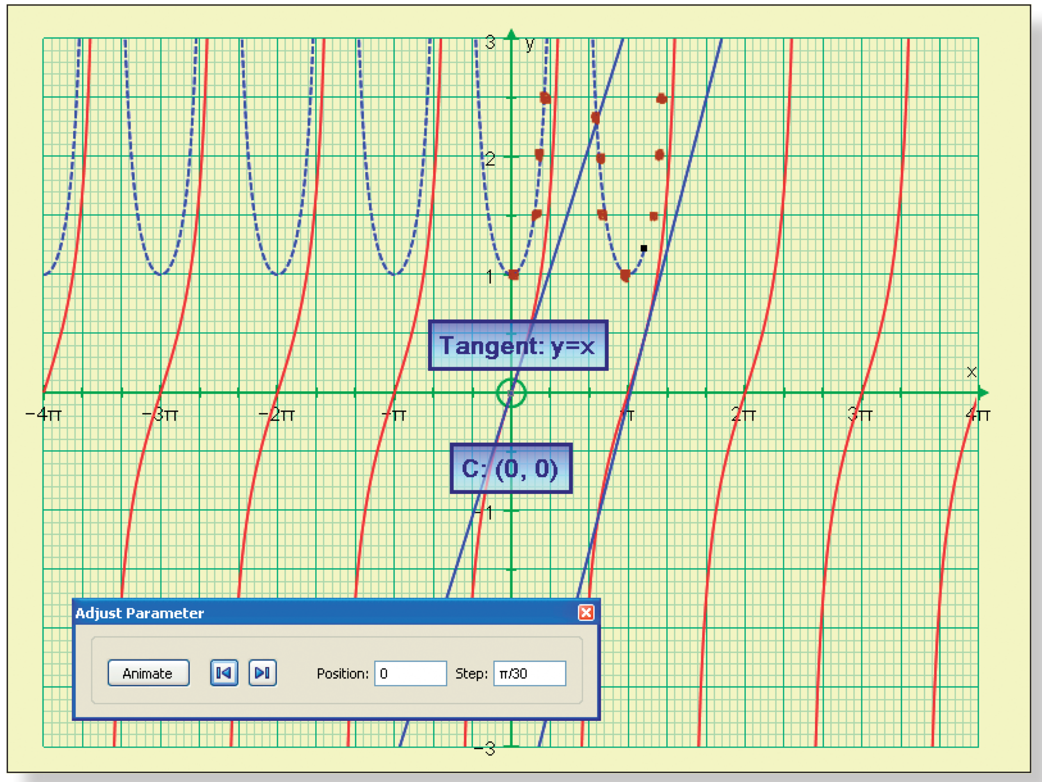
Left-click on the curve (it should turn black).

Select **Gradient Function** from the toolbar at the top and click **OK**.

The **Gradient Function** should begin to be drawn on the screen, hopefully going through the student's points.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Teacher: And the name of the **Gradient Function** of $y = \tan(x)$ is… $y = \sec^2(x)$.

Ideas for Further Work

- If you have covered the Quotient Rule, it might be nice to derive the **Gradient Function** for $y = \tan(x)$ by expressing $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$.
- Using the **Chain Rule** for trigonometric functions also follows directly on from this activity. See Teacher Demonstration T6: The Chain Rule.

Teacher Demonstration T6

Discovering the Chain Rule

Learning Objectives

- To obtain a graphical representation of how the chain rule works using the function y = sin(bx + c).
- To be able to apply this representation to find the gradient functions of functions in the form y = acos(bx + c), including fraction and negative values for the constants.
- To reinforce the transformations of trigonometric functions that result from adjusting the values of the constants

Required Pre-Knowledge

- To know the shape of the trigonometric functions: y = sin(x) and y = cos(x).
- To know the gradient functions of y = sin(x) and y = cos(x) – See Teacher Demonstration T5: Discovering the Gradient Functions of Trigonometric Functions.
- To be comfortable working in radians.
- To understand the concepts of tangents and gradient.
- To understand recognise transformations in the form f(x − c) and af(x).
- To be comfortable with the concept of the gradient function and its role in differentiation.

Pre-activity Set-up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

For this activity, you will need to set up two pages as follows:

Page - 1:

Select **Whiteboard Mode**.

Ensure you are working in **Radians**.

Edit the axes as follows:

x:	Minimum: −4π	Maximum: 4π	Numbers: π	Pips: π/3
y:	Minimum: −4	Maximum: 4	Numbers: 1	Pips: 0.5

Note: To enter the π symbol, press "**alt p**" at the same time, or type "**pi**".
Remove all of the green ticks underneath **Auto**.

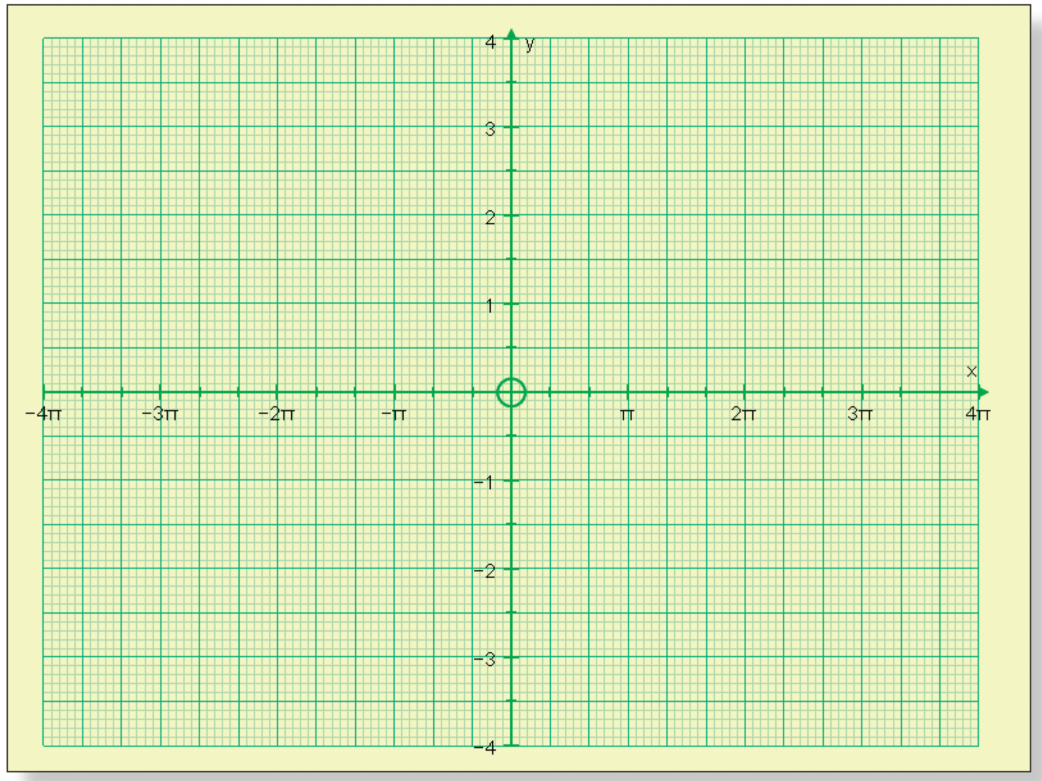
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Still in the **Edit Axes** menu:
Click on the **Appearance** tab.
Open the **drop-down** menu underneath **Themes.**
Select **Graph Paper**.
Click **OK.**

From the **Axes** menu on the very top toolbar, select **Show Key**.

This should make the key disappear.

Your screen should look something like this:



Page - 2:

Open another blank **2D Graph Page**.

Ensure you are working in **Radians**.

Edit the axes as follows:

x:	Minimum: −4π	Maximum: 4π	Numbers: π	Pips: π/3
y:	Minimum: −8	Maximum: 8	Numbers: 1	Pips: 1

Remove all of the green ticks underneath **Auto**.

Still in the **Edit Axes** menu:
Click on the **Appearance** tab.
Open the **drop-down** menu underneath **Themes**.

Select Graph Paper.
Click OK.

From the Axes menu on the very top toolbar, select Show Key.

This should make the Key at the bottom of the page disappear.

Enter the equation: $y = a\cos(bx - c)$
Still in the Add Equation box, click on Edit Constants.
Set the values for the constants as follows: $a = 1$, $b = 1$ and $c = 0$.
Click OK twice.

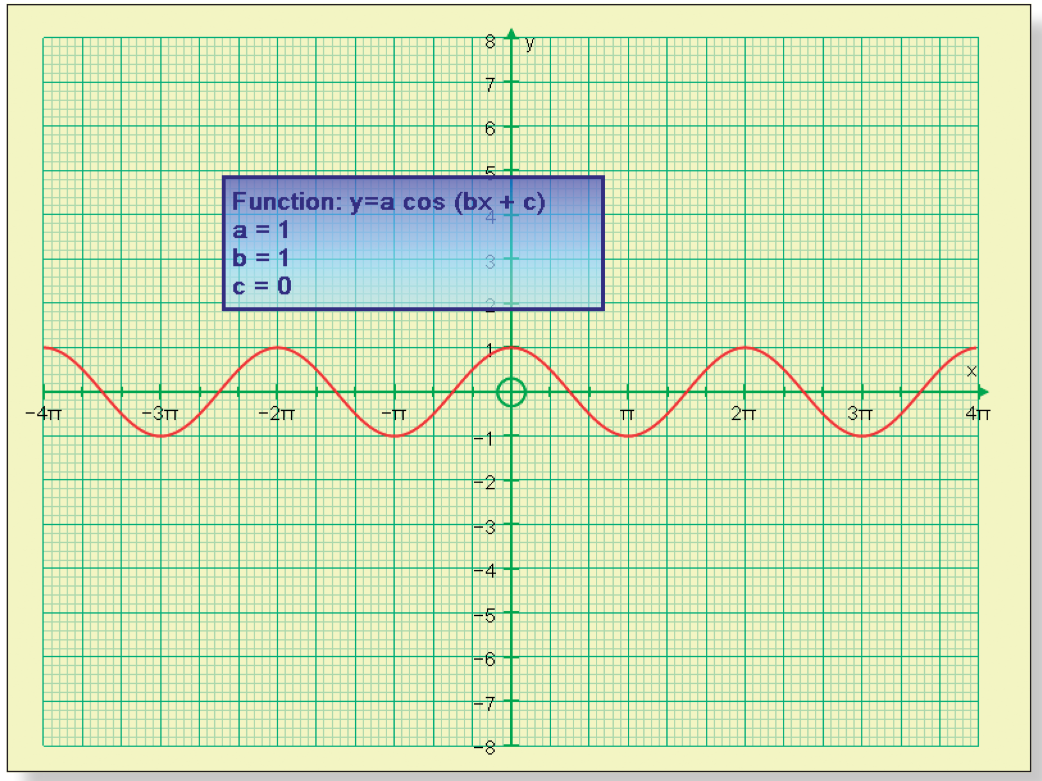
The graph of $y = \cos(x)$ should now appear on the screen.

Left-click on the curve to select it (it should turn black).

Click on Text Box.
Change the words "Equation 1" to "Function".
Tick the box next to Show Detailed Object Text.
Click OK.

The equation of the function, along with the current values of a, b and c should now be displayed.

Your screen should look something like this:



The two pages are now available as Tabs on the top of the screen and can be accessed any time by simply clicking on them.

We will use Page - 2 later on in this demonstration.

Step-by-Step Instructions

Activity 1: Visualising the Chain Rule

Open Page - 1 by clicking on the Tab near the top of the screen.

Teacher: Okay, to warm you up, can anybody come to the front and mark on a few key points that the graph of $y = \sin(x)$ would go through.

Prompt: Notice we are working in Radians. Does the graph go through the origin? What is the period of the graph? What is the amplitude? Where does it cross the axes?

Encourage students to come to the front to mark on their points using the Scribble Tool.

Use the Erase tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > Select All Scribbles, and press delete on the keyboard (or Right-click on the graph itself and select Delete Objects from the menu).

When you are ready:

Click on Slow Plot mode.

Enter the equation: $y = \sin(bx - c)$

Teacher: If I wanted to draw the graph of $y = \sin(x)$, what would the value of constants b and c need to be?

Ideal Response: $b = 1$ and $c = 0$.

Still in the Add Equation box, click on Edit Constants.
Set the values: $b = 1$, $c = 0$.
Click OK twice.

The curve should begin to be drawn on the screen, hopefully going through the students' points.

Press Pause Plotting both to stop the process and to resume it to focus on the key features of the graph.

Note: The Spacebar can also be used to serve this function.

Teacher: Now I am just going to label the equation of this curve so we can keep an eye on it.

Left-click on the curve (it should turn black).

Click on Text Box.

Change the words "Equation 1" to "Function".
Tick the box next to Show Detailed Object Text.
Click OK.

The equation of the function, along with the current values of b and c should now be displayed.

Click on the Constant Controller.
The **drop-down** menu allows you to select each constant.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the value of the step.

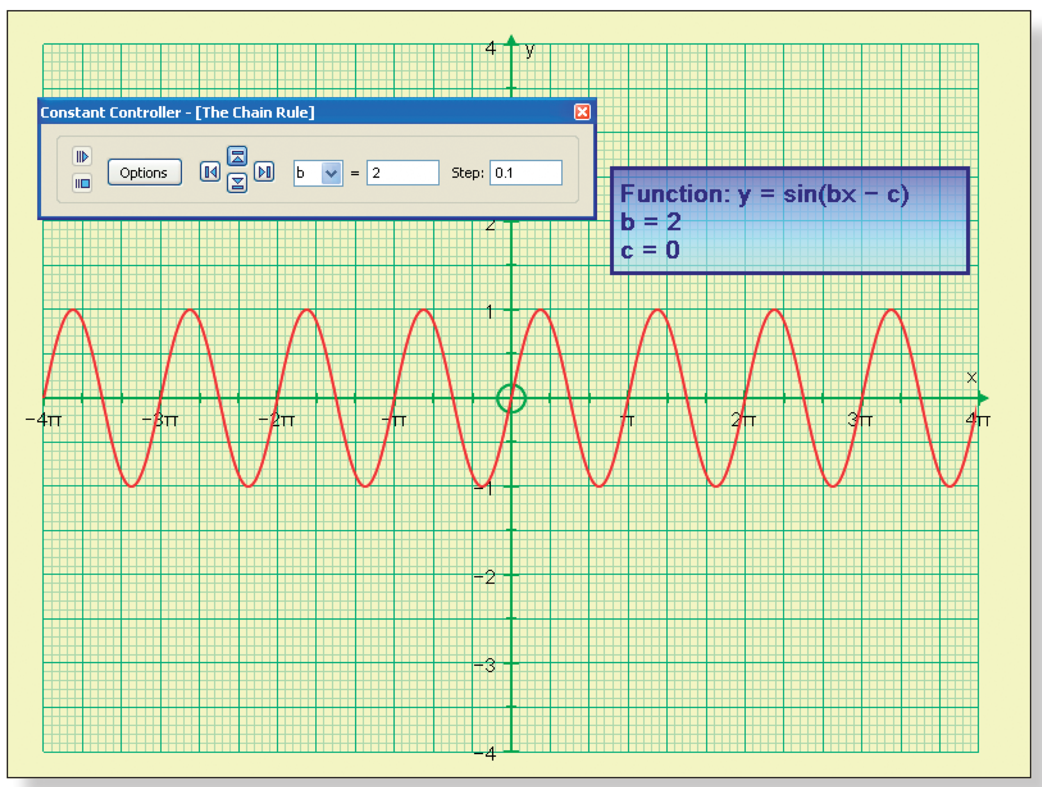
Teacher: What would happen to the shape and position of this graph if I started *increasing* the value of b?

Prompt: What effect does the constant b have on the graph? Think back to your work on Transformations. Imagine I increased the value of b to 2, what equation would I have then? What would the graph of that equation look like?

Ideal Response: Increasing the value of b increases the frequency of the graph, and reduces the period. If you increased the value of b to 2, the graph would complete a cycle every π radians instead of every 2π radians. The x values of all the axes crossing points, maximums and minimums would all be halved. The graph would become more squashed from the left and right!

Use the **up-down** buttons to confirm the students' answers, showing values of b between 0 and 1 as well as greater than 1.

Your screen should look something like this:



When ready, return the value of b back to 1.

Teacher: Good. Now, can anybody remember what the Gradient Function of y = sin(x) looks like?

Prompt: Remember the work we did on the Gradient Function. What does the Gradient Function actually mean? What is the Gradient Function of y = sin(x)? When we differentiate y = sin(x), what do we get?

Ideal Response: y = cos(x)

Teacher: Good. Now, let's add the Gradient Function to our graph.

Ensure Slow Plot mode is still turned on.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the graph of y = sin(bx − c) (it should turn black).

Select Gradient Function from the toolbar at the top.

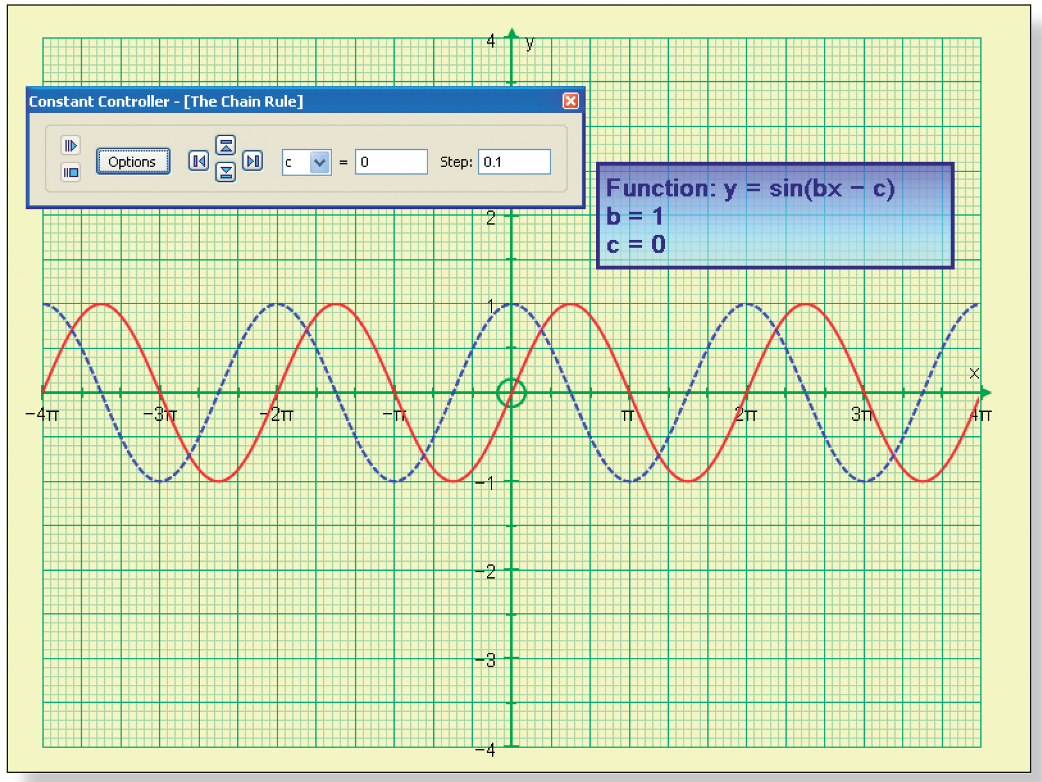
Click OK.

The Gradient Function, y = cos(x), should now begin to be drawn on the graph.

Press Pause Plotting (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Note: The plot will automatically stop at maximum and minimums as well as intersections with the x-axis.

Your screen should look something like this:



Teacher: Now comes the tricky part. What will happen to the Gradient Function of y = sin(bx + c) when I increase the value of b?

Prompt: Think about what happened to the function itself. How is the gradient of the function at each x value affected as the graph becomes more "squashed"? How will this affect the graph of the Gradient Function? Again, thinking about the graph when b = 2 might help you. Think about the frequency of the Gradient Function. Think about the amplitude of the Gradient Function.

Ideal Response: The frequency of the Gradient Function will increase to match the function itself. However, because the frequency has increased, the curve itself is steeper at any given x value. For example, when b = 2, the graph is twice as steep, and so the maximum value of the gradient also doubles, meaning the amplitude of the graph of the gradient function increases to 2, so the entire graph now lies between the values of y = 2 and y = −2.

Teacher: Good. Now, let's watch that in action…

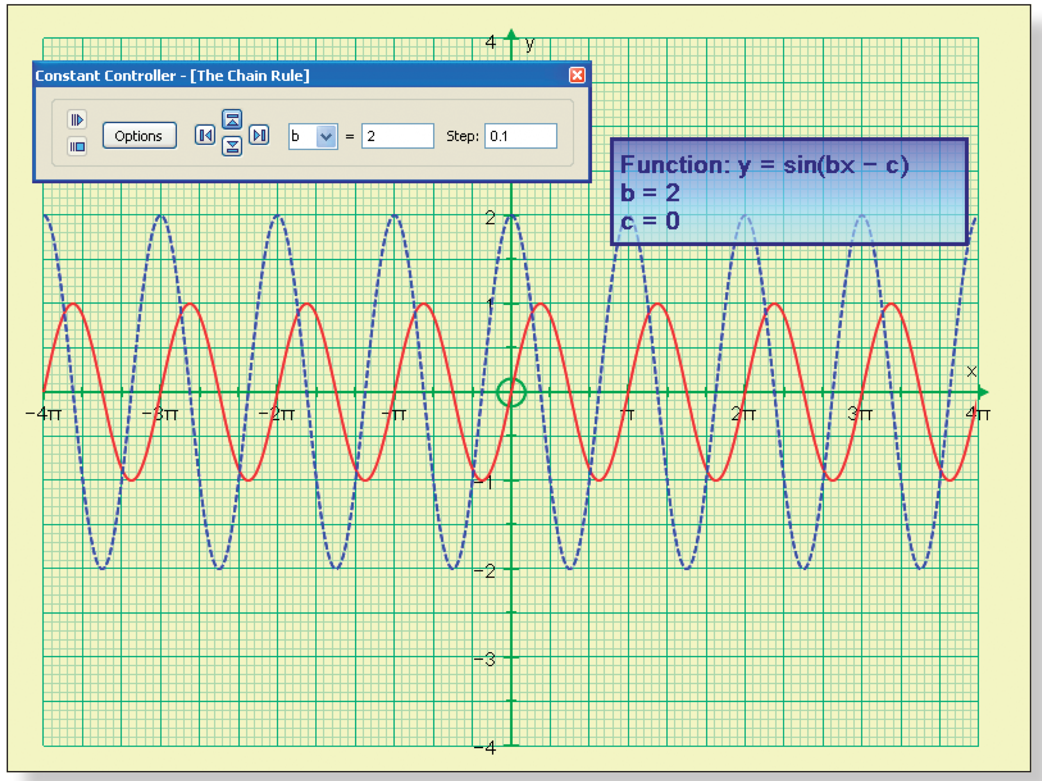
Use the up-down buttons to confirm the students' answers.

Pay particular attention to the case when b = 2 as this perhaps illustrates the concept most clearly.

Point out that as the frequency increases, the function gets stepper and steeper at each x value (apart from the turning points), and hence the value of the gradient at each point also increases, thus increasing the amplitude of the Gradient Function.

Again, showing values of b between 0 and 1, paying particular attention to when b = 0.5 illustrates this concept strongly.

Your screen should look something like this:



Teacher: So, if the equation of our function is y = sin(2x), what is the equation of our Gradient Function?

Ideal Response: y = 2cos(2x)

Teacher: And if the equation of our function is y = sin(0.5x), the equation of our Gradient Function is…

Ideal Response: y = 0.5cos(0.5x)

Teacher: And how about if the equation of our function is y = sin(bx)? What is the equation of our Gradient Function now?

Ideal Response: y = bcos(bx)

Give the students a few minutes to think about this and discuss it, before moving on.

Return the value of b back to 1.

Teacher: Now comes the really tricky part. I am now going to change the value of c. How will that affect the graph of our original function, and also the graph of our Gradient Function?

Prompt: Think about what effect subtracting the c would have on the function itself. Think back to the work we did on Transformations. f(x − c). What type of a transformation is that? What happens to the graph of the function? How does this affect the gradient of our original function? So, how will it affect the graph of the **Gradient Function**?

Ideal Response: The effect of subtracting the constant c is to *translate* the graph by c units. This will mean that the gradient at any given x value along the curve is now different, and so the **Gradient Function** is also translated by c units in the same direction. Crucially, the upper and lower limits for the gradient do not change during the translation, and so the amplitude of the **Gradient Function** remains the same.

Teacher: Good. Now, let's watch that in action…

Select c from the **drop-down** menu on the **Constant Controller**.

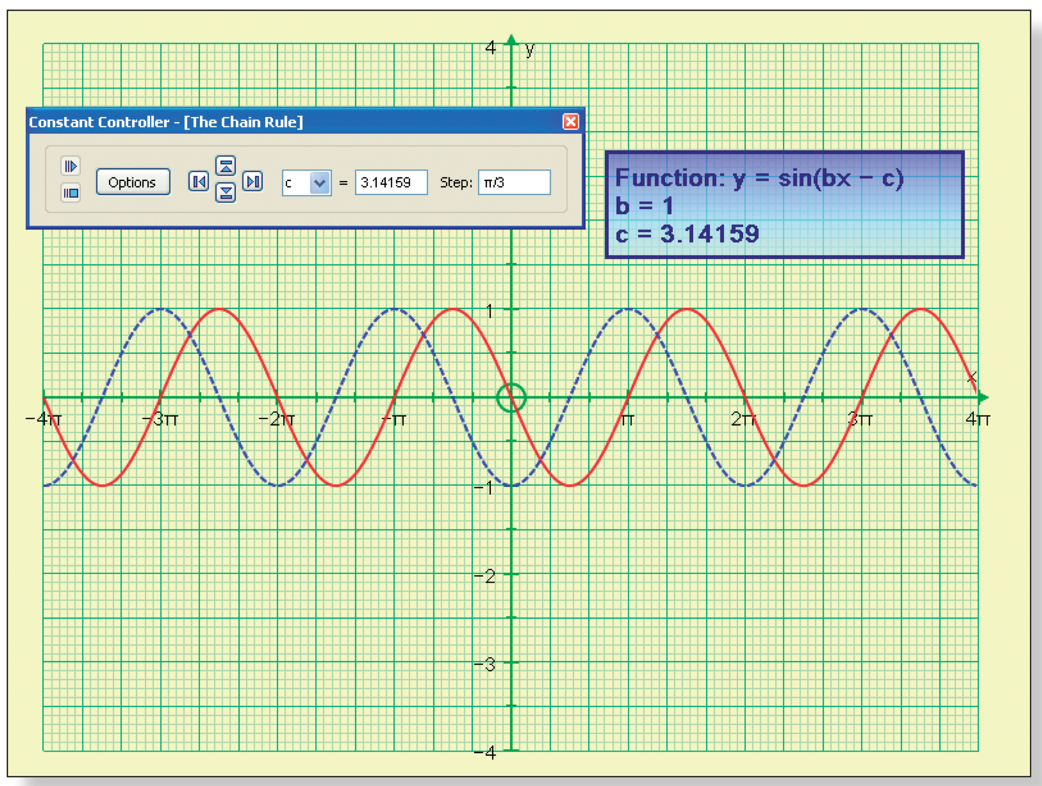
Type in "pi/3" in the step box to adjust the value of c in steps of $\frac{\pi}{3}$.

Use the **up-down** buttons to confirm the students' answers.

Point out that as the original function is translated, given gradients are just shifted along from one x value to another. This is probably most clearly observed at the maximum and minimum points. As the gradients themselves are being shifted, so too is the **Gradient Function** as a whole.

Because the upper and lower limits of the gradient remain the same, so does the amplitude of the **Gradient Function**.

Your screen should look something like this:



Teacher: So, if the equation of our function is y = sin(x − π), what is the equation of our **Gradient Function**?

Ideal Response: y = cos(x − π)

Teacher: And if the equation of our function is y = sin(x + 2π), the equation of our **Gradient Function** is…

Ideal Response: y = cos(x + 2π)

Teacher: And how about if the equation of our function is y = sin(x − c)? What is the equation of our **Gradient Function** now?

Ideal Response: y = cos(x − c)

Give the students a few minutes to think about this and discuss it, before moving on.

Return the value of c back to 0.

Teacher: Okay, prediction time. I am going to adjust the value of the constants so I get the equation: y = sin(2x − π). What will be the equation of the **Gradient Function**?

Prompt: Think about what specific effect the 2 has on the **Gradient Function**. Think about what effect the π has. Put this altogether!

Ideal Response: The **Gradient Function** of y = sin(2x − π) is y = 2cos(2x − π).

Checking the answer:

Adjust the values of the constants so that: b = 2, c = π.

To really reinforce this to the students, plot the equation:

Ensure **Slow Plot** mode is still on.

Enter the equation: y = 2cos(2x − π)

The equation will be plotted on top of the gradient function.

Clicking **Undo** is the easiest way to clear this equation, ready for another example.

It is up to you and the needs of the class how many examples like this you try.

Continue doing examples until the students are able to generalise the rule as follows:

Teacher: And how about if the equation of our function is y = sin(bx − c)? What is the equation of our **Gradient Function** now?

Ideal Response: y = bcos(bx − c)

Teacher: And now let's really put you to the test…

Activity 2: Testing and Extending with acos(bx − c)

Open Page - 2 by clicking on the Tab.

Close the Constant Controller.

Teacher: Here is the curve of y = acos(bx + c), with the values set as shown in the Text Box, so at the moment we simply have the graph of y = cos(x). First task, can somebody just remind us what the Gradient Function of y = cos(x) is?

Prompt: Is it just y = sin(x), or is it something slightly different? Think about the gradient at each point, and decide if it is increasing or decreasing.

Ideal Response: y = −sin(x)

Teacher: Good. Let's have a look at that…

Ensure Slow Plot mode is still on.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the graph of y = acos(bx − c) (it should turn black).

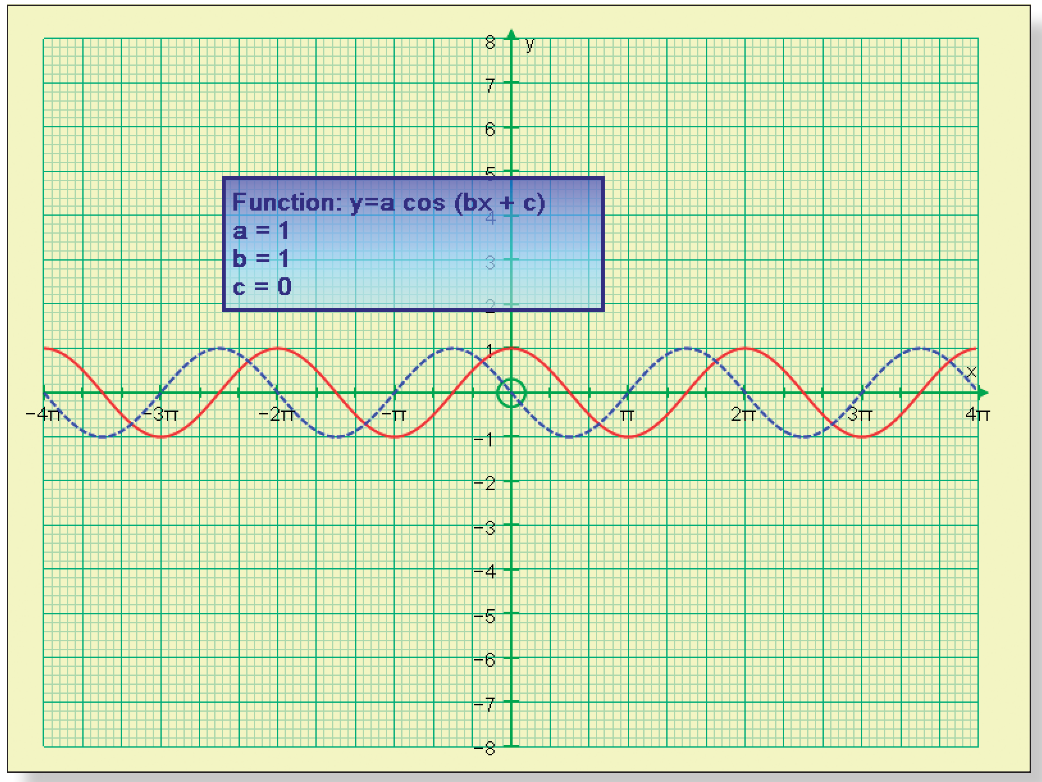
Select Gradient Function from the toolbar at the top.

Click OK.

The Gradient Function, y = −sin(x), should now begin to be drawn on the graph.

Press Pause Plotting (or the Spacebar) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Teacher: So, now I am going to give you a selection of functions, and I want you to use the work we have just done to predict the equation of the Gradient Function each time.

So, your first function is: y = cos(3x − 2π/3).

Ideal Response: y = −3sin(3x − 2π/3)

Checking the answer:

Click on the Constant Controller.

Adjust the values of the constants using the left-right buttons so that: a = 1, b = 3 and c = 2π/3.

Note: Again, you will need to alter the size of the step for c to "pi/3"

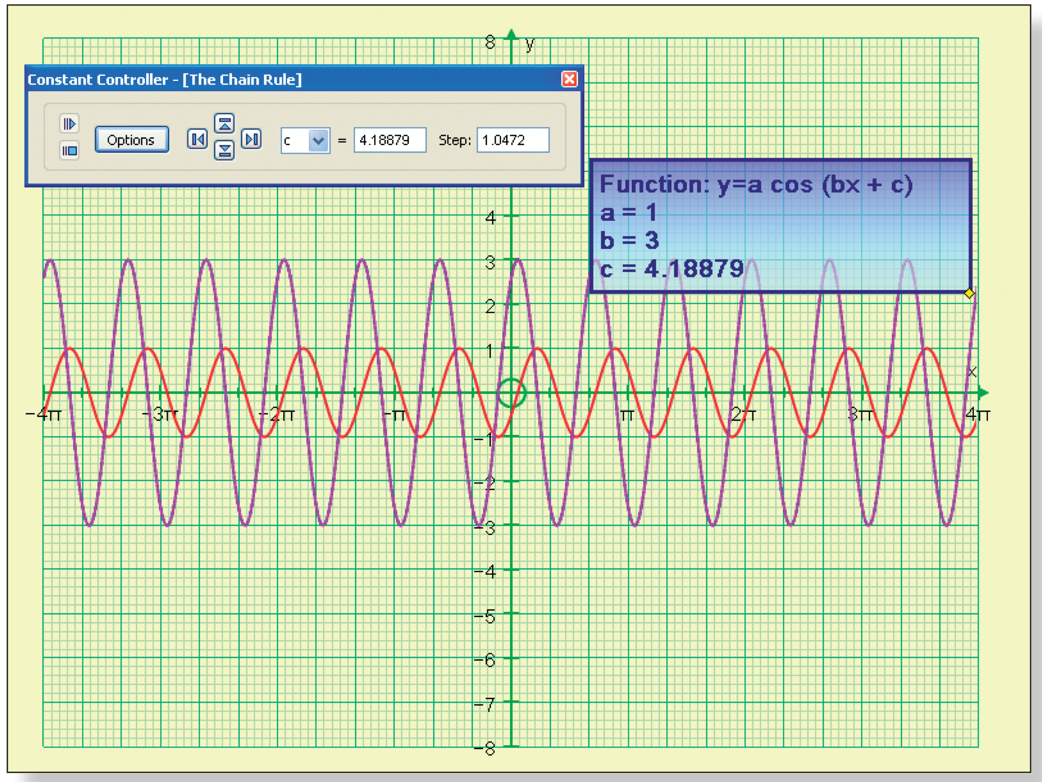
To really reinforce this to the students, plot the equation:

Ensure Slow Plot mode is still on.

Enter the equation: y = −3sin(3x − 2π/3)

The equation will be plotted on top of the gradient function.

Your page should look something like this:



Clicking **Undo** is the easiest way to clear this equation ready for another example.

Again, it is up to you how many of these examples you give the class. Here is a brief list, which is intended to follow a natural progression towards the most difficult of examples:

List of possible examples:

$y = \cos(-4x + \pi)$

$y = 2\cos(3x - \pi)$

$y = -8\cos(0.5x - 2\pi)$

$y = -4\cos(-2x + \pi/3)$

Alternatively, the students themselves could suggest the values of the constants and challenge each other to work out the equation of the **Gradient Function**.

By the end of the demonstration, the students should be able to answer the following:

Teacher: If the equation of our function is $y = a\cos(bx - c)$, what is the equation of our **Gradient Function**?

Ideal Response: $y = -ab\sin(bx - c)$

Ideas for Further Work

- Following on from this activity it might be nice to use Autograph in a similar way to find the **Gradient Functions** of other trigonometric functions such as $y = a\sin(bx^2 + cx + d)$.
- Alternatively, you could show the students the general form of the chain rule.
- The chain rule could then be applied to differentiate functions such as: $y = (3x - 2)^3$, and $y = 5^{2x+1}$.

Discovering the Reciprocal Functions

Learning Objectives

- To interactively discover the shape and the key features of the following reciprocal functions: y = sec(θ), y = cosec(θ) and y = cot(θ).
- To use knowledge of transformations to illustrate the link between the graphs of y = cos(θ) and y = sin(θ).
- To interactively identify the key features of functions in order to determine their shape.

Required Pre-Knowledge

- To know the shape and key points of the trigonometric functions: y = sin(θ), y = cos(θ) and y = tan(θ).
- To understand the concept of an asymptote and undefined regions on graphs, and how they relate to the shape of the function.
- To be aware of transformations in the form f(x − a).

Pre-Activity Set-Up

Open up Autograph in Advanced Mode and ensure you have a blank 2D Graph Page.

Select Whiteboard Mode.

Ensure you are working in Degrees.

Note: This demonstration could be carried out exactly the same way but working in Radians by adjusting the axes accordingly.

Edit the axes as follows:

x: Minimum: −540 Maximum: 540 Numbers: 90 Pips: 30
y: Minimum: −4 Maximum: 4 Numbers: 1 Pips: 0.5

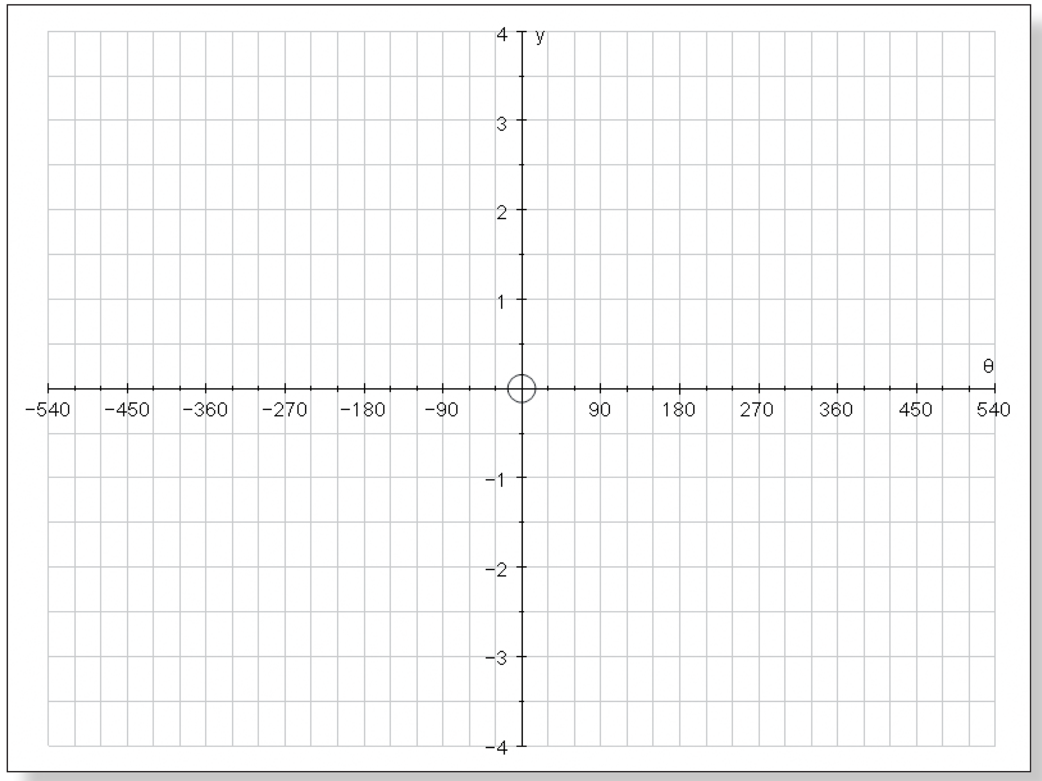
Remove all of the green ticks underneath Auto.
Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Still in the Edit Axes menu:
Click on the Labels tab.
Change the x under Variable and the x under Label to θ.
Note: To enter θ press "alt t" together and then click OK twice.

From the Axes menu on the very top toolbar, untick Show Key.

This should make the Key disappear.

Your screen should look something like this:



Step-by-Step Instructions

Activity 1: y = sec(θ)

Teacher: Okay, to warm you up, can anybody come to the front and do a quick sketch of y = cos(x).

Prompt: Notice we are working in Degrees. Does the graph go through the origin? What is the period of the graph? What is the amplitude? At what values of theta does the graph reach its maximum and minimum points? Where does it cross the axes?

Encourage students to come to the front to sketch their curves using the Scribble Tool, possibly marking key points on first before attempting to join them up.

Use the Erase tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > Select All Scribbles, and press delete on the keyboard (or Right-click on the graph area itself and select Delete Objects from the menu).

When you are ready:

Click on Slow Plot mode.

Enter the equation: $y = \cos(\theta)$
Note: To enter θ, again you can press "alt t" together, or just use the little theta button.

Note: It is not necessary to use the brackets when entering trigonometric equations in Autograph. The above equation could simply be entered as $y = \cos\theta$ if you prefer.

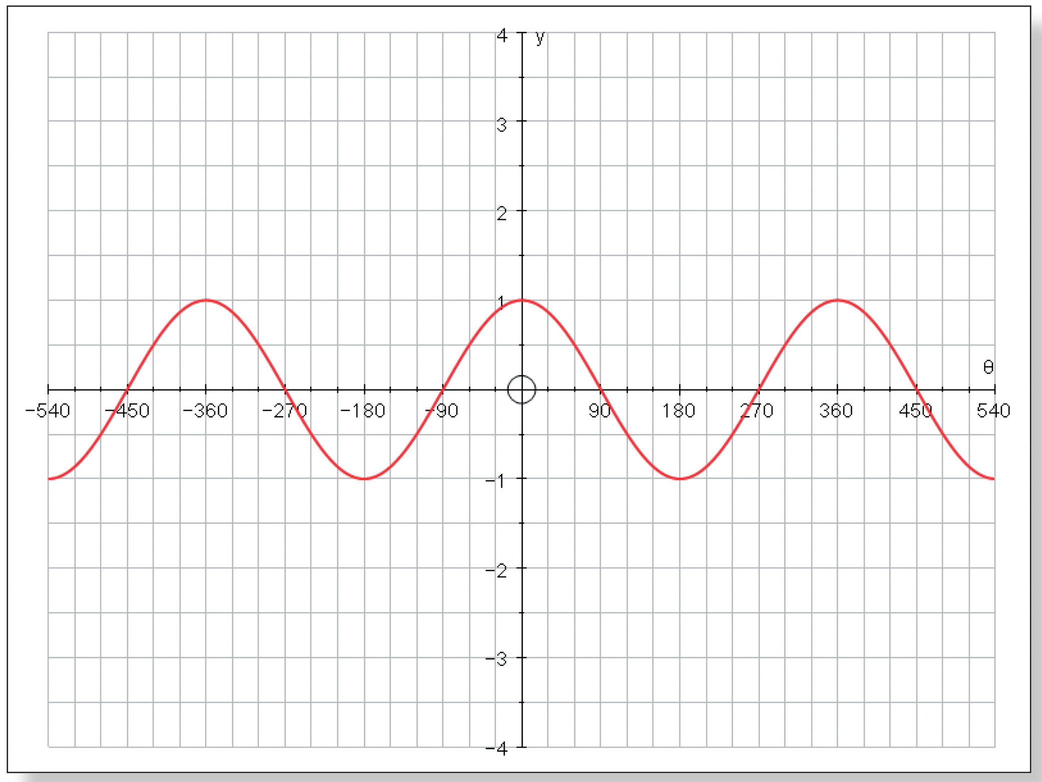
The curve should begin to be drawn on the screen.

Press Pause Plotting both to stop the process and to resume it to focus on the key features of the graph.

Note: The Spacebar can also be used to serve this function.

When ready, *clear all scribbles* from the screen as described above.

Your screen should look like this:



Teacher: Okay, what I want to do is to build up a picture of the graph $y = \frac{1}{\cos(\theta)}$, and I want to do it by thinking of some points which we know definitely lie on the graph, and then trying to build up it's shape from there. Can anybody think of a few nice points which would lie on the graph?

Prompt: What about at the maximum and minimum points? What would be the value of $y = \frac{1}{\cos(\theta)}$ when, say, $\theta = 0$, or $\theta = 180$?

Ideal Response: Whenever the value of $y = \cos(\theta)$ is 1 or −1, so too is the value of $y = \frac{1}{\cos(\theta)}$, so we can mark points on when $\theta = -540, -360, -180, 0, 180, 360, 540$.

Encourage students to come to the front to mark on these points.

Use the Erase tool to rub out any mistakes.

Teacher: Good. And looking at the graph of $y = \cos(\theta)$, are there any points on the curve $y = \frac{1}{\cos(\theta)}$ that might cause us problems when plotting?

Prompt: Remember, we have to do one divided by the value of $y = \cos(\theta)$ to get the value of $y = \frac{1}{\cos(\theta)}$. Which y values of $y = \cos(\theta)$ will give us problems? Where will there be any asymptotes on the graph? Where will the graph be undefined?

Ideal Response: Whenever the value of $y = \cos(\theta)$ is 0, the graph is undefined / has an asymptote, because we cannot divide things by zero. So, there are asymptotes at $\theta = -450, -270, -90, 90, 270, 450$.

Teacher: Good. Now, just to help us visualise that, I am going to draw on those vertical asymptotes with dashed lines… But because there are quite a few to draw, I am going to use a little short cut. What is the equation of the first asymptote on our screen (from the left)?

Ideal Response: $\theta = -450$

Teacher: Good. Now, I am going to enter the equation $\theta = a$, and define the constant a to be −450.

Turn off Slow Plot mode.

Enter the equation: $\theta = a$
Still in the Add Equation screen, click on Edit Constants.
Change the value of constant a from 1 to −450 and click OK.
Still in the Add Equation screen, click on Draw Options.
Select a dashed line from the drop-down menu, and click OK twice.

The line $\theta = -450$ should now be on your screen.

Click on the Constant Controller.
Click on Options, and select Family Plot.

Teacher: Now, if we want to tell Autograph to draw in all the asymptotes, what parameters do we need to enter here?

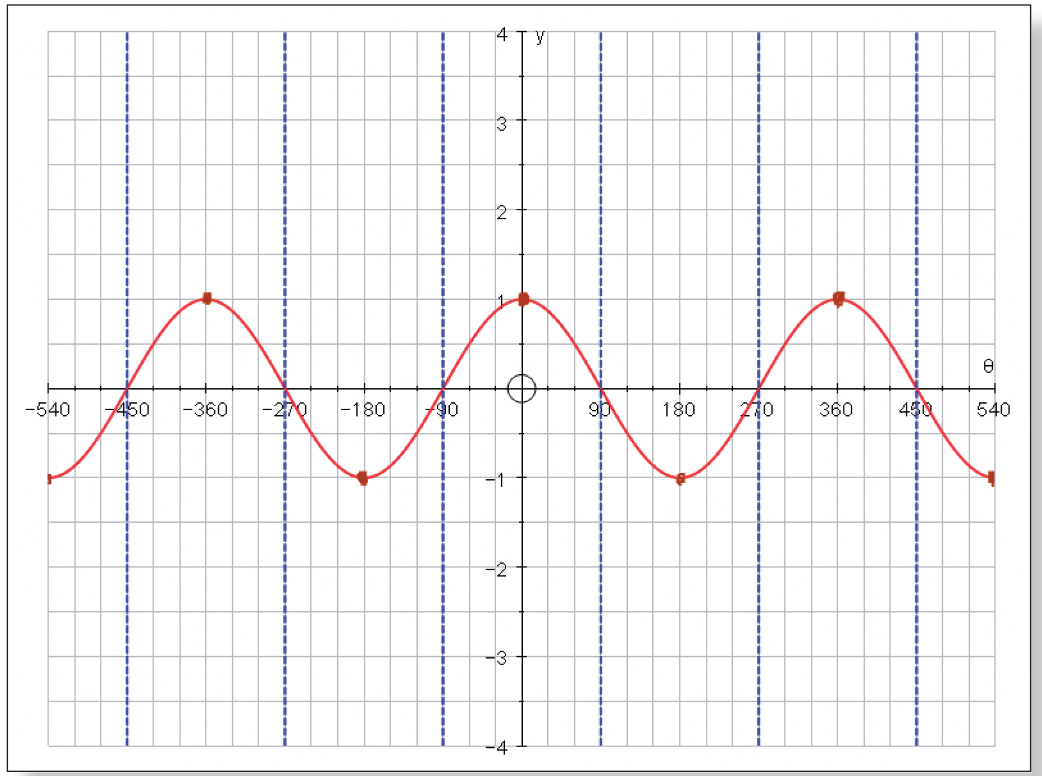
Prompt: What is the largest value of theta we need? What is the size of the interval between each asymptote?

Ideal Response: Start: −450 Finish: 450 Step: 180

Enter in those values and click OK.

Close the Constant Controller.

Your screen should look something like this:



Teacher: Okay, so we know a few points which the graph passes through, and we know where the asymptotes are, so now the only question is what is happening between the asymptotes? Does anybody have any ideas what shape the graph takes?

Prompt: Think about what is happening to the value of y in $y = \cos(\theta)$. What does that mean for the value of y in $y = \frac{1}{\cos(\theta)}$.

Ideal Response: Between the asymptotes, the value of y is always between −1 and 1. This means that when we do 1 divided by this value, the value will always be bigger than one. So, the graph must form U shapes for the maximums of $y = \cos(\theta)$, and upside down U shapes for the minimums.

Teacher: Good. Let's see if we can get Autograph to help us plot some of those points to make things a little clearer. I'm going to plot a point with a theta value of b.

Enter a point with x co-ordinate b, but leave the y co-ordinate blank for a moment.

Teacher: Now, if θ takes a value of b, what would the y value be on the curve $y = \frac{1}{\cos(\theta)}$?

Ideal Response: $y = \frac{1}{\cos(b)}$

Teacher: Excellent. So, let's plot that co-ordinate:

Enter y co-ordinate as 1/cos(b) and press **OK**.

Click on the **Constant Controller**.
The **drop-down** menu allows you to select each constant.
The **up-down** buttons adjust the value of the constant.

The **left-right** buttons adjust the value of the step.

Change the starting value to −540, and the step to 10.

A point should now be marked at (−540, −1), right on top of one of the earlier scribbles.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on this point (it should have a square around it).

Click on **Text Box** and click **OK**.

The co-ordinates of the point should now be displayed.

Make sure the point (and nothing else) is still selected.

Right-click and select **Trace Point** from the menu.

This will keep a record of the path of the point as it moves.

Teacher: Okay, so we are now going to see exactly what is going on between the asymptotes and see if it is as we predicted.

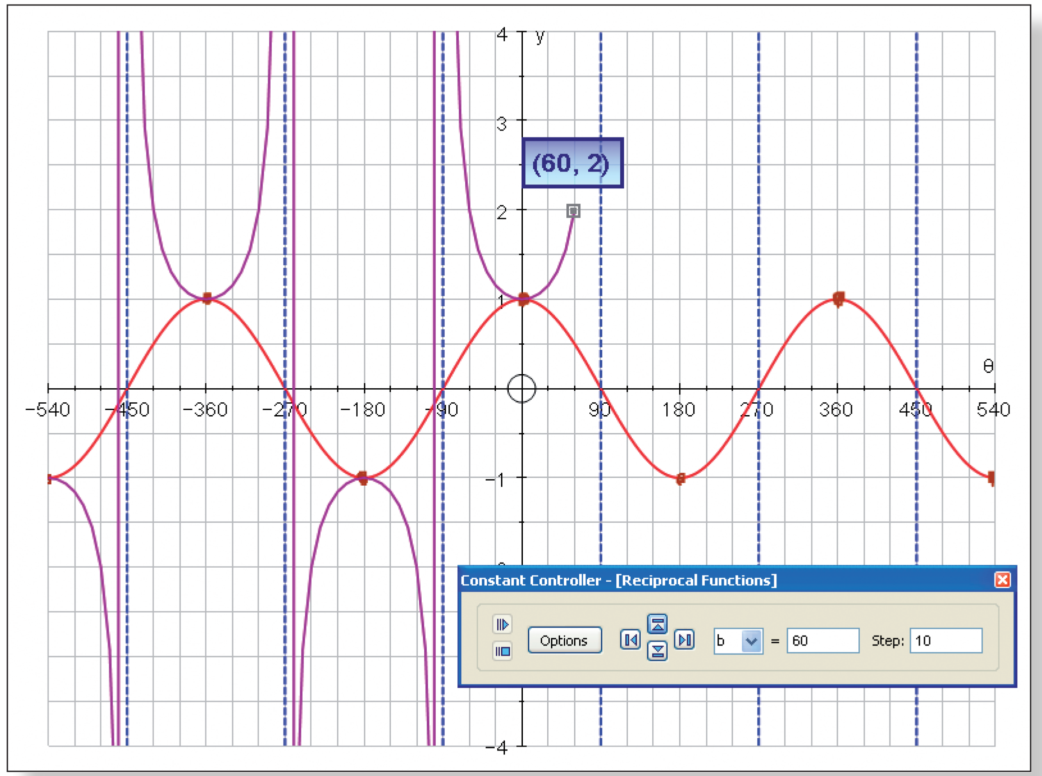
Use the **left button** to increase the value of θ.

The path of the point should be marked.

Draw the students' attention to what happens to the point as cos(θ) gets smaller as you approach the asymptotes.

Continue right along the curve, pausing at the important points, going quicker as they students begin to notice the pattern.

Your screen should look something like this:



Note: You may wish to briefly explain that the vertical lines created by the trace do not represent the asymptotes of the function, but are simply the computer's attempts to join up points which should never really be joined up!

Teacher: And there you have it! The graph of y = $\frac{1}{\cos(\theta)}$. But this graph has another name… y = sec(θ).

Close the **Constant Controller**.

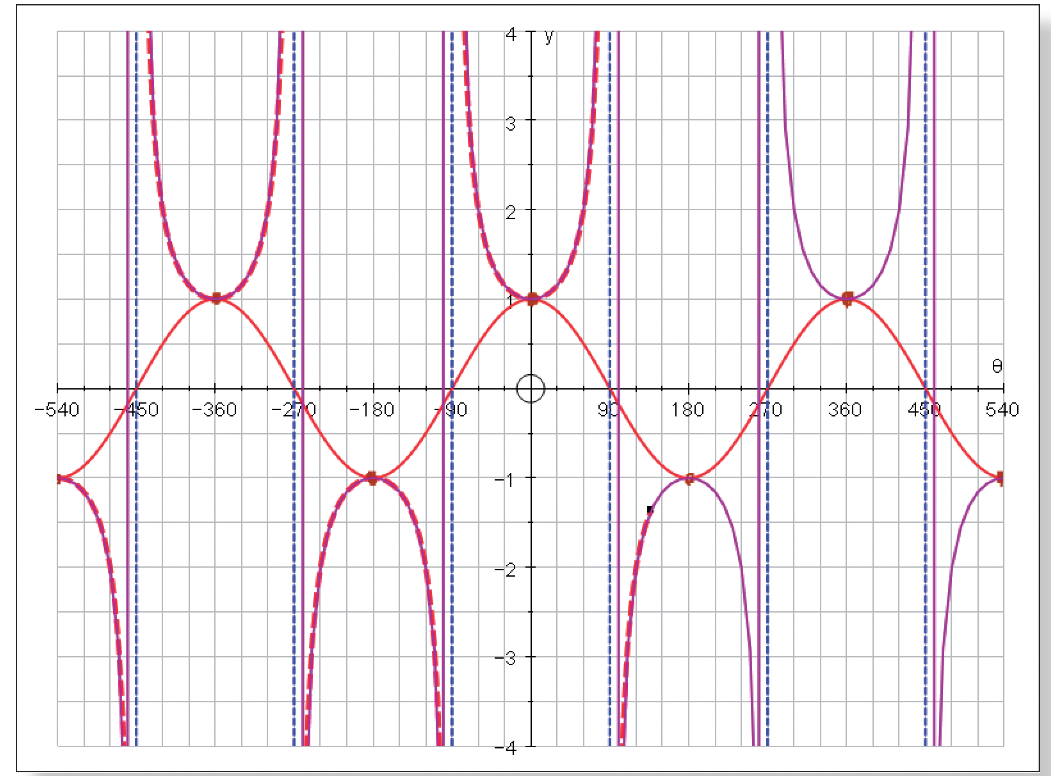
Click on **Slow Plot** mode.

Enter the equation: y = sec(θ)

The graph should begin to plot over the top of the trace, making the curve much smoother.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process, and to resume it to focus on the key features of the graph.

Your screen should look something like this:



ACTIVITY 2: y = cosec(θ)

Teacher: Now, of course, we could do exactly the same thing to find the graph of y = $\frac{1}{\sin(\theta)}$, but I wonder if there is a quicker way… Can anybody tell me what is the relationship, in terms of transformations, between the graph of y = cos(θ) and y = sin(θ)?

Prompt: Look at the screen and try to picture where the graph of y = sin(θ) would go. How do you get from y = cos(θ) to y = sin(θ)? How do you describe this as a transformation?

Ideal Response: sin(θ) = cos(θ − 90). The cos curve is translated 90 degrees to the right to give the sin curve.

Make sure **Slow Plot** mode is still on.

Enter the equation: y = sin(θ)

Teacher: So, how does this allow us to quickly figure out what the graph of y = $\frac{1}{\sin(\theta)}$, looks like?

Prompt: Look at the shape of the sin curve and the cos curve. Think about how we built up the graph of y = $\frac{1}{\cos(\theta)}$. Think about the asymptotes and the maximum and minimum points.

Ideal Response: The graph of y = $\frac{1}{\sin(\theta)}$ must also be translation of 90 degrees to the right from the graph of y = $\frac{1}{\cos(\theta)}$.

Teacher: Sounds good, but we better just check…

Click on **Manage Equation List**.

Left-click on the equation θ = a from the menu and click the red cross in the top corner and click **OK**.

Click on **Edit** > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph area itself and select **Delete Objects** from the menu).

Note: Ignore the warning message. This is just to let you know that both the trace and the textbox are tied to the point, but we don't need them any more anyway!

Your screen should now only contain the graphs of: y = cos(θ), y = sec(θ), and y = sin(θ).

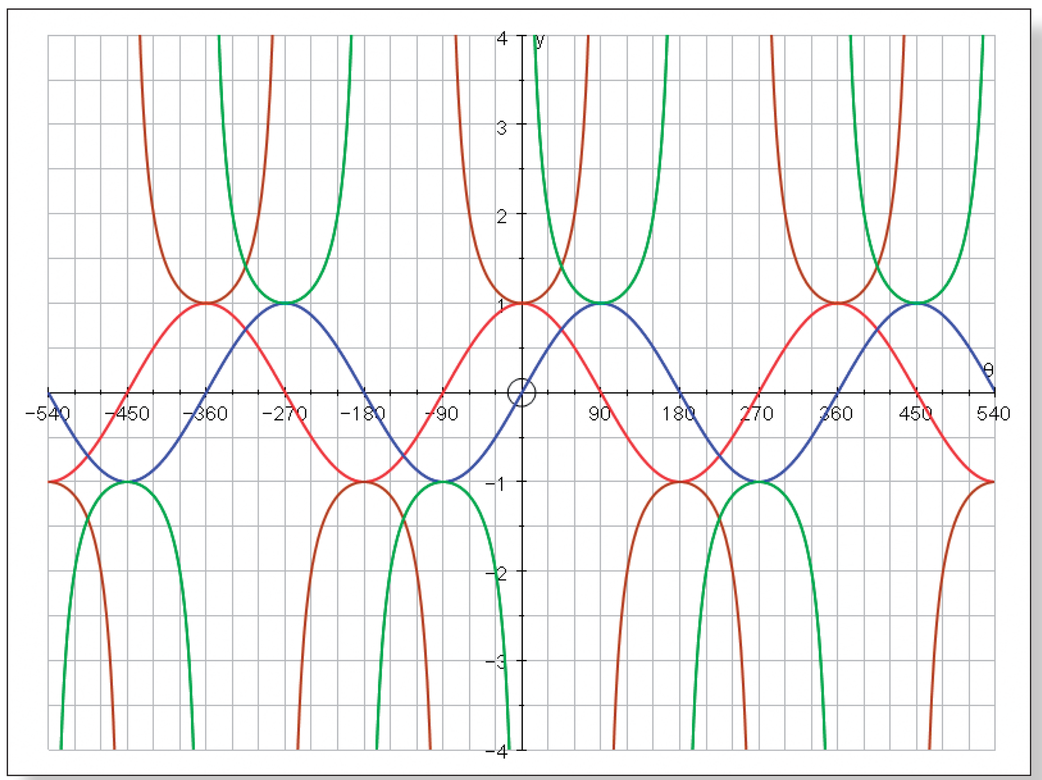
Make sure **Slow Plot** mode is still on.

Enter the equation: y = 1/sin(θ)

The graph should now begin to plot.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Teacher: And this function too has a special name… it is called y = cosec(θ).

Activity 3: y = cot(θ)

Click on **Edit** > **Select All** and press **delete** on the keyboard (or **Right-click** on the graph area itself and select **Delete Objects** from the menu).

This should clear the screen and leave you just with your set of axes.

Teacher: Now, of course, we are left with one question… what does the graph of y = $\frac{1}{\tan(\theta)}$ look like? But before we do that, we need to know what the graph of y = tan(θ) looks like, so any ideas? Anyone brave enough to come up and sketch it?

Prompt: At what points does the graph cross the x-axis? At what points is the graph undefined / have asymptotes?

Encourage students to come to the front to sketch their curves.

Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph itself and select **Delete Objects** from the menu).

When you are ready:

Ensure **Slow Plot** mode is turned on.

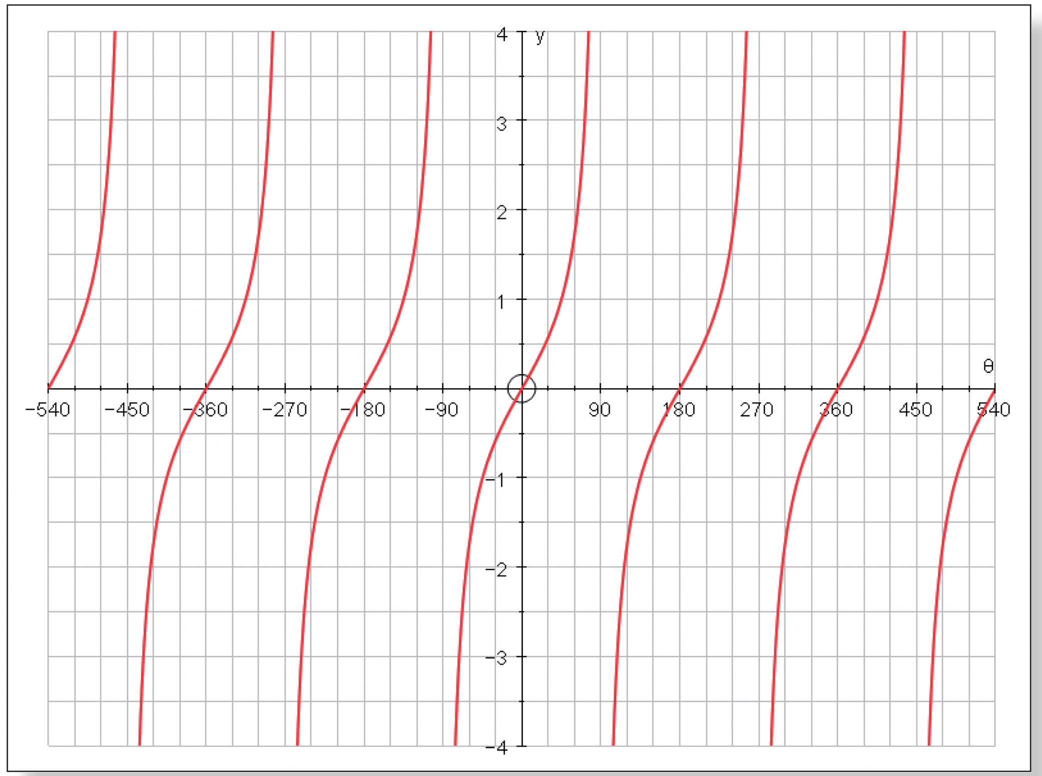
Enter the equation: y = tan(x)

The graph should now begin to plot.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Delete all scribbles as described above

Your page should look something like this:



Teacher: Now, thinking about how we derived the shape of the first couple of graphs, can anybody describe to us the shape of the graph of $y = \frac{1}{\tan(\theta)}$? Can you come up and mark on any points?

Prompt: At what points is the graph undefined / have asymptotes? At what points do the two graphs intersect? Where does the graph cross the x-axis? Do the two graphs have a line of symmetry?

Encourage students to come to the front to sketch their curves.

Use the Erase tool to rub out any mistakes.

Ideal Response: The graph of $y = \frac{1}{\tan(\theta)}$ must have asymptotes when $y = \tan(x)$ is equal to 0. So, the asymptotes must be at $\theta = -540, -360, -180, 0, 180, 360, 540$. The graphs must intersect whenever $y = \tan(x)$ is equal to 1. The graph of $y = \frac{1}{\tan(\theta)}$ must have a value of zero when $y = \tan(x)$ is undefined, so it must cross the x-axis when $\theta = -450, -270, -90, 90, 270, 450$. The two graphs have a vertical line of symmetry through their points of intersection at: $\theta = -495, -315, -135, 45, 225, 405$.

Teacher: Good. Now, let's trace the co-ordinates of points that lie on the curve as we did before to build up a picture of it.

Enter a point with x co-ordinate c, and a y co-ordinate of 1/tan(c).

Click on the Constant Controller.

Change the starting value to −450, and the step to 10.

A point should now be marked at (−450, 0).

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point (it should have a square around it)

Click on Text Box and click OK.

The co-ordinates of the point should now be displayed.

Make sure the point (and nothing else) is still selected.

Right-click and select Trace Point from the menu.

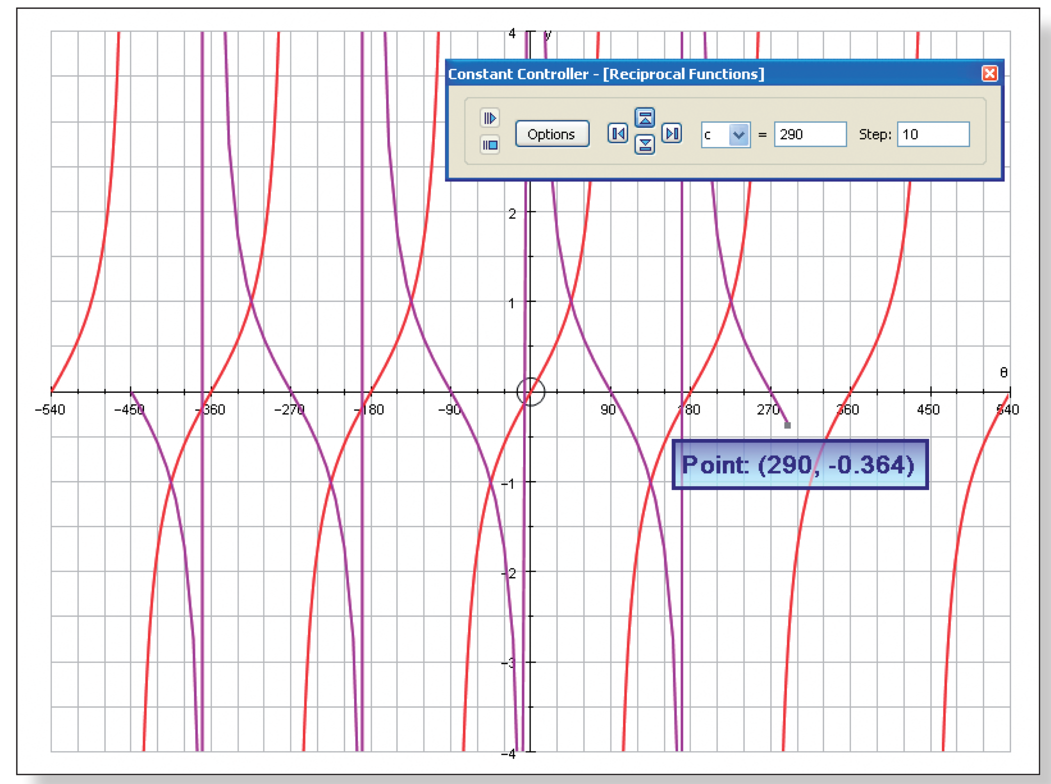
This will keep a record of the path of the point as it moves.

Use the **left button** to increase the value of θ.

The path of the point should be marked, hopefully fitting the points marked on by the students.

Draw the students' attention to what happens to the point as tan(θ) gets both bigger and smaller.

Your screen should look something like this:



Teacher: And there you have it! The graph of $y = \frac{1}{\tan(\theta)}$. But this graph has another name… $y = \cot(\theta)$.

Close the Constant Controller.

Click on **Slow Plot** mode.

Enter the equation: y = cot(θ)

The graph should begin to plot over the top of the trace, making the curve much smoother.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:

Teacher: And there you have it! We have managed to derive the shape of the tree trigonometric reciprocal functions!

Ideas for Further Work

- Trigonometric Identities involving reciprocal functions. See Teacher Demonstration T11: Trig Identities.
- Solving equations involving the reciprocal functions.
- Transformations involving the reciprocal functions.

Teacher Demonstration

T8 Investigating Trigonometric Identities

Learning Objectives

- To get a graphical representation of the following two major trigonometric identities to deepen understanding of why they work:

  $\sin(\theta)/\cos(\theta) = \tan(\theta)$

  $\sin^2(\theta) + \cos^2(\theta) = 1$

- To interactively identify the key features of the functions: $y = \sin^2(\theta)$ and $y = \cos^2(\theta)$ in order to determine their shape.

Note: The methods shown in this demonstration can also be used to derive the following trigonometric identities:

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$\cot^2(\theta) + 1 = \mathrm{cosec}^2(\theta)$

See Ideas for Further Work at the end of this demonstration for more information.

Required Pre-Knowledge

- To know the shape of the trigonometric functions: $y = \sin(\theta)$, $y = \cos(\theta)$ and $y = \tan(\theta)$.
- To understand the concept of an asymptote and undefined regions on graphs, and how they relate to the shape.

Note: It not necessary that the students already know each of the identities beforehand. This demonstration could either be used as a way of reviewing and consolidating knowledge of the identities, or as a way of introducing them.

Pre-activity Set-up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

Ensure you are working in **Radians**.

Note: This demonstration could be carried out exactly the same way but working in **Degrees** by adjusting the axes accordingly.

Edit the axes as follows:

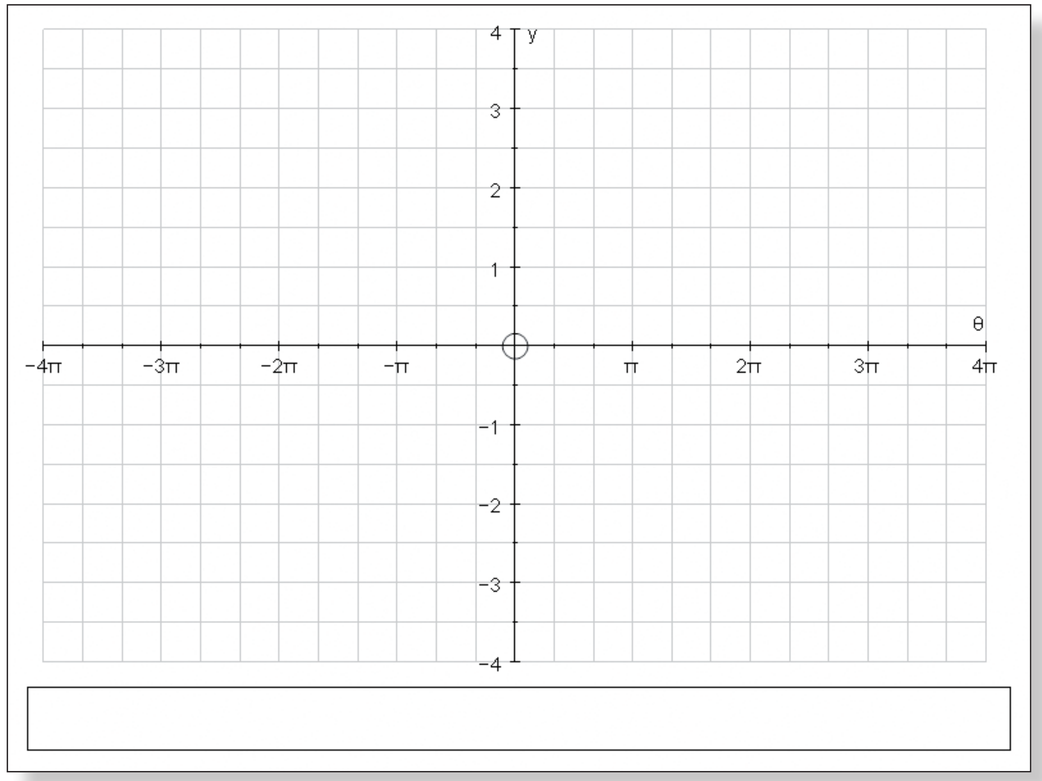
x: Minimum: -4π Maximum: 4π Numbers: π Pips: $\pi/3$

y: Minimum: −4 Maximum: 4 Numbers: 1 Pips: 0.5

Note: To enter the π symbol, press "**alt p**" at the same time, or type "pi".
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Still in the **Edit Axes** menu:
Click on the **Labels** tab.
Change the x under **Variable** and the x under **Label** to θ and click **OK** twice.
Note: To enter θ press "**alt t**" together.

Your page should look something like this:



Step-by-Step Instructions

Activity 1: sin(θ)/cos(θ) = tan(θ)

Teacher: Okay, to warm you up, can anybody come to the front and do a quick sketch of y = sin(θ).

Prompt: Notice we are working in **Radians**. Does the graph go through the origin? What is the period of the graph? What is the amplitude? Where does it cross the axes?

Encourage students to come to the front to sketch their curves using the **Scribble Tool**.

Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph itself and select **Delete Objects** from the menu).

When you are ready:

Click on **Slow Plot** mode.

Enter the equation: y = sin(θ)
Note: To enter θ, again you can press "**alt t**" together, or just use the little theta button.

Note: It is not necessary to use the brackets when entering trigonometric equations in Autograph. The above equation could simply be entered as y = sinθ if you prefer.

The curve should begin to be drawn on the screen.

Press **Pause Plotting** both to stop the process and to resume it to focus on the key features of the graph.

Note: The **Spacebar** can also be used to serve this function.

Teacher: Good, and can somebody just remind us of the transformation that maps y = sin(θ) onto y = cos(θ)?

Prompt: Think about the points on the graph that y = cos(θ) goes through. How does this relate to y = sin(θ)? What type of transformation is this?

Ideal Response: y = cos(θ) represents a translation of $\frac{\pi}{2}$ radians to the left from the graph of y = sin(θ).

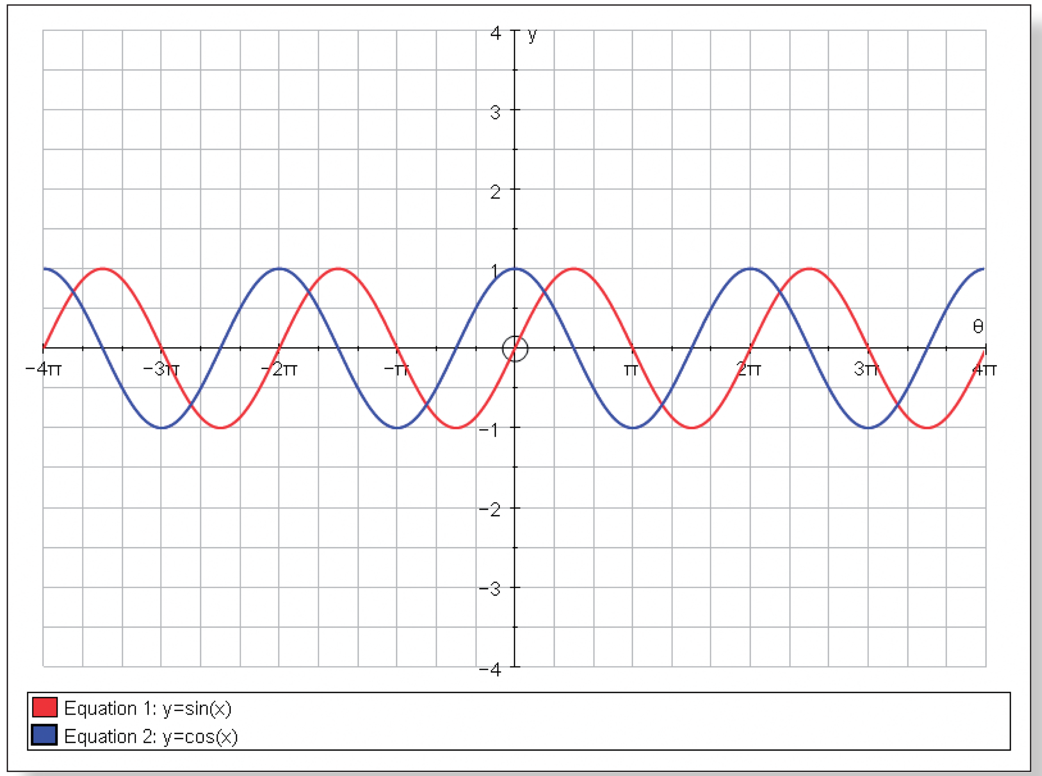
Teacher: Excellent. Now, let's just add the graph of y = cos(θ) to our page…

Enter the equation: y = cos(θ)

The curve should begin to be drawn on the screen.

Delete any scribbles off the page as explained above.

When finished, your screen should look something like this:



Teacher: Okay, now what I want us to think about is the function that would be formed if we were to divide $\sin(\theta)$ by $\cos(\theta)$. Can anybody identify any points which would lie on that graph, or any special features of the graph?

Prompt: Think about when $y = \cos(\theta)$ has a value of 1. What does that mean for the graph of $y = \frac{\sin(\theta)}{\cos(\theta)}$? How about when $y = \cos(\theta)$ has a value of 0? How about when $y = \sin(\theta)$ has a value of 0? When will the graph be negative? When will the graph have high positive values or high negative values?

Ideal Response: The graph of $y = \frac{\sin(\theta)}{\cos(\theta)}$ will have the same values as $y = \sin(\theta)$ when $y = \cos(\theta)$ has a value of 1. The graph will be undefined when $y = \cos(\theta)$ has a value of zero, meaning there will be vertical asymptotes at $\theta = \pm\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$. The function will cross the x-axis when $y = \sin(\theta)$ has a value of 0.

If you like, encourage students to come to the front to mark on points / sketch their curves.

Use the Erase tool to rub out any mistakes.

Teacher: Okay, so let's use Autograph to build up a picture of the function $y = \frac{\sin(\theta)}{\cos(\theta)}$…

Enter a point on the curve with x co-ordinate θ, and y co-ordinate $\sin(\theta)/\cos(\theta)$.

A point should now appear on the page.

Teacher: If I start off with a value of θ of -4π, where will the point be?

Ideal Response: When $\theta = -4\pi$, $\sin(\theta) = 0$, so $y = \frac{\sin(\theta)}{\cos(\theta)}$ also equals 0, so the point will be at $(-4\pi, 0)$.

Click on the Constant Controller.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the value of the step.

Change the starting value to -4π, and the step to $\pi/12$.
Note: To enter the π symbol, just type "**pi**".

A point should now be marked at $(-4\pi, 0)$.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point (it should have a square around it).

Click on Text Box and click OK.

The co-ordinates of the point should now be displayed.

Make sure the point (and nothing else) is still selected.

Right-click and select Trace Point from the menu.

This will keep a record of the path of the point as it moves.

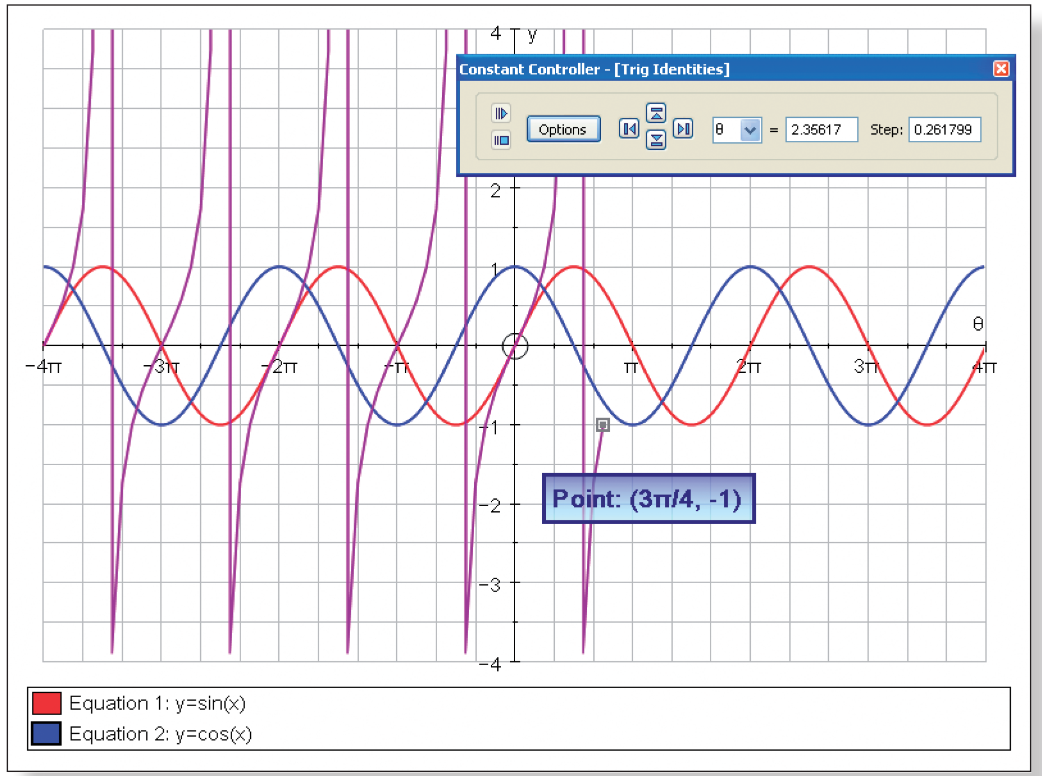
Teacher: Okay, now we are going to trace the path of points on the graph $y = \frac{\sin(\theta)}{\cos(\theta)}$ to get a picture of the shape of the function.

Use the **up button** to increase the value of θ.

The path of the point should be marked.

Draw the students' attention to what happens to the important points on the graph, particularly around the x-axis and approaching the asymptotes.

Your screen should look something like this:



Note: You may wish to briefly explain that the vertical lines created by the trace do not represent the asymptotes of the function, but are simply the computer's attempts to join up points which should never really be joined up!

Teacher: And of course, this graph has another name… which is?

Ideal Response: $y = \tan(\theta)$

Close the Constant Controller.

Ensure you are still in Slow Plot mode.

Enter the equation: $y = \tan(\theta)$

The graph should begin to plot over the top of the trace, making the curve much smoother.

Teacher: And so what is the identity that links together $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$?

Ideal Response: $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$

ACTIVITY 2: $\sin^2(\theta) + \cos^2(\theta) = 1$

Click on Edit > Select All and press delete on the keyboard (or Right-click on the graph itself and select Delete Objects from the menu).

This should clear the screen leaving you with just the set of axes.

Enter the equation: $y = \sin(\theta)$

Teacher: Okay, let's look at a different identity. To begin with, can anybody come tell us what the graph of $y = \sin^2(\theta)$ would look like?

Prompt: Look at the graph of $y = \sin(\theta)$. What does the squared symbol actually mean? What effect does squaring all the y values have? Does the amplitude change? Does the slope of the graph change? What happens to the negative values? What is the period of the graph?

Ideal Response: Squaring each of the y values will mean that there will be no negative values, and so the graph will never go below the x-axis. The amplitude will now be between 0 and 1, and the period of the graph will be halved as there are no negative values. y values of 0 and 1 will still be the same, but y values inbetween will get smaller when squared, altering the slope of the graph.

Teacher: Good. Can somebody come up to the front and do a quick sketch of the graph?

Encourage students to come to the front to sketch their curves.

Use the Erase tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > Select All Scribbles, and press delete on the keyboard (or Right-click on the graph area itself and select Delete Objects from the menu).

When you are ready:

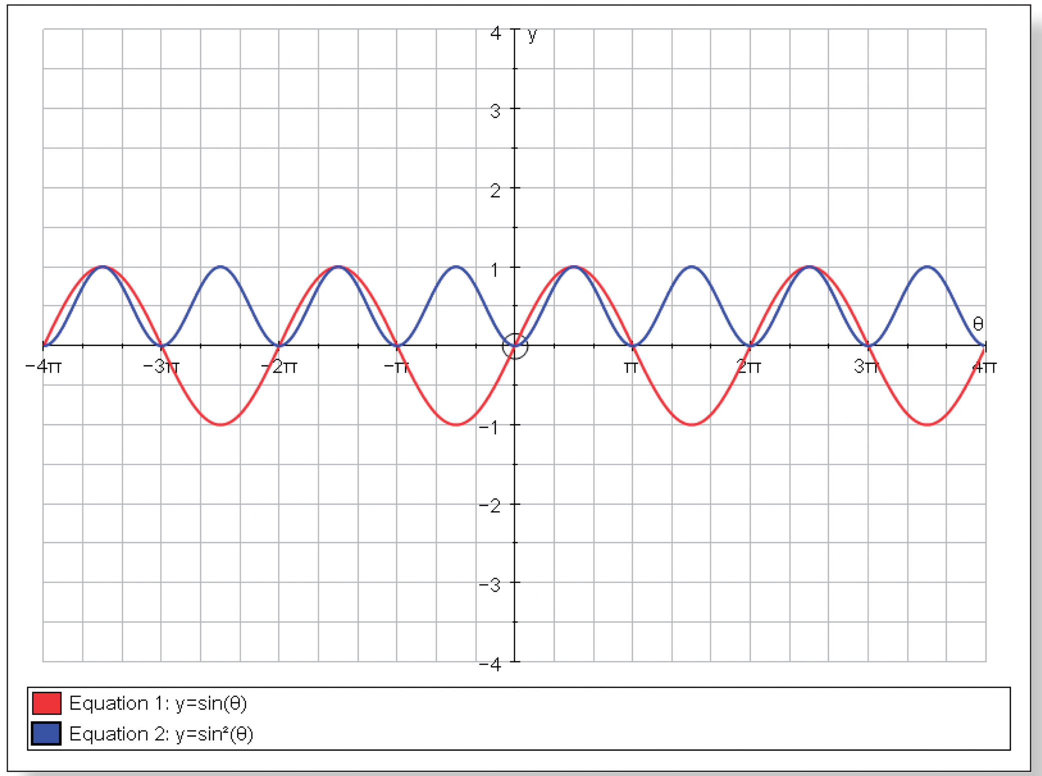
Ensure Slow Plot mode is turned on.

Enter the equation: $y = \sin^2(\theta)$
Note: To enter the squared term, either use the little two, or press "alt 2" together.
Note: The equation can also be written as $y = (\sin(\theta))^2$.

The curve should begin to be drawn on the screen.

Press Pause Plotting (or the Spacebar) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Turn off Slow Plot mode.

Click on Manage Equation List.

Left-click on the equation y = sin(θ) from the menu and click the red cross in the top corner.

On the same screen, add the equation y = cos(θ), and click OK.

The graph of y = cos(θ) should have replaced y = sin(θ).

Teacher: Thinking about what we have just done, can somebody come up and do a quick sketch of the graph of $y = \cos^2(\theta)$.

Encourage students to come to the front to sketch their curves.

Use the Erase tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > Select All Scribbles, and press **delete** on the keyboard (or **Right-click** on the graph area and select Delete Objects from the menu).

When you are ready:

Click on Slow Plot mode.

Enter the equation: $y = \cos^2(\theta)$

The curve should begin to be drawn on the screen.

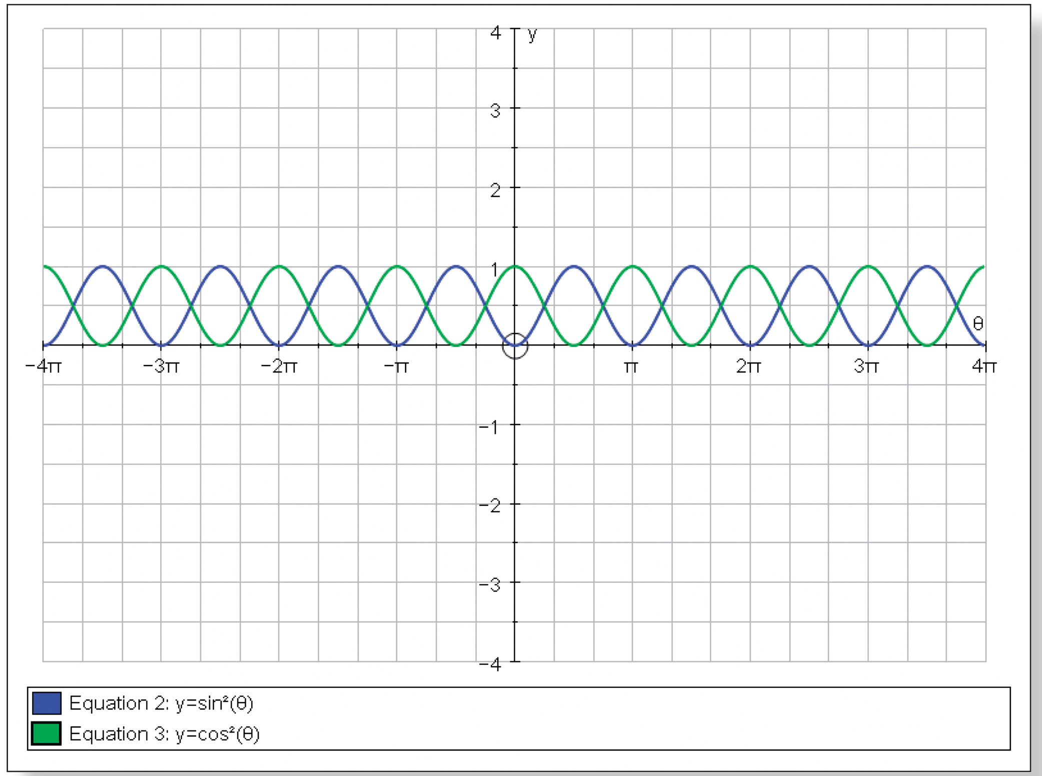
Press Pause Plotting (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Click on Manage Equation List.

Left-click on the equation y = cos(θ) from the menu and click the red cross in the top corner

Delete all scribbles from the screen as described above.

Your screen should look something like this:



Teacher: Now, looking at the graphs, clearly there seems to be some relationship between them, and I wonder if we could express it as an identity? Let me add on a couple of points to help you…

Enter the equation: x = a

Still in the Add Equation box:

Click on Edit Constants and change the value of a to −3π, and click OK.

Click on Draw Options and from the **drop-down** menu select a dashed line.

Teacher: What will this line look like?

Ideal Response: A vertical line going through the point x = −3π.

Click OK twice, and a vertical line should be on your screen.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the dashed line and the graph of $y = \sin^2(\theta)$ (they should both turn black).

Right-click and select **Solve Intersection** from the menu.

The point where the line crosses the curve should now be marked.

Left-click on an unoccupied part of the graph area to *de-select everything*.

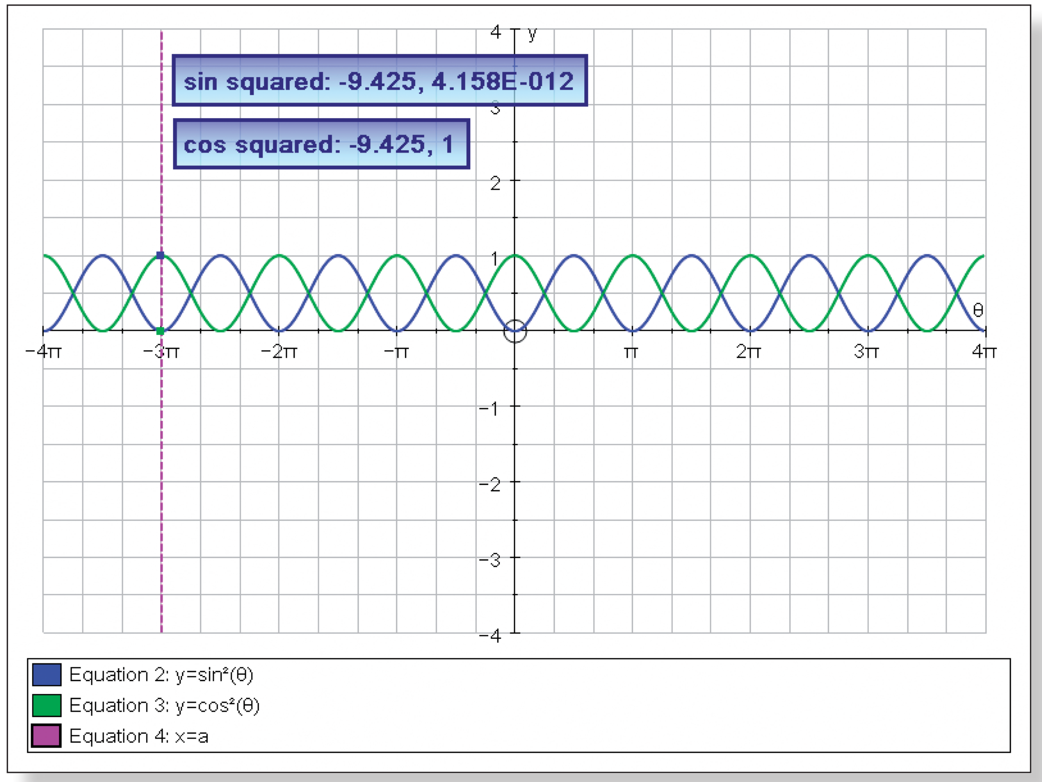
Left-click on the point marking the intersection (it should turn black).

Click on **Text Box**, change the words "intersection solver" to "sin squared", and click **OK**.

The co-ordinates of the point of intersection should now be displayed.

Repeat the above instructions to label the point of intersection of the line and $y = \cos^2(\theta)$.

When completed, your screen should look something like this:



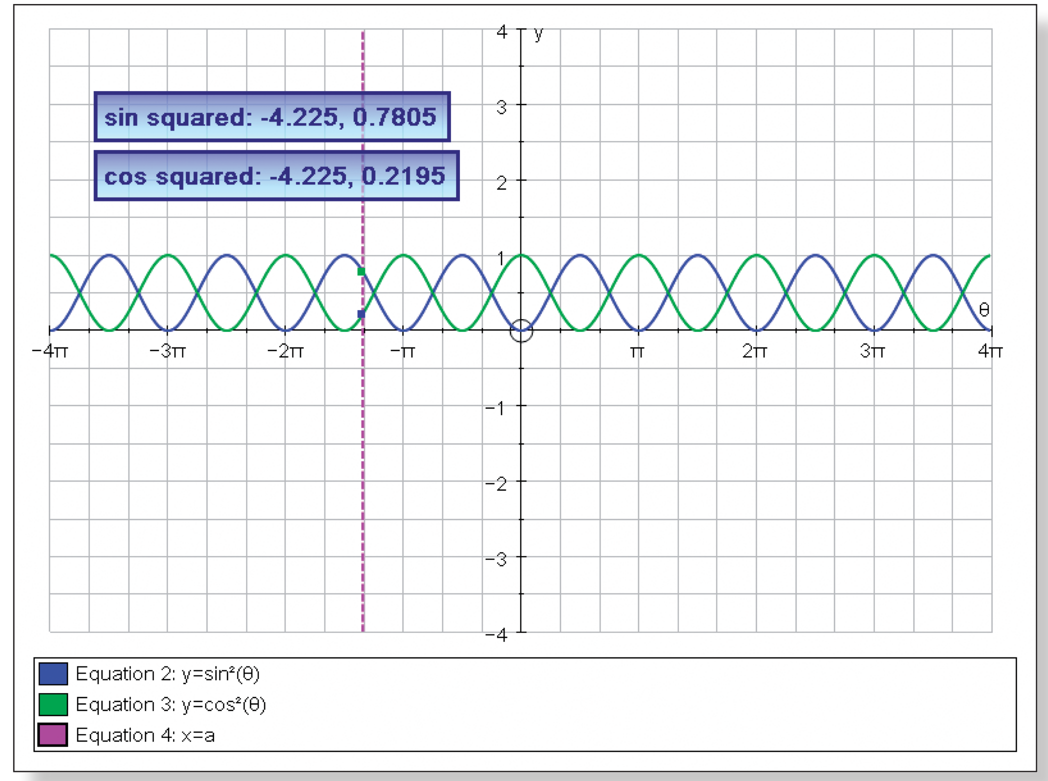
Click on the **Constant Controller**.

Change the value of the step to $\pi/12$.

Use the **up button** to increase the value of a.

The vertical line should move across the screen, adjusting the co-ordinates of the points of intersection.

Your page should look something like this:



Teacher: Now, keeping your eye on the co-ordinates, can anybody give me an identity linking $\sin^2(\theta)$ and $\cos^2(\theta)$?

Prompt: Pay particular attention to the y values.

Ideal Response: $\sin^2(\theta) + \cos^2(\theta) = 1$

Note: Another nice way to show that this always adds up to 1 is to place a point at $x = a$ on the graph of $y = \sin^2(x)$, then add a vector $\begin{pmatrix} 0 \\ \cos^2(a) \end{pmatrix}$ to the point, and then use the constant controller to adjust.

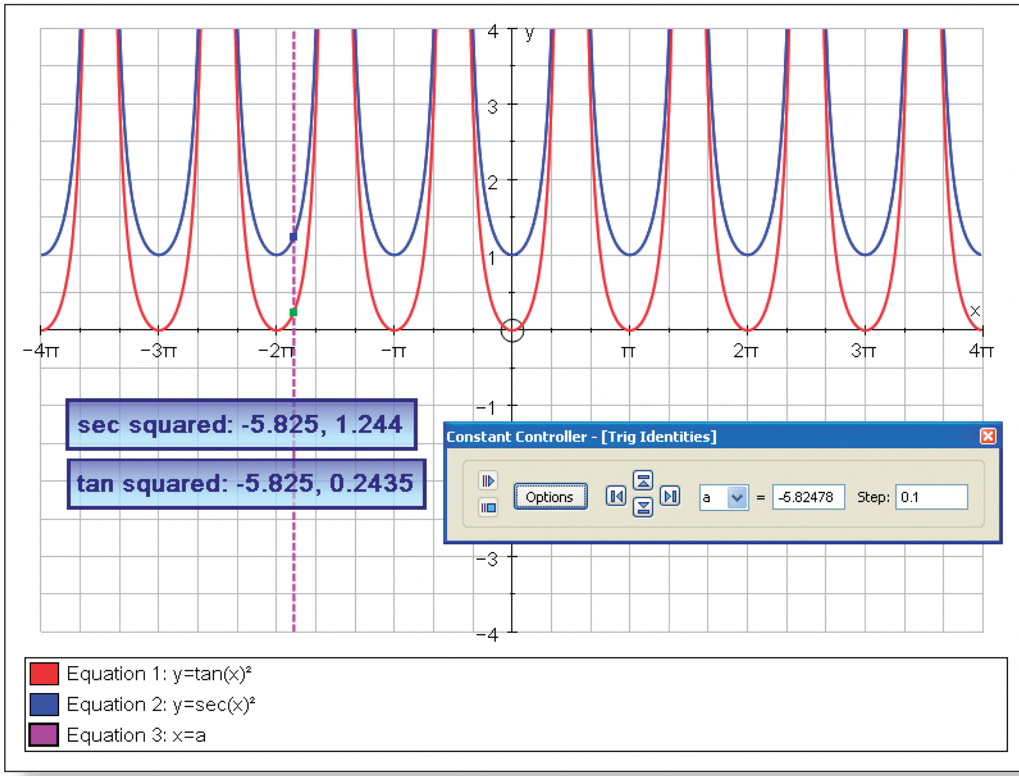
Ideas for Further Work

- Deriving these identities using right-angled triangles and the unit circle.
- Solving trigonometric equations which rely on these identities.
- Deriving the shape of the reciprocal trigonometric functions: see Teacher Demonstration T7: Reciprocal Functions.
- Proving further trigonometric identities.

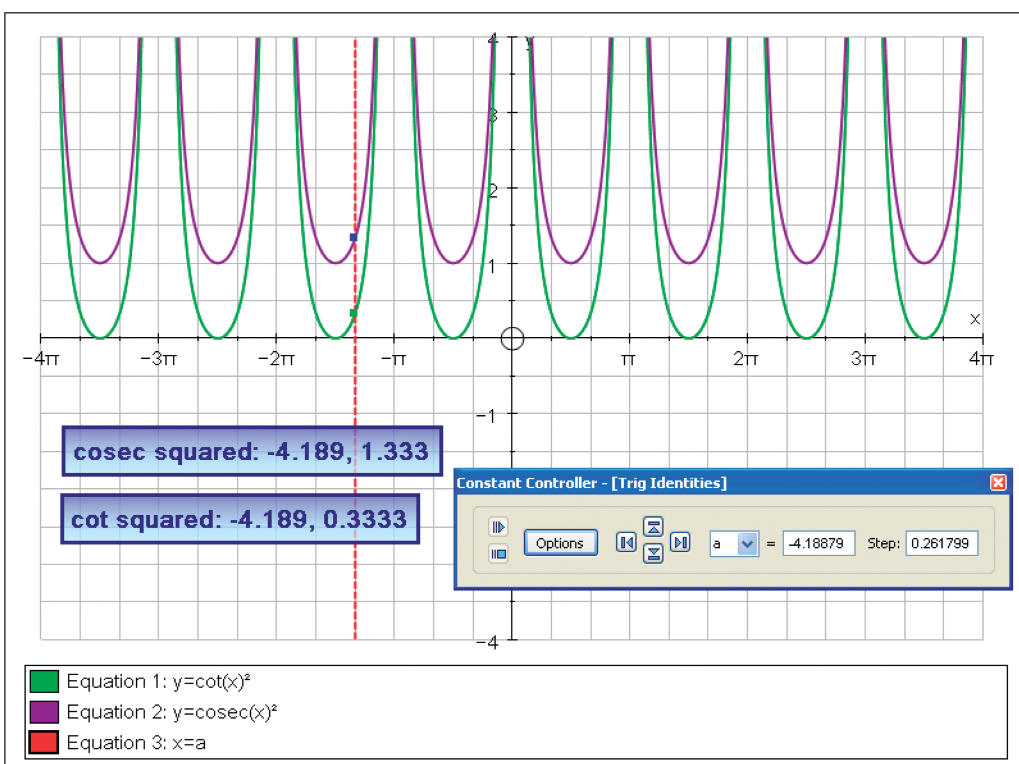
Note: A really nice activity that could follow directly from this demonstration would be to use Autograph and its constant controller to investigate identities in the form: $y = \sin^n(x) + \cos^n(x)$. Sure, when $n = 1$ and 2, life is pretty easy, but how about when $n = 3, 4, 5\ldots$

Note: Autograph can also be used in a similar way to get a graphical representation of the reciprocal identities. For example:

1. $\tan^2(\theta) + 1 = \sec^2(\theta)$



2. $\cot^2(\theta) + 1 = \mathrm{cosec}^2(\theta)$



Teacher Demonstration

T9 Completing the Square: A Graphical Approach

Learning Objectives

- To get a visual representation of what the "completed square" form of a quadratic function actually means graphically.
- To understand that the completed square form of a quadratic equation enables us to:
  1. Find the co-ordinates of the maximum or minimum point.
  2. Sketch a curve using horizontal and vertical translations from the curve $y = x^2$.
- To be able to deduce important features about the graph of a function when in the completed the square form.
- To be able to use the completed square form to derive the equation of quadratic functions, both positive and negative, given the co-ordinates of the maximum or minimum points.

Note: This demonstration activity focuses solely on quadratic equations which can be expressed in the from $y = (x + q)^2 + r$, and what we can learn from equations in this form. It is intended to expand students' understanding of curve sketching and how the equations of functions relate to their graph, and not to cover expressing equations in the from $y = p(x + q)^2 + r$.

Required Pre-Knowledge

- To be able to express quadratic equations: $y = x^2 + ax + b$ in the "completed square" form: $(x + q)^2 + r$.
- To know the shape of quadratic functions.
- To be able to factorise quadratic functions.

Note: Prior knowledge of transformations in the form $f(x + a)$, $f(x) + a$, and $-f(x)$ would be useful for this activity, but is not essential. Indeed, it is possible to use this activity to introduce, or re-cap on transformation work whenever it occurs in the course. Student Investigation 1 – Transforming Graphs would be one way of introducing this topic.

Pre-activity Set-up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

For this activity, you will need to set-up *two* blank 2D Graph Pages.

Page - 1:

Edit the axes as follows:

x: Minimum: −20 Maximum: 20 Numbers: 2 Pips: 1

y: Minimum: −12 Maximum: 12 Numbers: 2 Pips: 1

Remove all of the green ticks underneath Auto.

Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Still in the Edit Axes menu:
Click on the Appearance tab.
Open the **drop-down** menu underneath Themes.
Select Graph Paper.

Click on Equal Aspect Mode.

This alters the x scale so that the axes are square.

Enter the equation: $y = x^2 + 20x + 92$

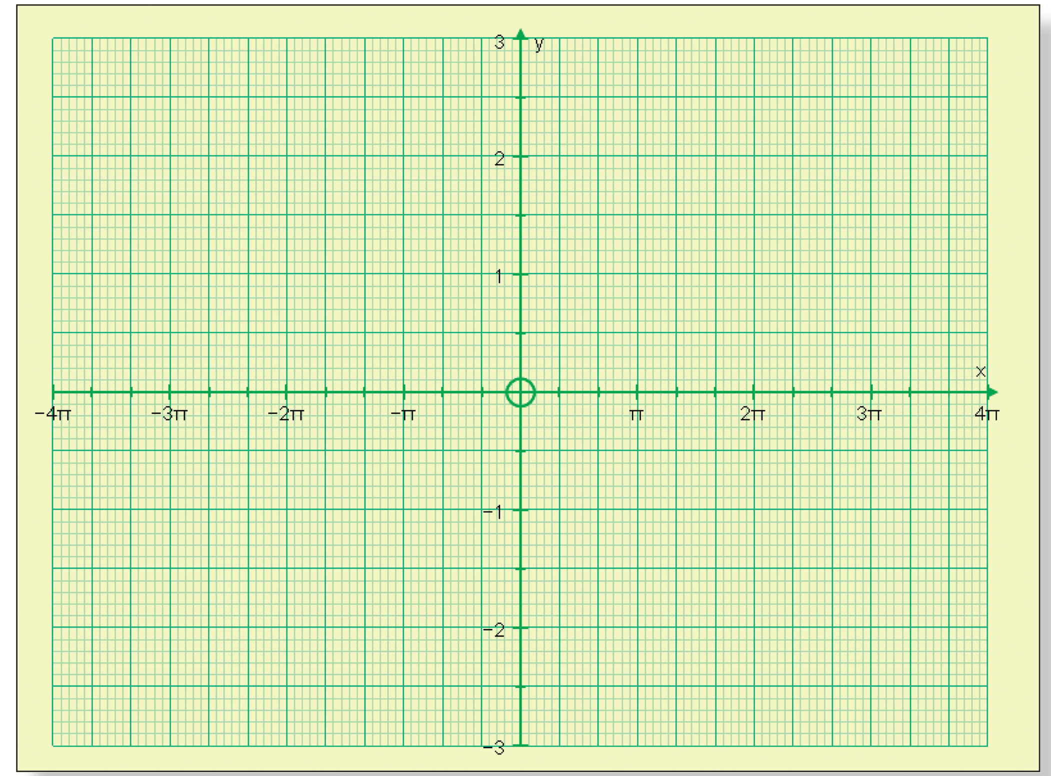
Enter the equation: $y = -x^2 + 8x - 10$

At the top of the screen go to Axes > Show Key.

This should make the key at the bottom of the screen disappear.

Note: This can also be done by **right-clicking** on the Key towards the bottom of the screen where it says "Equation 1: $y = 8$", and from the menu **left-click** on Show Key.

Your page should look something like this:



Left-click on an unoccupied part of the graph area to *de-select everything*.

We will use *Page - 1* later.

Open up a New 2D Graph Page.

Page - 2:

Edit the axes as follows:

x: Minimum: −12 Maximum: 12 Numbers: 1 Pips: 1

y: Minimum: −6 Maximum: 6 Numbers: 1 Pips: 1

Remove all of the green ticks underneath Auto.

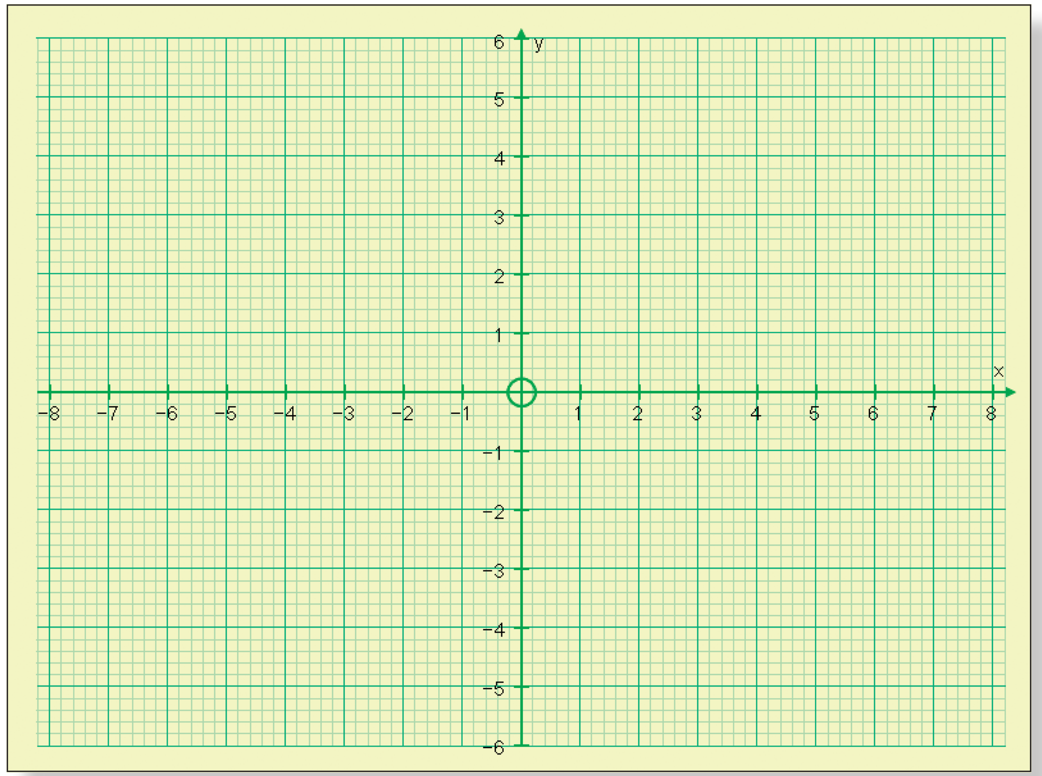
Still in the Edit Axes menu:
Click on the Appearance tab
Open the **drop-down** menu underneath Themes
Select Graph Paper

Click on Equal Aspect Mode.

From the Axes menu, select Show Key.

This should make the Key disappear.

Your screen should look like this:



The start of this demonstration uses *Page - 2.*

Note: Pages 1 and 2 are available as Tabs at the top of the screen, and each page can be viewed by simply clicking on the relevant tab.

Step-by-Step Instructions

Activity 1: Finding the Minimum Point

Teacher: Okay, to warm you up, if you were asked to sketch the graph: $y = x^2 + 3x - 2$, can anybody tell me something about its key features? For example, its shape, where it crosses the axes?

Prompt: **Shape:** is the x^2 term positive or negative? Is it a U shape or an upside down U?

x-axis: what is the value of x when y is 0? Can we factorise? Do we need to use the quadratic formula? Can we tell what side of the y-axis the crossing points are going to lie on?

y-axis: what is the value of y when x is 0?

Ideal Response: **Shape:** the x^2 term is positive, so the graph must be U shaped.

x-axis: $x^2 + 3x - 2$ doesn't factorise, so we would have to use the quadratic formula ($x = -3.56$ and 0.56 to 2 decimal places, just in case any keen students worked it out). Because the final term (-2) in the expression is negative, we can also say, without working them out, that one solution must be positive and one solution must be negative, so the curve crosses the x-axis at either side of the y-axis.

y-axis: when $x = 0$, $y = -2$, so the y-intercept is at $(0, -2)$.

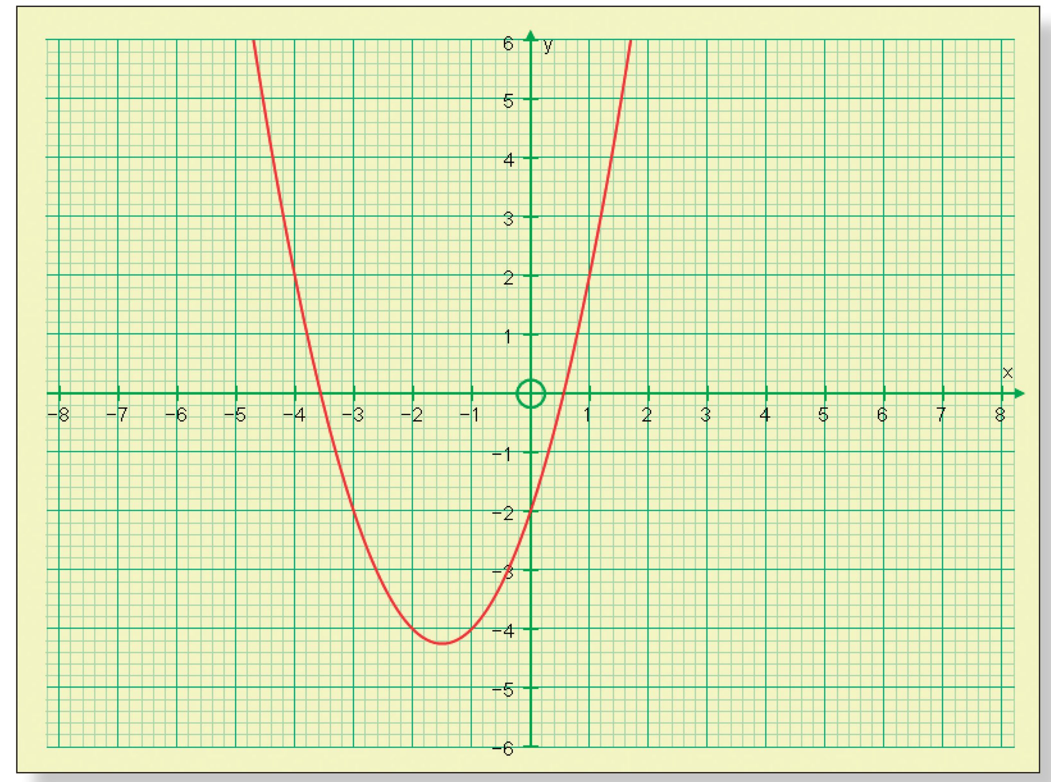
Click on Slow Plot mode.

Enter the equation: $y = x^2 + 3x - 2$

Note: To enter x^2, either use the little 2 button, or press "**alt 2**" together, or press "**xx**"

The curve should begin to be drawn on the screen.

Your screen should look something like this:



Press Pause Plotting both the stop the process, or to resume it to focus on the key features of the graph.
Note: The **Spacebar** on the keyboard can also be used for this function.

Teacher: Okay, so now, can you express the same equation, $y = x^2 + 3x - 2$, in "completed square" form?

Prompt: Remember, completed square form means: $(x + q)^2 + r$.

Ideal Response: $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 2$

$y = (x + 1.5)^2 - 4.25$

Teacher: Good. Now, the question is, how could the completed the square form have helped us when we were asked to sketch the curve?

Prompt: Think about what the completed square form can tell us about the curve. Think about the smallest possible value that y can take according to the completed square form. Try a few x values out in your head, and see what you get for y. What value of x gives us this minimum value of y? Why will all other values of x give us a greater value of y? What important point on the curve does this knowledge immediately give us?

Ideal Response: The completed square form shows us that the smallest value y can take is -4.25, and this occurs when x is equal to -1.5. Any other x value would mean that the bracket has a value other than zero, and hence when it is squared, this value will be positive, thus increasing the value of y. And so, the minimum value of y (-4.25) occurs when $x = -1.5$, which must give us the co-ordinates of our minimum point of the curve: $(-1.5, -4.25)$.

Teacher: Excellent. Right, let's check that using a special function on Autograph which finds us the maximum and minimum values of any function.

Left-click on the curve (it should turn black).

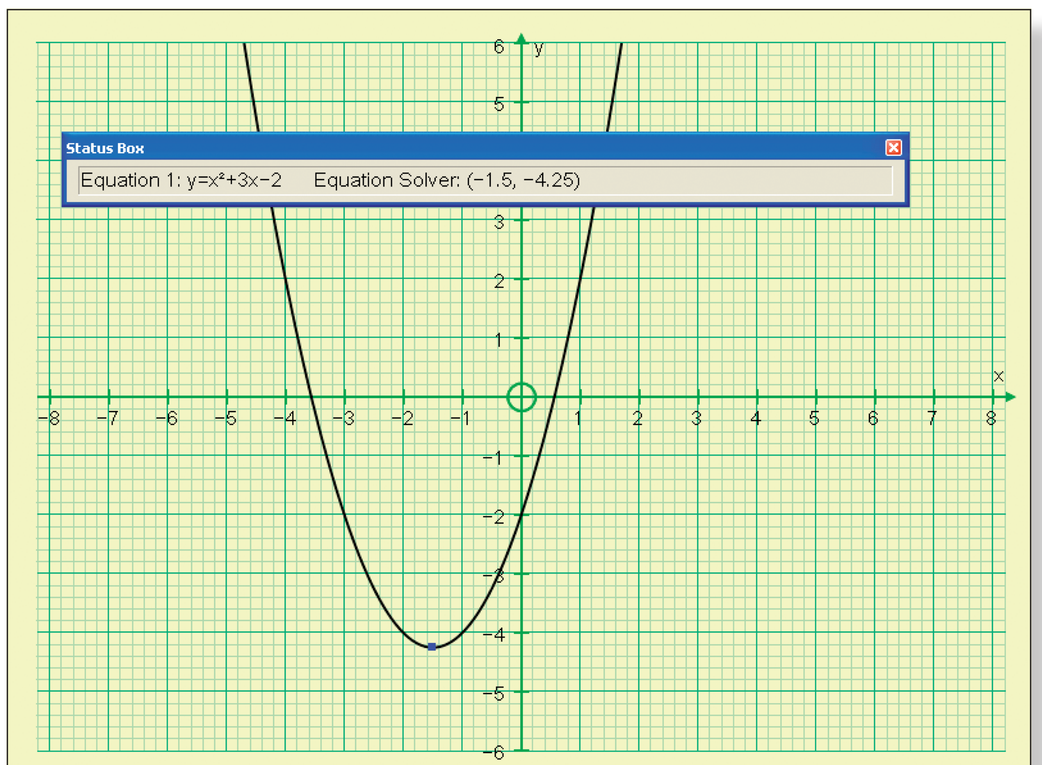
Right-click and select **Solve f'(x)=0** from the menu.

The minimum point should now be marked on the curve, and co-ordinates displayed on the bottom of the screen.

On the top toolbar, click on **View** > **Status Box** to see this result more clearly.

Note: The **Status Box** will only ever display information about object currently selected. If the **Status Box** ever goes blank, or displays the wrong information, simply **left-click** on your chosen object again.

Your page should look something like this:



Activity 2: Sketching Curves using Translations

Teacher: But, believe it or not, the completed square form is even more useful than that! So long as we know how to complete the square, and we can remember what the graph of $y = x^2$ looks like, then we can draw any quadratic function really quickly.

Teacher: Can somebody quickly remind us what the graph of $y = x^2$ looks like?

Make sure **Slow Plot** is still turned on.

Enter the equation: $y = x^2$

Click **OK** and the curve should begin plotting on the screen.

Teacher: Can anybody describe in terms of transformations, how to get from the graph of $y = x^2$ to the graph of $y = x^2 + 3x - 2$.

Prompt: Think about movements **left-right**, and movements up-down. What type of transformations are these movements?

Ideal Response: The graph of $y = x^2$ has been translated 1.5 units to the left, and 4.25 units down. Or, in terms of a vector, it has been translated: $\begin{pmatrix} -1.5 \\ -4.25 \end{pmatrix}$.

Teacher: Which is exactly what the completed square form of the equation told us would happen! Watch…

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve $y = x^2$ (it should turn *black*).

Right-click and select **Delete** from the menu, or simply press **delete** on the keyboard.

Click on the cross in the top right corner of the **Status Box** to close it.

Enter the equation: $y = (x + q)^2 + r$
Still on the **Add Equation** screen, click on **Edit Constants**.

Teacher: What values of q and r will we need in order to have the graph of $y = x^2$?

Ideal Response: $p = 0$ and $r = 0$.

Still in **Edit Constants**, change the values of both q and r to 0.

Click **OK**, and the graph should be exactly where $y = x^2$ used to be.

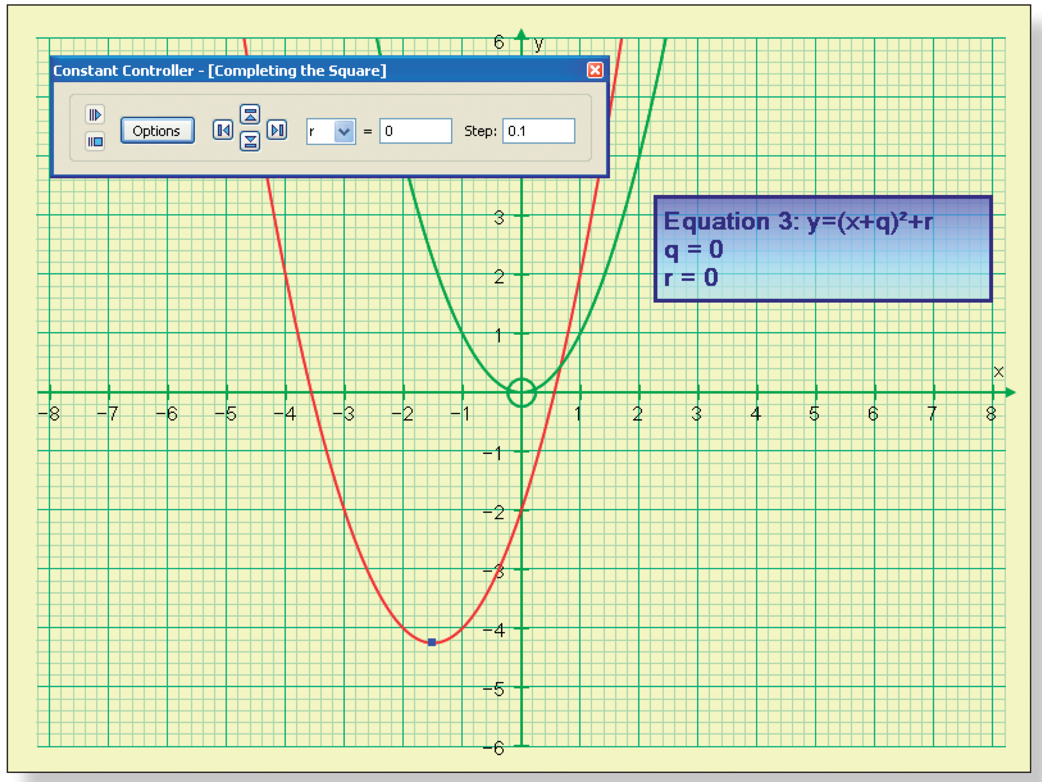
Ensure the graph of $y = (x + q)^2 + r$ is selected.

Click on **Text Box**.
Place a tick in the box next to **Show Detailed Object Text** and click **OK**.

The equation of the curve, together with the values of q and r should now be labelled.

Click on the **Constant Controller**.
You can now easily adjust the value of q and r.
The **drop-down** menu switches between the constants.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the size of the step.

Your page should look something like this:



Teacher: Now, at the moment, the values of q and r are both set to zero, so we just have the graph of $y = x^2$. But now I am going to alter them in turn to match the completed square form of our equation, which was $y = (x + 1.5)^2 - 4.25$.

Teacher: What do you think will happen as I adjust the value of q?

Select constant q, and begin adjusting it in steps of 0.1 until the value is equal to 1.5.

Draw the students' attention to the fact that the curve moves in the *opposite direction* than the sign in front of the q implies.

Note: If you have covered transformations, you could link this into the $f(x + a)$ transformation. If you have not covered it yet, remind students of this activity when the time comes.

Teacher: What do you think will happen as I adjust the value of r?

Now select constant r, and begin adjusting it in steps of 1, then 0.1, and then 0.01 until the value is equal to −4.25.

Note: Again, link to $f(x) + a$ if you have covered it.

Teacher: Can anybody summarise what each part of the completed square form tells us about the location of the curve relative to $y = x^2$?

Prompt: What did q tell us? What did r tell us?

Ideal Response: r tells us how far up or down to translate the curve. q tells us how far left or right, but we must remember to switch the sign.

Teacher: That's right. And I can now show you that this is true for every single point on the curve.

Activity 3: A Link to Vectors...

Adjust the values of q and r back to 0 so the curve returns to $y = x^2$.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve $y = x^2$ (it should turn black).

Enter a point on the curve with x co-ordinate 0.
The point (0, 0) should now appear on the curve.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point at (0, 0) (it should have a square around it).

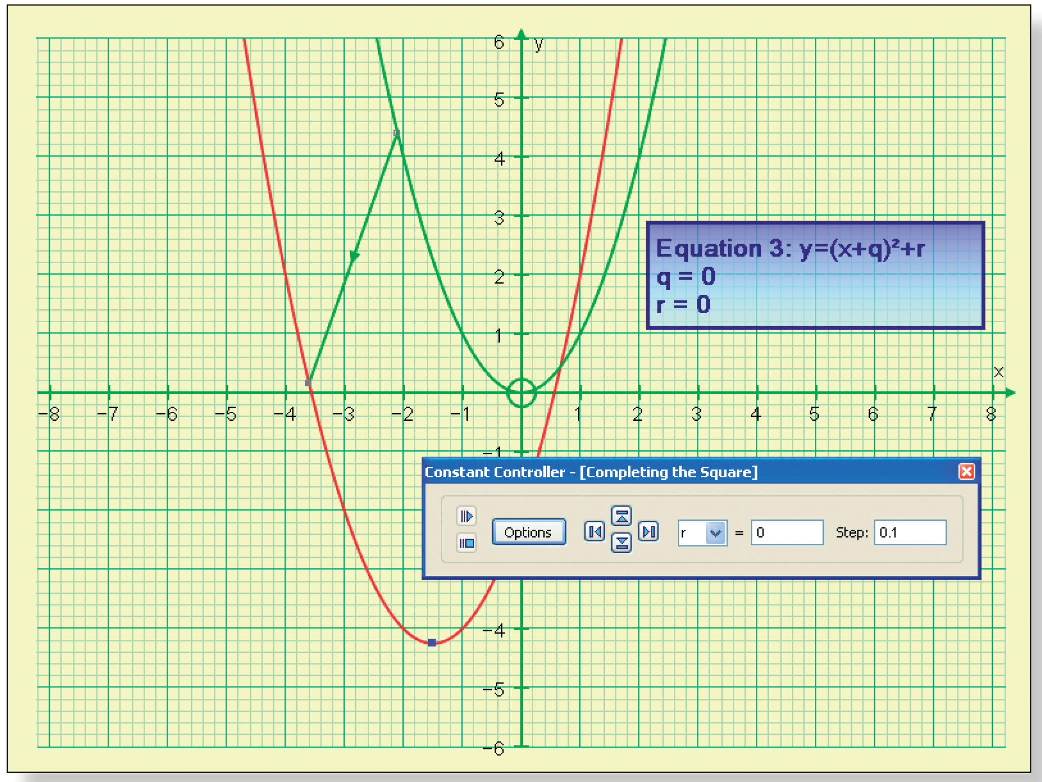
Right-click and select **Vector** from the menu.

Enter in the values −1.5 and −4.5 and click **OK**.

A vector should now appear connecting the two minimum points together.

Use the **left-right arrows** on your keyboard to move the point around, drawing the students' attention to the fact that the vector remains the same.

Your page should look something like this:



You can then really emphasise this concept by *tracing the point* as follows:

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve $y = x^2 + 3x - 2$ (it should turn black).

Right-click and select **Delete** from the menu.

Left-click on the point at the end of the vector (not the point on the curve $y = x^2$).

Right-click and select **Trace Point** from the menu.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point on the curve $y = x^2$.

Now use the **left-right arrows** on the keyboard to move the point along the curve, and as it goes it leaves a trace of the curve $y = x^2 + 3x - 2$ behind.

Teacher: And can anybody generalise this? How far must we translate each point on the curve $y = (x + q)^2 + r$, if we start at $y = x^2$?

Ideal Response: We must translate this by the vector: $\binom{-q}{r}$

Teacher: Good. And so not only does the completed square form of a quadratic equation tell us the value of the minimum point, it also tells us exactly how to sketch the curve starting from $y = x^2$. Now to put you to the test…

Activity 4: Putting it all Together

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point on the curve $y = x^2$.

Right-click and select **Delete** from the menu, or simply press delete on the keyboard, ignoring the warning.

You should now be left with one curve.

Question 1: Can somebody come up and sketch the graph of: $y = x^2 - 5x + 2$, and tell us the co-ordinates of the minimum point?

Prompt: Think about what the completed the square form tells us.

Answer 1:

$y = x^2 - 5x + 1$
$\quad = (x - 2.5)^2 - 6.25 + 1$
$\quad = (x - 2.5)^2 - 5.25$

So, the minimum point is at $(2.5, -5.25)$, and the curve is a translation of $y = x^2$ by 2.5 units to the right and 5.25 units down.

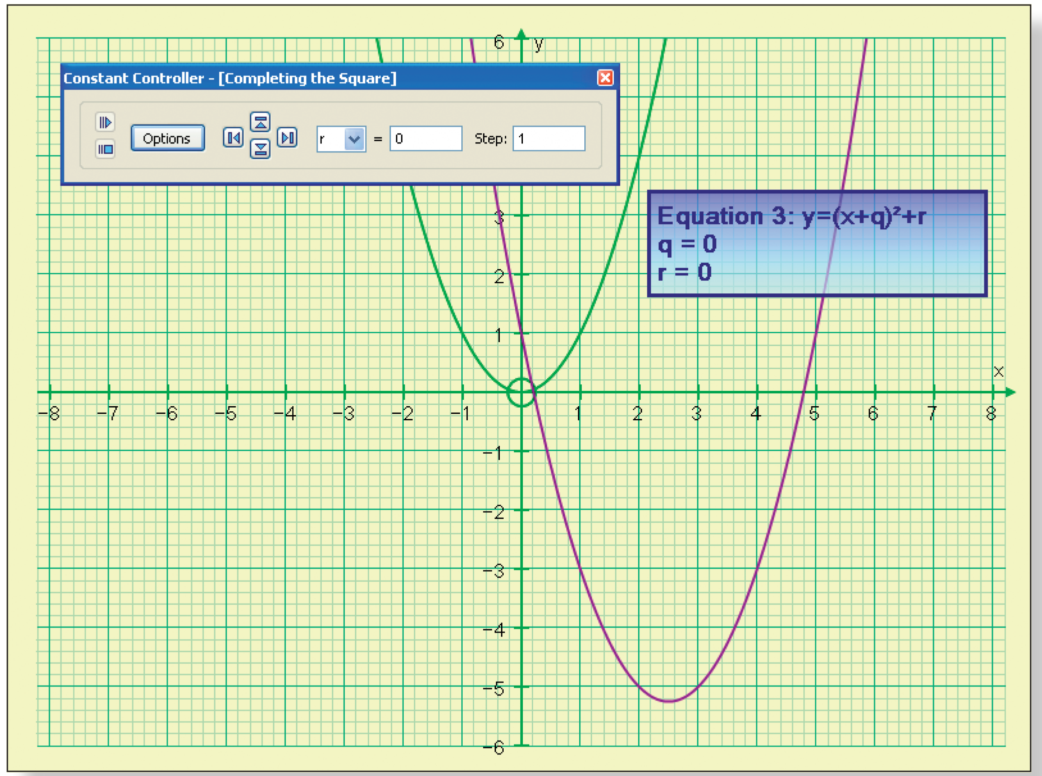
Encourage students to come to the front to sketch their curves and use the **Scribble Tool**.

Use the **Erase** tool to rub out any mistakes.
If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press delete.

Ensure **Slow Plot** mode is still on.

Enter the equation: $y = x^2 - 5x + 1$ to show the students that they are correct.

Your page should look something like this:



Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve $y = x^2 - 5x + 1$ (it should turn black).

Right-click and select Delete from the menu, or simply press delete on the keyboard.

Question 2: Without using the quadratic formula, show that the curve $y = x^2 + 10x + 27$ never crosses the x-axis.

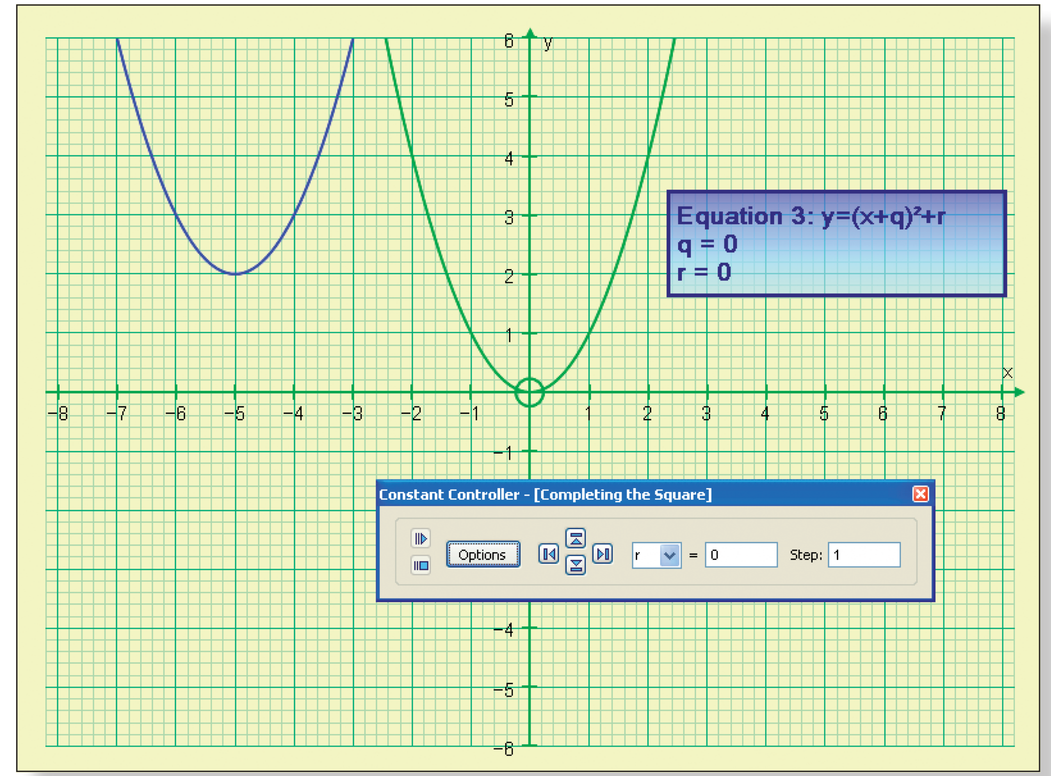
Prompt: Think about the minimum point on the curve and the shape of the curve.

Answer: $y = x^2 + 10x + 27$
$= (x + 5)^2 - 25 + 27$
$= (x + 5)^2 + 2$
So, the minimum point is at $(-5, 2)$, which is above the x-axis, and seeing as this is the minimum point, the curve will never go below this.

Ensure Slow Plot mode is still on.

Enter the equation: $y = x^2 + 10x + 27$ to show the students that they are correct.

Your page should look something like this:



Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve $y = x^2 + 10x + 27$ (it should turn black).

Right-click and select Delete from the menu, or simply press delete on the keyboard.

Question 3: Point A lies on the curve $y = x^2$ at $(-1, 1)$. The curve is translated to become $y = x^2 + 8x + 11$. What are the co-ordinates of Point A following the translation?

Prompt: Think about how the complete the square form helps us sketch curves relative to $y = x^2$.

Answer: $y = x^2 + 8x + 11$
$y = (x + 4)^2 - 16 + 11$
$y = (x + 4)^2 - 5$
So, the minimum point is at $(-4, -5)$, meaning the *whole curve/every point* has been translated 4 units to the left, and 5 units down.

So, the Point A must now be at $(-5, -4)$.

Enter the equation: $y = x^2 + 10x + 27$

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve $y = x^2$ (it should turn black).

Enter a point on the curve with x co-ordinate -1.

The point $(-1, 1)$ should now appear on the curve.

Left-click on an unoccupied part of the graph area to *de-select everything*.

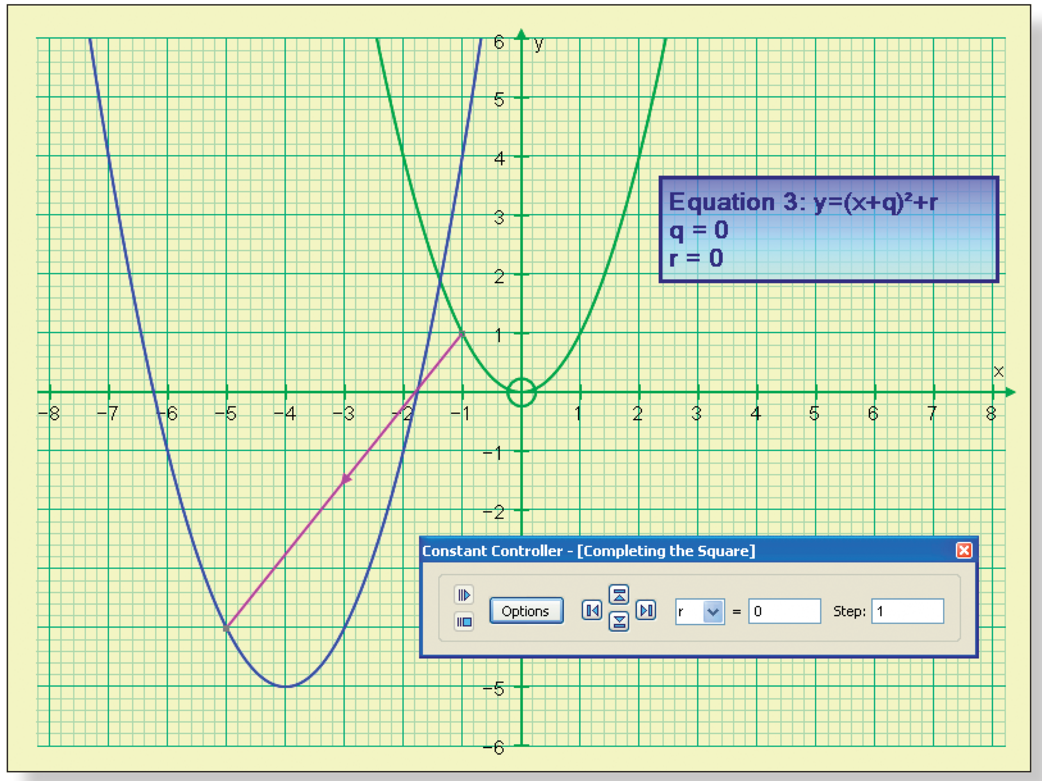
Left-click on the point at (−1, 1) (it should have a square around it).

Right-click and select Vector from the menu.

Enter in the values −4 and −5 and click OK.

The vector should show the path of the point and show the students they are correct.

Your page should look something like this:



Note: Again, you can also use the left-right arrows on the keyboard to move the point along the curve and illustrate how every single point is translated in the same way.

Click on the *Page - 1* tab to open the page you prepared at the start of the lesson.

Question 4: What are the equations of these two curves in the form $y = x^2 + ax + b$?

Red Line:

Prompt: You know the minimum point, so try working backwards!

Answer: Minimum Point: $(-10, -8)$
So, using completed the square form: $y = (x + 10)^2 - 8$
Expand brackets: $y = x^2 + 20x + 100 - 8$
Simplify: $y = x^2 + 20x + 92$

Left-click on the red curve (it should turn black).

On the top toolbar, click on View > Status Box.

The correct equation of the curve should now be displayed.

Blue Line:

Prompt: This is quite a tricky one and a good point of discussion. One way to do it is to treat the curve as a positive U shaped graph with a minimum point at (−10, −8), and work out the equation of that [$y = x^2 + 8x + 22$], and then think about reflecting that curve in the line $y = 6$. Another way is to remember to multiply the terms inside the square brackets by −1 (change all the signs) and proceed as normal to get the curve you require.

Answer: Maximum Point: $(4, 6)$
So, complete the square form: $y = -1\,[\,(x - 4)^2\,] + 6$
Expand brackets: $y = -1\,[\,x^2 - 8x + 16\,] + 6$
Expand second brackets: $y = -x^2 + 8x - 16 + 6$
Simplify: $y = -x^2 + 8x - 10$

Left-click on the blue curve and the equation should appear in the Status Box.

Ideas for Further Work

- This demonstration would link nicely into any of the following topics:
  1. Transformations of curves – See Student Investigation 1
  2. Co-ordinate Geometry
  3. Harder completing the square
  4. Further algebraic manipulation
  5. Differentiation as an alternative way to find maximum and minimum points

LEARNING OBJECTIVES

- To be introduced to the concept of parametric equations, and to be able to visualise and understand their construction.
- To understand how parametric equations involving trigonometric terms relate to their original normal Cartesian equations by using identities.
- To understand the effect that the range of a set of parametric equations may have on the graph of the resulting normal Cartesian function.

REQUIRED PRE-KNOWLEDGE

- To be comfortable working in radians.
- To be aware of the following trigonometric identities: $\sin^2(t) + \cos^2(t) = 1$ and $\cos(2t) = 1 - 2\sin^2(t)$.
- To be comfortable with the concept of a constant.
- To understand that the equation of a circle with centre (a, b) and radius r can be expressed as follows: $(x - a)^2 + (y - b)^2 = r^2$
- To be familiar with the graphs of $y = \sin(ax)$ and $y = \cos(ax)$, where a is a constant.
- To understand the concept of the range of a function.

PRE-ACTIVITY SET-UP

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

Edit the axes as follows:

x:	Minimum: −6	Maximum: 6	Numbers: 1	Pips: 1
y:	Minimum: −3	Maximum: 3	Numbers: 1	Pips: 1

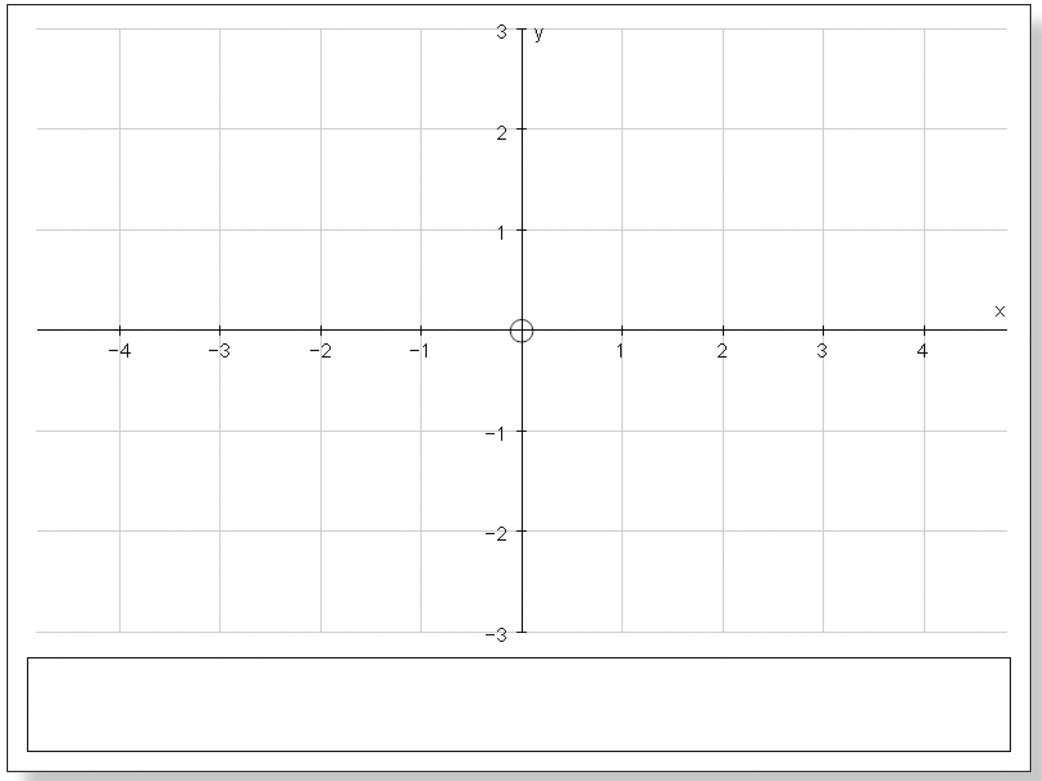
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Select **Equal Aspect Mode**.

This will automatically adjust the x-axis so the axes are square.

Ensure you are working in **Radians**.

Your screen should now look something like this:



STEP-BY-STEP INSTRUCTIONS

ACTIVITY 1: INTRODUCING THE CONCEPT

Teacher: Right, a tricky one to start with. Does anybody have any idea what the graph of this function (or even functions) would look like: $x = \cos(t)$, $y = \sin(t)$.

Prompt: Unless students have experienced parametric equations before, it is unlikely that they will be able to determine the nature of the graph at this stage. Subtle prompts along the following lines should get their brains ticking: Will there be one function or two? How many variables are there? How are we going to cope with that on our usual x-y set of axes?

Teacher: Okay, how about we try to find a few points which lie on this function, or functions. Can anybody think of any values of t which might give us some nice x and y co-ordinates?

Prompt: Remember, we are working in radians. How about 0? How about π? 2π? What about some of the special values in between?

Ideal Response: By trying different values of t, we can build up a set of co-ordinates that must lie on the function, or functions:

When $t = 0$	$\cos(t) = 1$	$\sin(t) = 0$	which gives us	$(1, 0)$
When $t = \pi$	$\cos(t) = -1$	$\sin(t) = 1$	which gives us	$(-1, 1)$
When $t = 2\pi$	$\cos(t) = 1$	$\sin(t) = 0$	which gives us	$(1, 0)$
When $t = \frac{\pi}{2}$	$\cos(t) = 0$	$\sin(t) = 1$	which gives us	$(0, 1)$

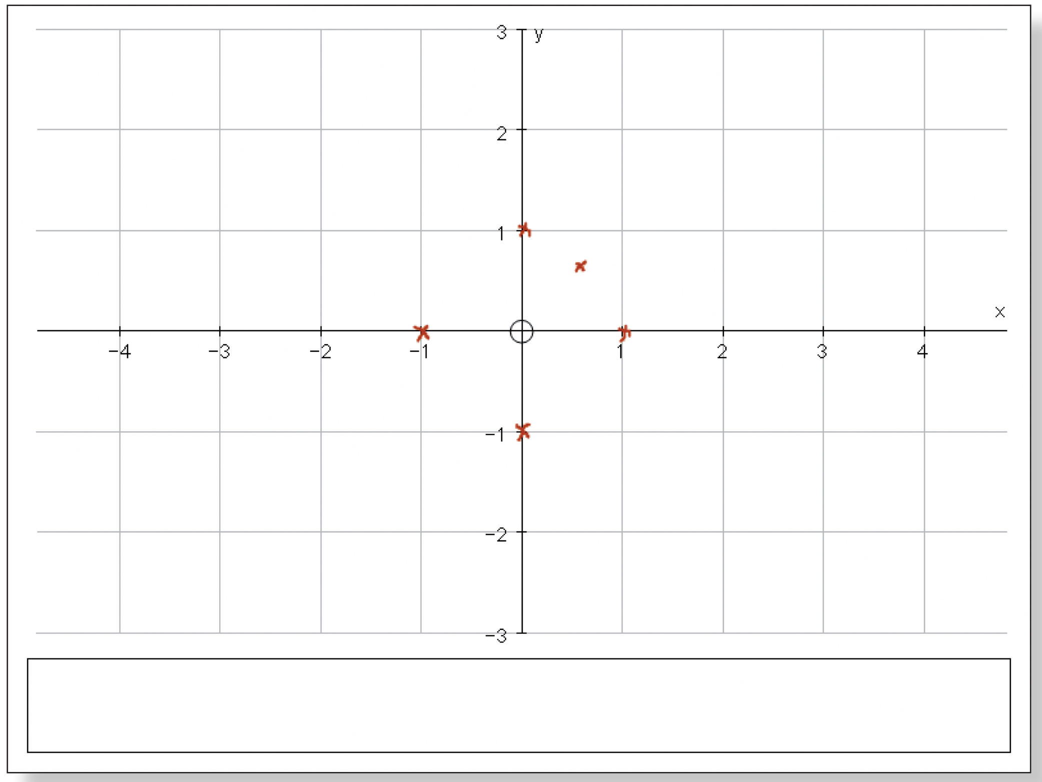
When t = $\frac{3\pi}{2}$ cos(t) = 0 sin(t) = −1 which gives us (0, −1)
When t = $\frac{\pi}{3}$ cos(t) = 0.5 sin(t) = $\frac{\sqrt{3}}{2}$ which gives us (0.5, $\frac{\sqrt{3}}{2}$)
etc…

Encourage students to come to the front to mark these points on the graph using the **Scribble Tool**.

Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on **Edit** > **Select all scribbles**, and press **delete** on the keyboard (or **Right-Click** on the graph area itself and select **Delete Objects** from the menu).

Your screen should look something like this:



Teacher: Okay, so can anybody describe the graph that seems to be forming here?

Prompt: Look at the shape of the points. Does it look like we have one or two functions? Be specific with your description.

Ideal Response: The points seem to form a circle, with centre (0, 0), and radius 1.

Teacher: Sounds good, so let's use Autograph to check:

Click on **Slow Plot** mode.

Enter the equation: x = cos(t), y = sin(t)
Note: The brackets are not necessary, and can be omitted it you prefer, but the comma is crucial!

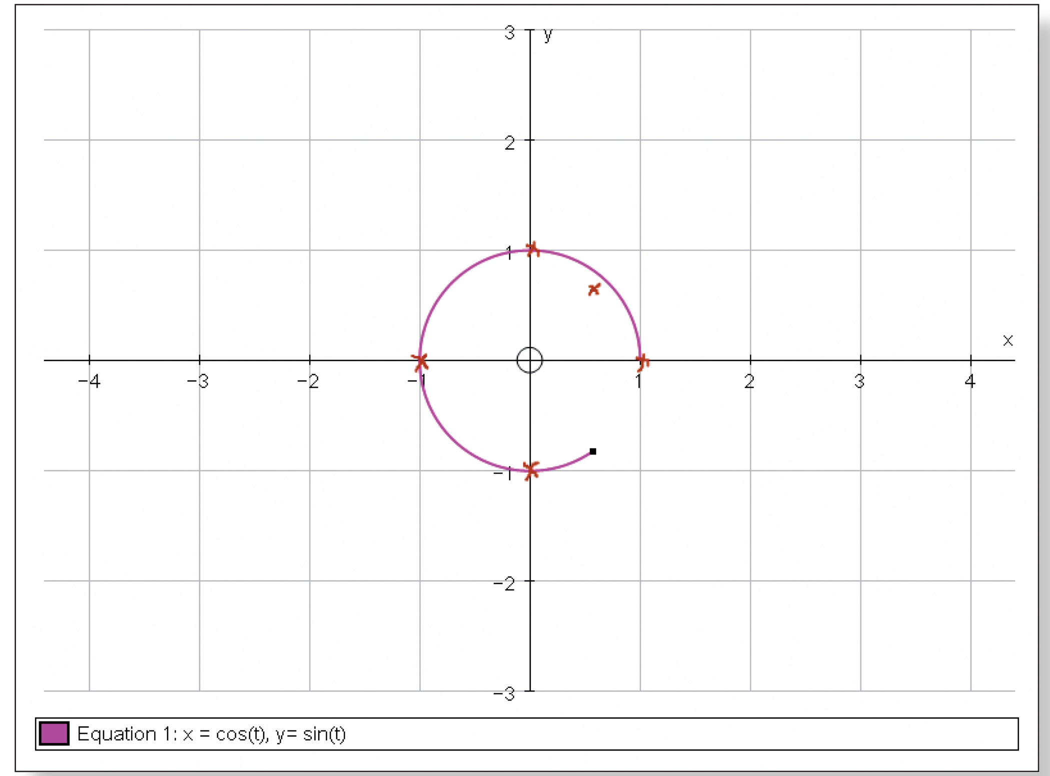
Click **OK**.

The curve should appear on screen, hopefully going through the points plotted by the students.

Press **Pause Plotting** both to stop the process and to resume it to focus on the key features of the graph.

Note: The **Spacebar** can also be used to serve this function.

Your screen should look something like this:



At this point it might be a good idea to clear all the scribbles off the screen.

Click on **Edit** > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph area and select **Delete Objects** from the menu).

You should now be left with just the graph of the circle.

Teacher: Now, the big question is why on earth do the equations x = cos(t), y = sin(t) give us the graph of a circle with centre (0, 0) and radius 1? Is there anyway we could have known that without plotting a load of points?

Prompts: Which variable does not appear on the graph? How can we eliminate t? Would a trigonometric identity help? Which trigonometric identities involve sin and cos? Can you use $\sin^2(t) + \cos^2(t) = 1$ to help eliminate t? What about if you start by squaring both sides? Then adding them together…

Ideal Response: 1. The variable t does not appear on the graph, so it must have been eliminated. We can use an identity to help us eliminate t.

2. Square both sides of the equations, giving: $x^2 = \cos^2(t)$ and $y^2 = \sin^2(t)$.
3. Now, add the corresponding sides of the equations together, giving: $x^2 + y^2 = \cos^2(t) + \sin^2(t)$.
4. Use the identity $\sin^2(t) + \cos^2(t) = 1$ to simplify the right hand side, giving: $x^2 + y^2 = 1$.

This is the equation of a circle, with centre (0, 0), and radius 1.

Teacher: Excellent! The equations $x = \cos(t)$, $y = \sin(t)$ are called **Parametric Equations**. They contain three variables, one of which (t) is called the **parameter**. What you have just managed to do, by eliminating the parameter, is to express them in normal, Cartesian form.

Give the students a few moments to digest this before moving on.

Activity 2: Extending the Circle

Make sure you are in **Select Mode**.

Left-click on the circle (it should turn black).

Right-click and select **Delete Object** from the menu, or simply press **Delete** on the keyboard.

You should now be left with just a set of axes.

Teacher: Okay, let's make life a little more difficult. I am now going to enter the equations: $x = a + r\cos(t)$, $y = b + r\sin(t)$. If I wanted to produce the exact same circle as before, what values of a, b and r would I need?

Ideal Response: $a = 0$, $b = 0$, $r = 1$

Teacher: Good, so here we go:

Make sure **Slow Plot** mode is still on.

Enter the equation: $x = a + r\cos(t)$, $y = b + r\sin(t)$
Still on the **Enter Equation** screen, click on **Edit Constants**, and set the values of the constants as follows: $a = 0$, $b = 0$, $r = 1$.
Click **OK** twice.

The circle should appear on the screen in the exact location as the previous one.

Left-click on the curve (it should turn black).

Click on **Text Box**.

Tick the box next to **Show Detailed Object Text** and click **OK**.

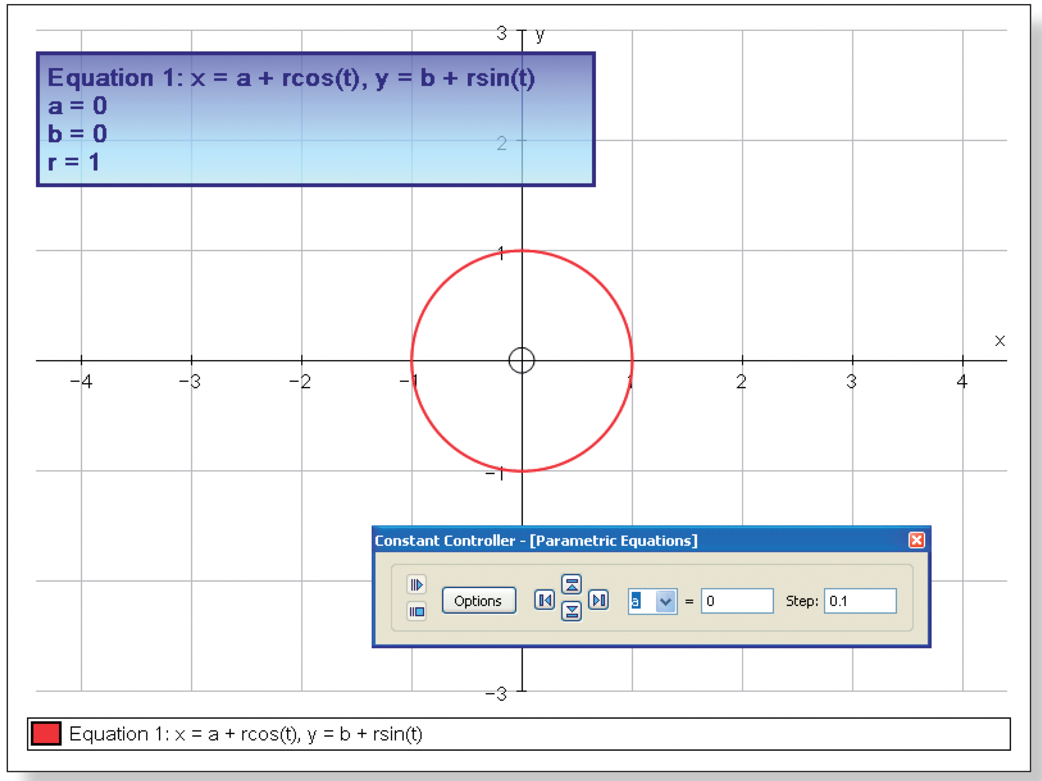
The equation of the circle, along with the current values of a, b and r should now be displayed.

Click on the **Constant Controller**.

The **drop-down** menu allows you to select each constant.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the value of the step.

Move both the **Text Box** and the **Constant Controller** to a convenient point on the page.

Your screen should look something like this:



Teacher: Now, the big question is, what do you think will happen to the graph of our function if we change the values of constants a, b and r?

Prompt: Again, more than likely this may not be immediately obvious to the students. Allow them a few minutes to make predictions and justify those predictions.

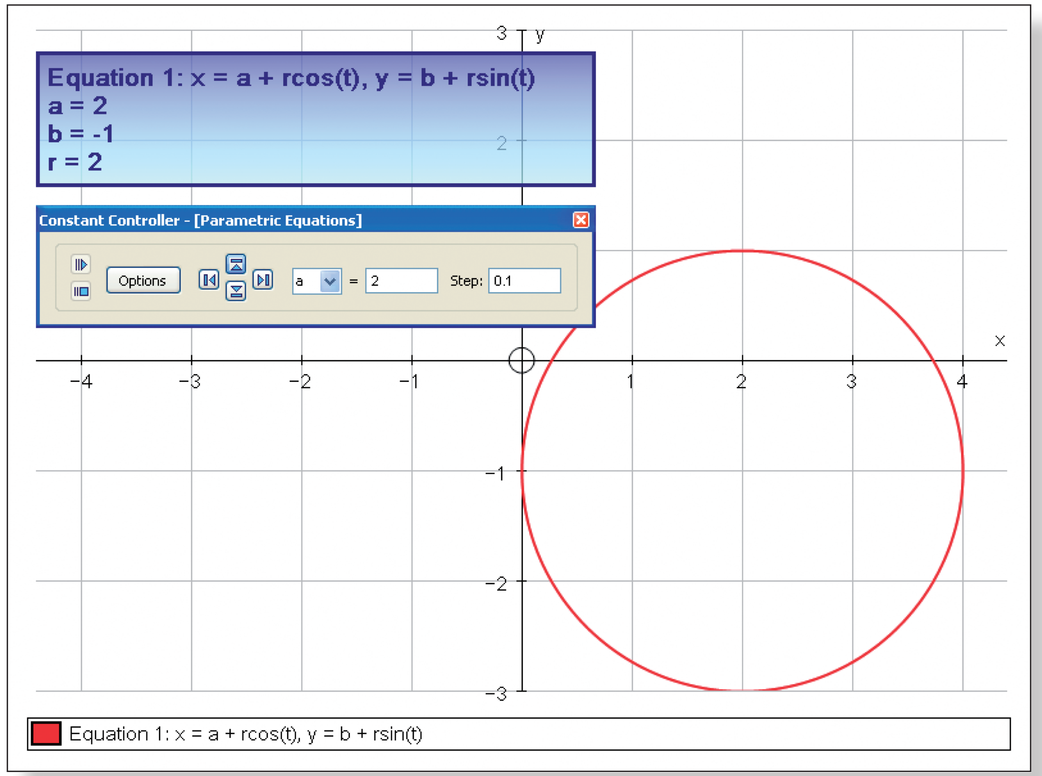
Manipulate one constant at time, perhaps adjusting the value of the step to 1, or 0.5.

Encourage the students to suggest different values for the constants.

Encourage them to predict the exact nature of the circle given certain values of constants.

Challenge them to give you values of constants that will result in specific circles.

After some manipulation, your screen might look something like this:



Teacher: What effect do a, b and r have on the circle, and can anybody express the equation of the circle in terms of a, b and r?

Prompt: Think about the location and size of the circle for different values of the constants. How does this relate to the equation of a circle? What is the general form for the equation of a circle? What do the constants represent in that equation?

Ideal Response: a and b effect the location of the centre of the circle, and r effects the size of the radius. The resulting circle can be expressed as follows: $(x - a)^2 + (y - b)^2 = r^2$, giving us a circle with centre (a, b) and radius r.

Teacher: Excellent. Now, using a similar technique to what we did before, can you show how we could have found that out just from the original parametric equations: $x = a + r\cos(t)$, $y = b + r\sin(t)$?

Prompt: Think about eliminating the parameter t again. Use the same identity as before, but re-arrange the equations first. Think a few steps ahead. What would you like to see on both sides of the equation at the end in order to use the identity and simplify?

Ideal Response:

1. Rearrange both equations, giving: $x - a = r\cos(t)$ and $y - b = r\sin(t)$.
2. Square both sides of the equations, giving: $(x - a)^2 = r^2\cos^2(t)$ and $(y - b)^2 = r^2\sin^2(t)$.
3. Now, add the corresponding sides of the equations together, giving: $(x - a)^2 + (y - b)^2 = r^2\cos^2(t) + r^2\sin^2(t)$.
4. Factorise the right hand side, giving: $(x - a)^2 + (y - b)^2 = r^2(\cos^2(t) + \sin^2(t))$.
5. Use the identity $\sin^2(t) + \cos^2(t) = 1$ to simplify the right hand side, giving: $(x - a)^2 + (y - b)^2 = r^2$.

This is the equation of a circle, with centre (a, b), and radius r.

Give the students a few moments to digest this before moving on.

Activity 3: Using another Identity

Close the **Constant Controller** by clicking the red cross in the corner.

Click on **Edit** > **Select All** and press **delete** on the keyboard (or **Right-click** on the graph area and select **Delete Objects** from the menu), ignoring the warning about deleting dependent objects.

You should now be left with only a set of axes.

Teacher: Okay, this time I would like to plot the following parametric equations: $x = \sin(t)$ and $y = \cos(2t)$. Would anybody like to make a prediction of what they think the resulting graph will look like?

If the students are particularly keen, invite them to the front to quickly sketch their predictions.

Use the **Erase** tool to rub out any mistakes.

If you want to clear all scribbles, click on Edit > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph itself and select **Delete Objects** from the menu).

Teacher: Once again, how about we try to find a few points which lie on this function? Can anybody think of any values of t which might give us some nice x and y co-ordinates?

Prompt: Remember, we are working in radians. How about 0? How about π? 2π? What about some of the special values in between? Remember, it is $\cos(2t)$.

Ideal Response: By trying different values of t, we can build up a set of co-ordinates that must lie on the function:

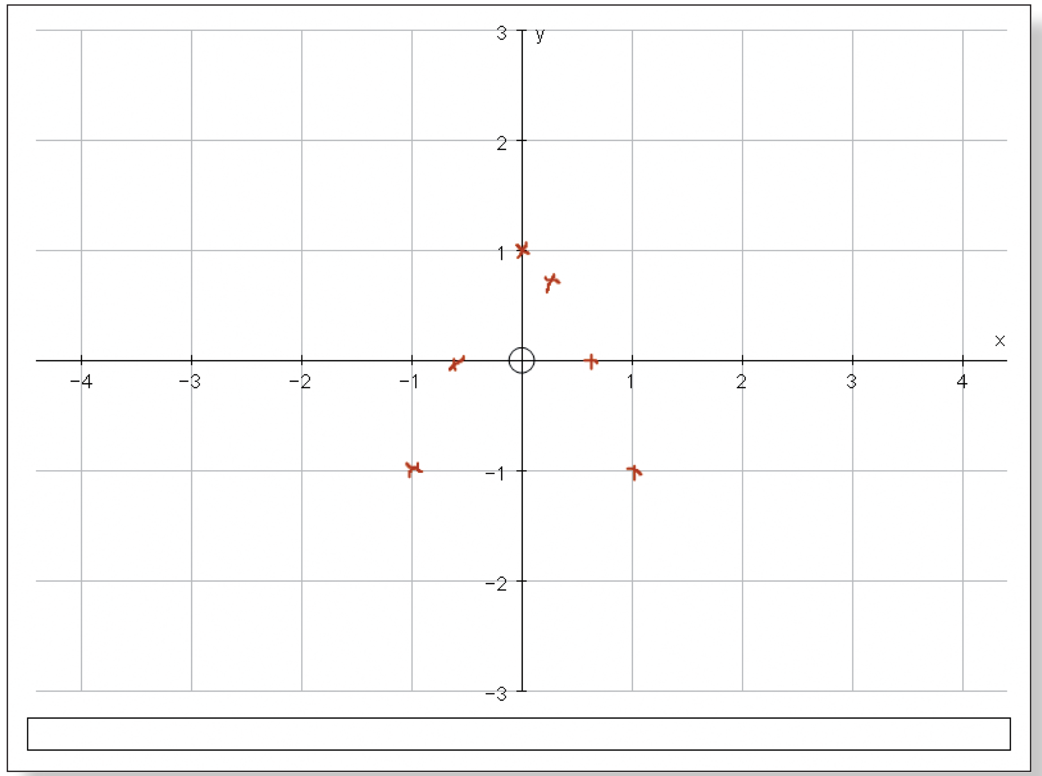
When $t = 0$	$\sin(t) = 0$	$\cos(2t) = 1$	which gives us	(0, 1)
When $t = \pi$	$\sin(t) = 0$	$\cos(2t) = 1$	which gives us	(0, 1)
When $t = 2\pi$	$\sin(t) = 0$	$\cos(2t) = 1$	which gives us	(0, 1)
When $t = \frac{\pi}{2}$	$\sin(t) = 1$	$\cos(2t) = -1$	which gives us	(1, −1)
When $t = \frac{3\pi}{2}$	$\sin(t) = -1$	$\cos(2t) = -1$	which gives us	(−1, −1)
When $t = \frac{\pi}{4}$	$\sin(t) = \frac{1}{\sqrt{2}}$	$\cos(2t) = 0$	which gives us	$(\frac{1}{\sqrt{2}}, 0)$

etc…

Encourage students to come to the front to mark these points on the graph.

Use the **Erase** tool to rub out any mistakes.

Your screen should look something like this:



Teacher: Okay, so can anybody describe to us the graph that seems to be forming here?

Prompt: Look at the shape of the points. What types of functions give us graphs like this?

Ideal Response: It seems to be an upside-down U-shape, which suggests a negative quadratic. It seems to have a maximum at (0, 1).

Teacher: Sounds good, so let's use Autograph to check:

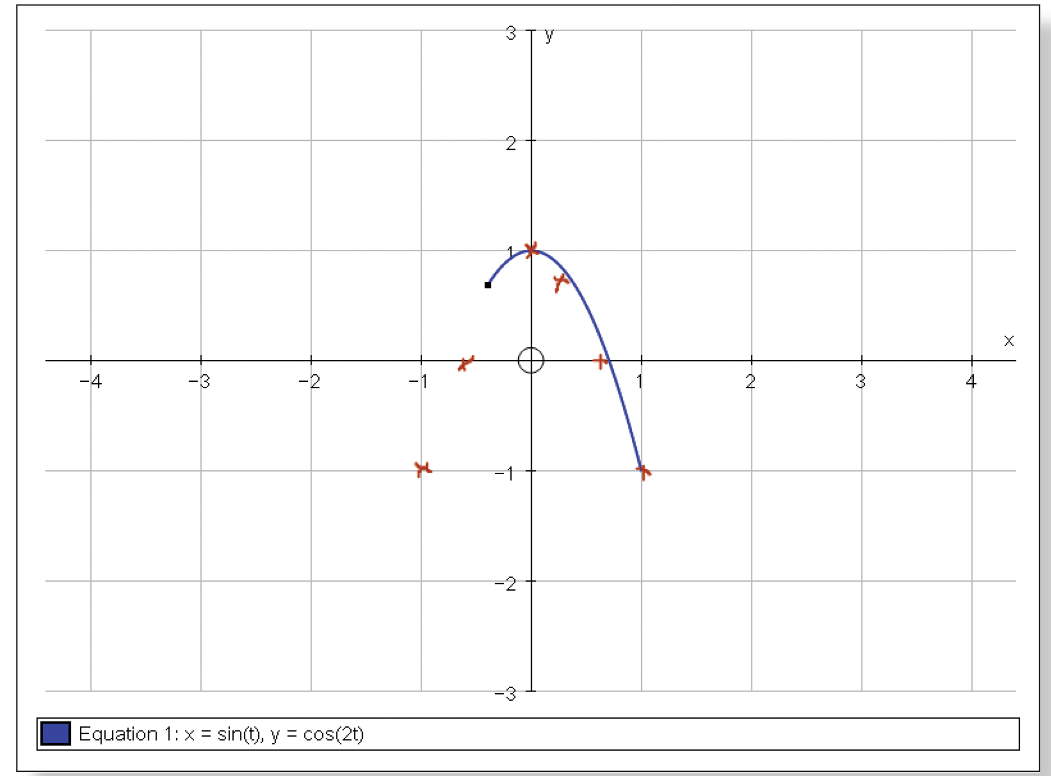
Click on **Slow Plot** mode.

Enter the equations: $x = \sin(t)$, $y = \cos(2t)$ and click **OK**.

The curve should appear on screen, hopefully going through the points plotted by the students.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Your screen should look something like this:



Teacher: Now, I saw two strange things going on when this curve was being plotted. Did anybody else notice them?

Click on **Re-Plot** if necessary.

Ideal Response: Autograph seems to go over the curve twice, and the curve doesn't go below the line $y = -1$.

Teacher: Excellent. Let's look at that again.

Delete all scribbles as explained above.

Click on **Re-Plot** to show the students the construction of the graph again.

Draw students' attention to the values of t changing at the bottom of the screen.

Point out that Autograph is inputting t values between -2π and 2π.

Teacher: Let's deal with the first strange thing first: why does Autograph seem to plot the graph twice?

Prompt: Think about the equations $x = \sin(t)$ and $y = \cos(2t)$. What are the periods of those graphs? But what range are we plotting over in this example? How does that help explain why Autograph seems to plot the curve twice?

Ideal Response: The curve is plotted twice because the period of the graph as a whole is 2π, and we are plotting it over 4π range.

Teacher: Excellent! Now, time to deal with the second strange feature: why does the graph suddenly stop at the line $y = -1$?...

Prompt: Again, it is unlikely the students will come up with the correct answer at this stage, but encourage any suggestions.

Teacher: Well, to help us answer this one we first need to express our parametric equations in normal Cartesian form as before. Any ideas how?

Prompt: We need another trigonometric identity. This time it must link $\cos(2t)$ with $\sin(t)$. Can you think of an indentify which allows us to express $\cos(2t)$ in terms of $\sin(t)$? What about a double angle formula? What about $\cos(2t) = 1 - 2\sin^2(t)$? How does that help us figure out the normal Cartesian equation of our curve?

Ideal Response:

1. We start with: $x = \sin(t)$, $y = \cos(2t)$.
2. Use the identity $\cos(2t) = 1 - 2\sin^2(t)$ to change the second parametric equation: $y = 1 - 2\sin^2(t)$.
3. We can now express y in terms of x: $y = 1 - 2x^2$.

This gives us the normal Cartesian equation of our curve.

Teacher: Sounds good, so let's use Autograph to check:

Click on **Slow Plot** mode.

Enter the equation: $y = 1 - 2x^2$
Note: To enter x^2, either use the little 2, press "**alt 2**" together, or type "**xx**"

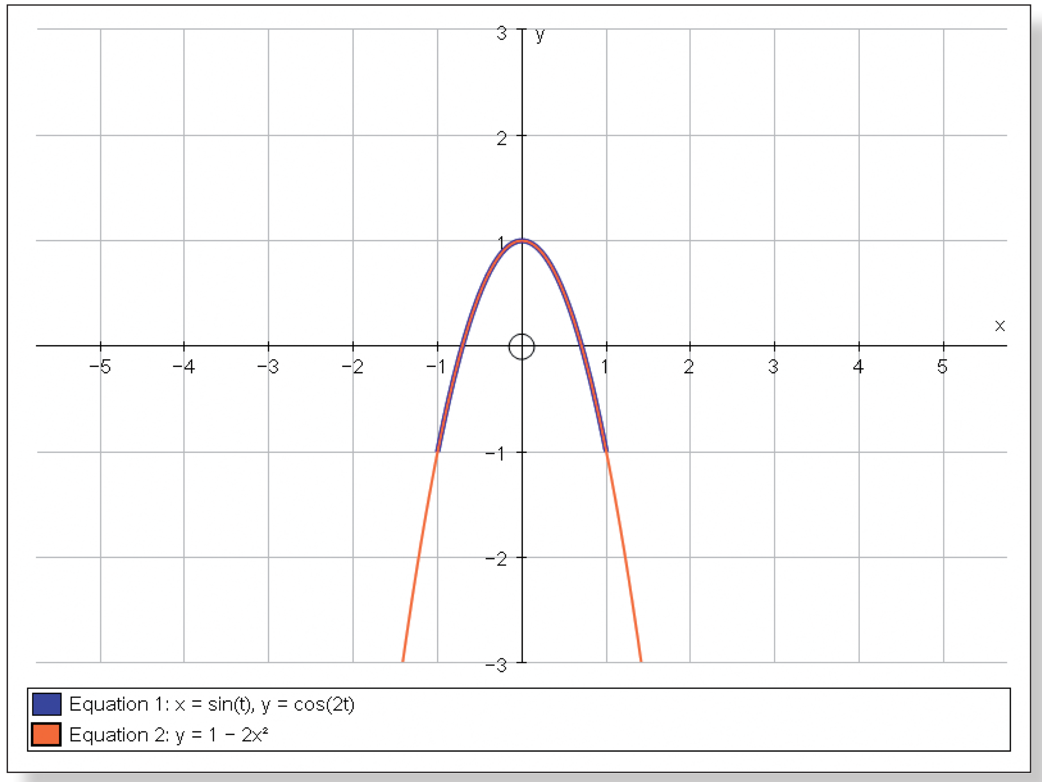
The curve should appear on screen, going on top of the parametric curve.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Teacher: Let's just make things a little clearer:

In the **Key** at the bottom of the screen, double **left-click** on Equation 1.
This should bring up the **Edit Equation** screen.
Click on **Draw Options**, and change the line thickness to 3 pt.
Click **OK**.

Your screen should look something like this:



Teacher: So, how come our Parametric Equations stop at $y = -1$, whereas when the equations are expressed in normal Cartesian form, they continue down into the negatives?

Prompt: Think about the graphs of $\sin(t)$ and $\cos(2t)$. Think about their range. What effect does this have on the Cartesian form of the curve on the screen?

Ideal Response: The curve stops at $y = -1$ because the values of both x and y are restricted by the ranges of $\sin(t)$ and $\cos(t)$ which only exist between -1 and 1.

Ideas for Further Work

- Further practice of converting parametric form into Cartesian form using trigonometric identities.
- Students should find the process of converting non-trigonometric parametric equations into Cartesian by eliminating the parameter easier than dealing with the trigonometric ones.
- Introduce parametric differentiation.
- Exam style questions on parametric equations.

Learning Objectives

- To be introduced to the concept of differential equations, and to be able to visualise and understand their construction.
- To understand the concepts of a general solution of first order differential equations and a family of curves.
- To understand the relationship between differential equations, gradient functions, and the processes or differentiation and integration.
- To be able to find the particular solution to a differential equation using the conditions given.

Required Pre-Knowledge

- To be aware of the concept of the gradient function, and the notation: $\frac{dy}{dx}$.
- To know how to differentiate and integrate functions involving positive powers of x.
- To understand the role of the constant in differentiation and integration.

Pre-Activity Set-Up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

Edit the axes as follows:

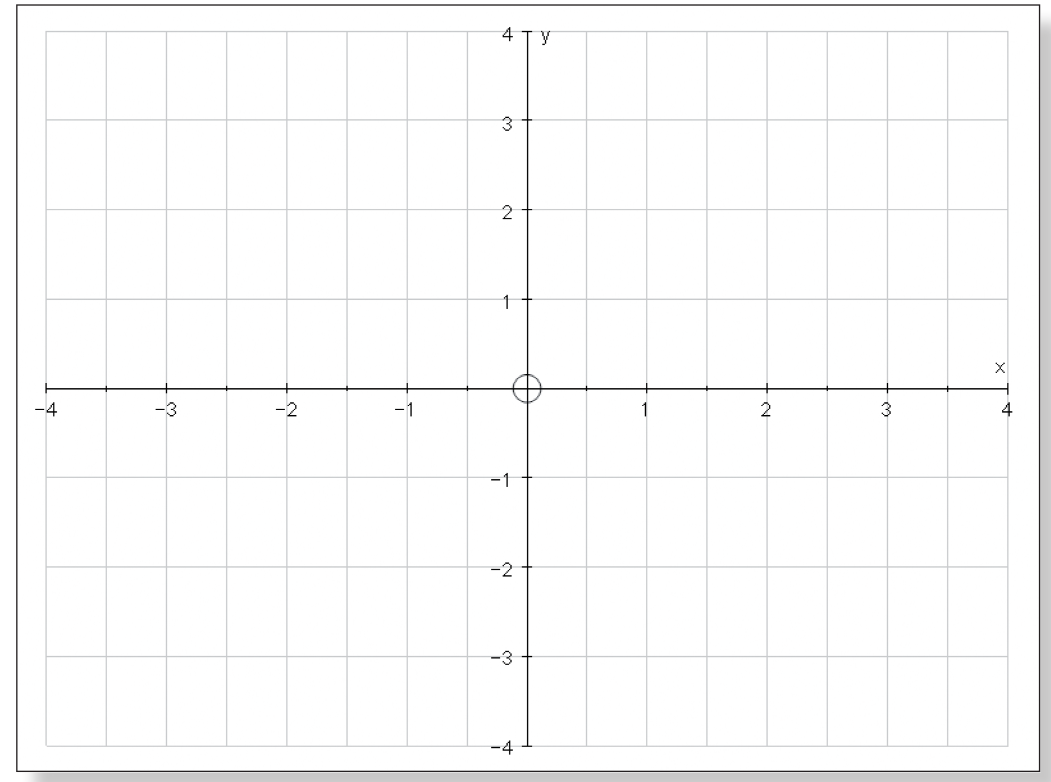
x:	Minimum: −4	Maximum: 4	Numbers: 1	Pips: 0.5
y:	Minimum: −4	Maximum: 4	Numbers: 1	Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

On the top toolbar click on **Axes**, and then **Show Key**.

This should remove the key from the bottom of the screen.

Your screen should now look something like this:



Step-by-Step Instructions

Activity 1: Introducing the Concept

Teacher: Okay, can anybody tell me what they think this means: $\frac{dy}{dx} = x$

Prompt: What does $\frac{dy}{dx}$ mean? What does that tell us about the gradient at every x co-ordinate on the function?

Ideal Response: $\frac{dy}{dx} = x$ means that the gradient function is equal to x. In other words, the gradient at every single x co-ordinate on our function is equal to x.

Teacher: Good. So, if we were to plot $\frac{dy}{dx} = x$, what would it look like?

Prompt: What would be the value of the gradient when x is 1? How about when x is −1, 2, 4, −3? What shape does this give us? Where would we draw this? Would it be a single, unique function?

Ideal Response: The gradient of the function is just equal to the x value, so when x is 1, the gradient is 1, when x is −3, the gradient is −3, and so on. This suggests that the function will be a curve. But it will be hard to plot as we don't know the corresponding y value of each x value.

At this point it might be a good idea to invite students to the front to roughly mark on these gradients using the **Scribble Tool**. This will emphasise the difficulty in drawing the function, as the students will not know where to position the line.

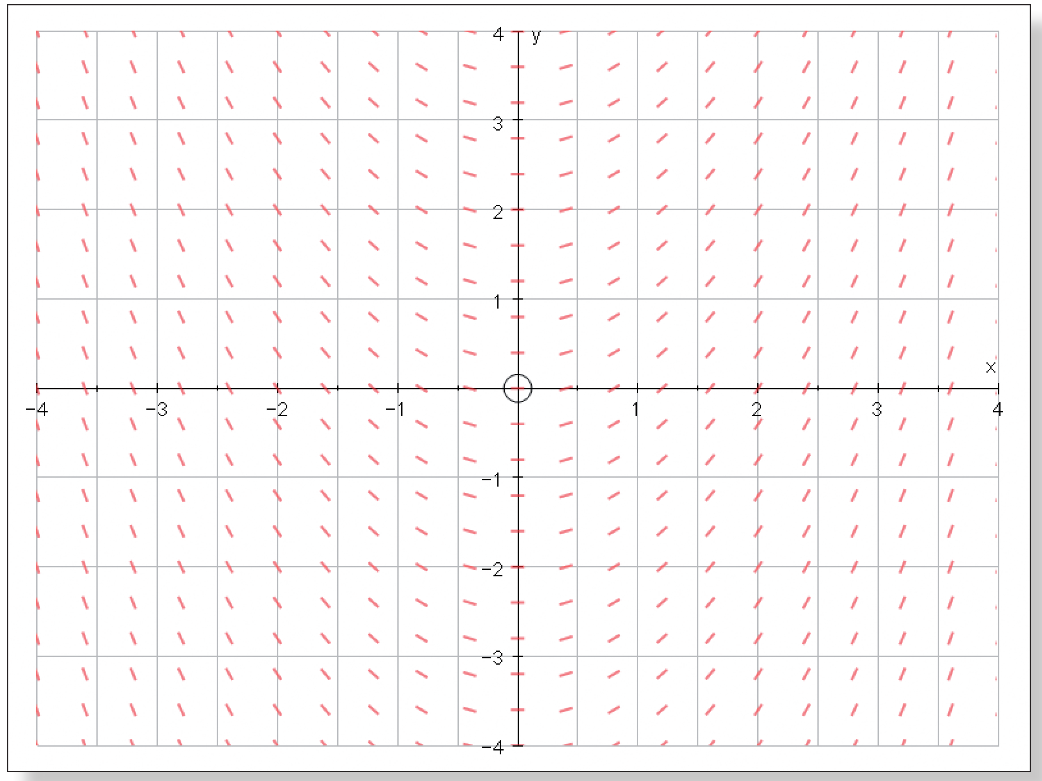
Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph area and select **Delete Objects** from the menu).

When you are ready (and all scribbles have been cleared):

Enter the equation: dy/dx = x
Note: If you prefer, this can be entered as y' = x.
Click **OK**.

Your screen should look something like this:



Teacher: Notice that all we, or the computer, can do is to mark on a series of lines to represent the gradient of the function at that point. We know exactly what the gradient of the curve is at each x value, we just don't know where to draw the curve.

Give the students a few moments to digest this before moving on.

Activity 2: A General Solution

Teacher: Now, what I can do is draw in a few of these functions so we can get a better idea of what is going on.

Click on **Slow Plot** mode.

Your cursor should be a cross.

Left-click on various points of the graph to form the family of curves.

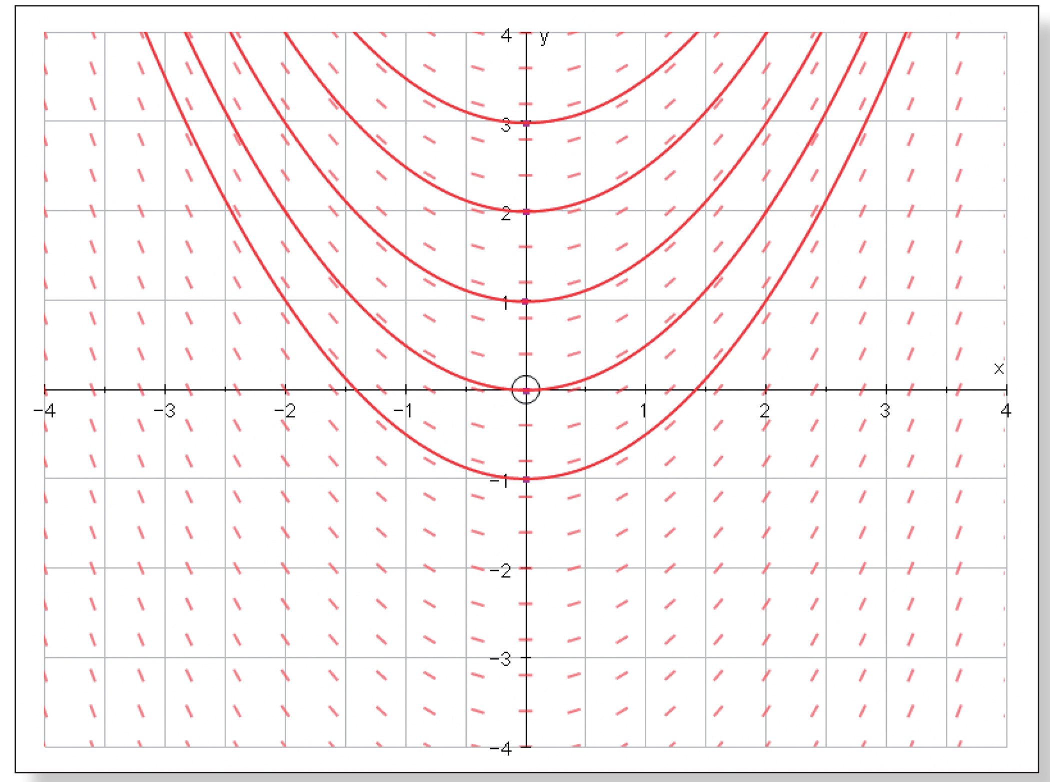
Press **Pause Plotting** both to stop the process and to resume it to focus on the key features of the graph.

Note: The **Spacebar** can also be used to serve this function.

Make sure you generate one of these curves by clicking on (0, 0).

Note: If your cursor ever returns to being an arrow, simply click on **D. E. Solution** to get the cross back!

Your screen should look something like this:



Draw the students' attention to the curve which passes through (0, 0).

Teacher: Can anybody tell me the equation of this curve?

Prompt: Think about the shape of the curve. Think about the points that lie on the curve. Think about the fact that this curve came from the equation: $\frac{dy}{dx} = x$ If the gradient function is x, what can we say about the power of the function itself? The equation of which curve has a gradient function of x, and passes through (0, 0)? So y = ? How do we turn the $\frac{dy}{dx}$ into a y? Maybe think about integration.

Ideal Response: The curve must be a positive quadratic. We can tell this from the shape, and by the fact that if the gradient function is x, the power of the original function must be x-squared. If we integrate the both sides of the gradient function with respect to x, we get back to the function itself, which gives us $y = \frac{1}{2}x^2 + c$. Because the curve passes through (0, 0), the value of c is 0, so we are just left with $y = \frac{1}{2}x^2$.

Teacher: Sounds good, but let's just check.

Make sure **Slow Plot** mode is still turned on.

Enter the equation: $y = \frac{1}{2}x^2 + c$
Note: To enter x^2, either use the little 2 button, or type "**xx**", or press "**alt 2**" together.
Still on the Enter Equation screen, click on **Edit Constants**, and alter the value of c to 0.
Click **OK** twice.

The curve should appear on screen, going directly over the curve which passes through the origin.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph.

Teacher: So, what are the equations of the other curves on the screen?

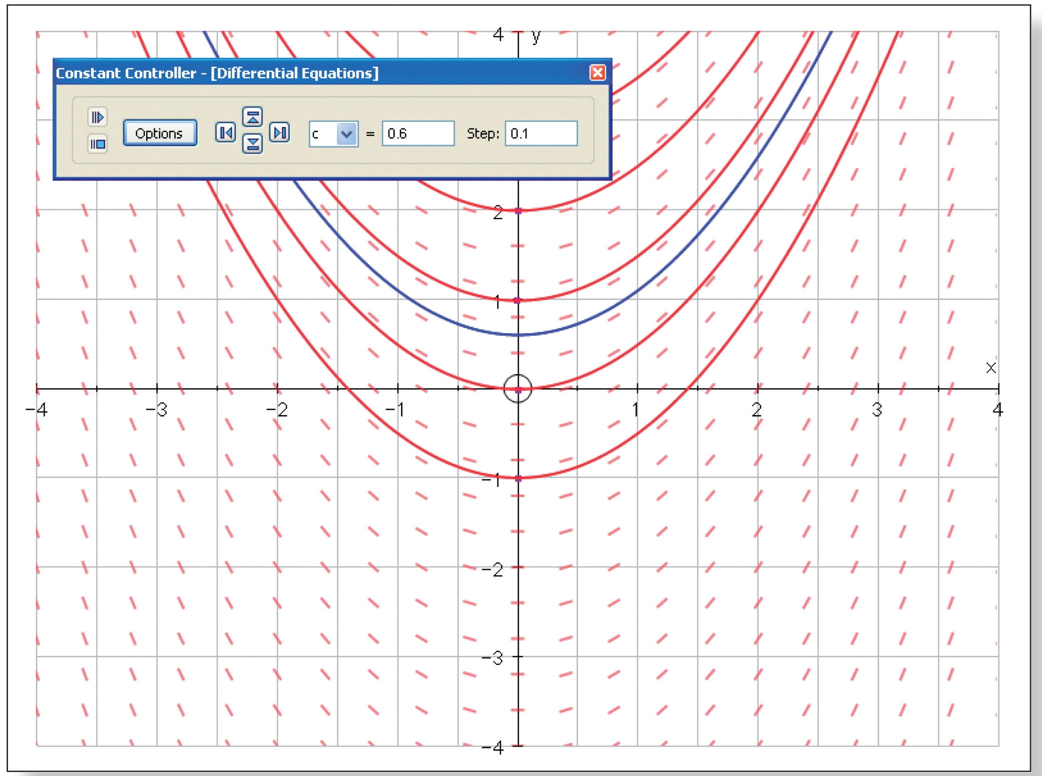
Prompt: Think about what we have just done. Think about the difference between these curves, and the curve which passes through the origin. What type of transformation is this? How does this help us figure out the equation of these curves?

Ideal Response: The other curves are simply translations up and down the y-axis, and so their equations are simply $y = \frac{1}{2}x^2 + c$, where c is the value of the y intercept.

Teacher: And we can easily check that…

Click on the **Constant Controller**.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the value of the step.
Adjust the value of c until it matches each of the curves you have drawn.

Your screen should look something like this:



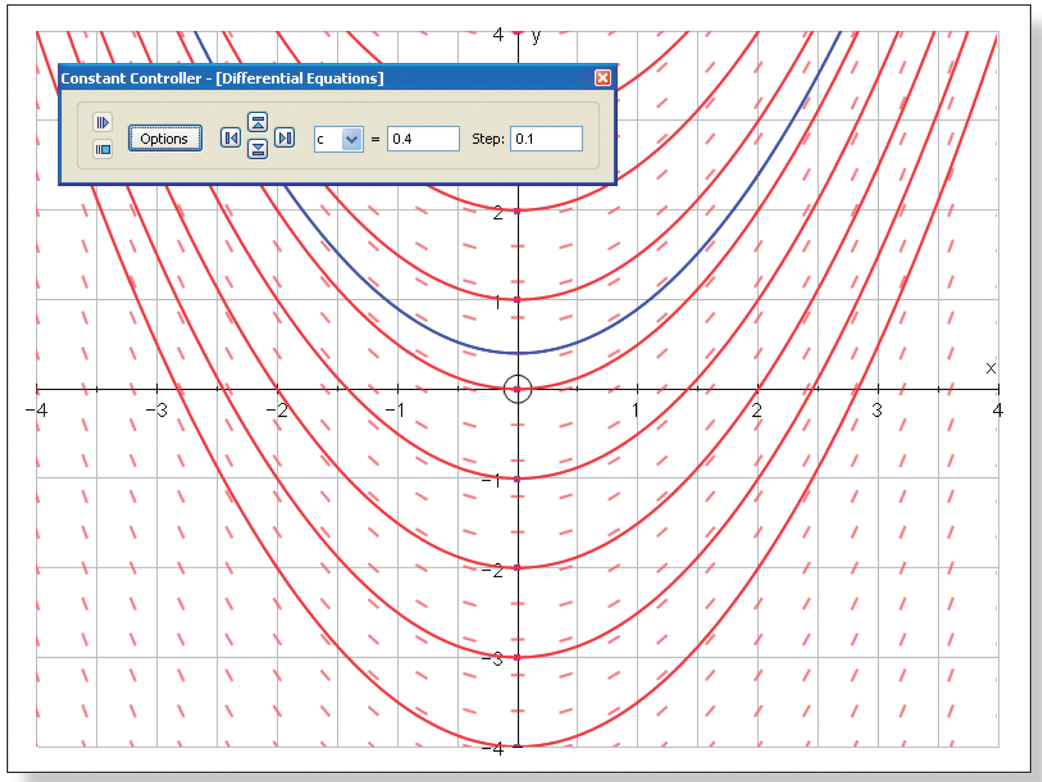
Teacher: Good. Now, before I ask you the tricky question, let's just make our page a bit clearer…

Turn off **Slow Plot** mode.

Left-click twice on one of the red family of curves.

This should bring up the **Edit Equation** screen.
Click on **Startup Options**.
Underneath **Initial Conditions**, click on **Point Set**.
Click on **Enter Start Points**.
Click on y-axis, and click **OK** twice.

Your screen should look something like this:



Teacher: Okay, so the question is: just what it going on here? Why do we have a "family of curves" from our original equation: $\frac{dy}{dx} = x$?

Prompt: Think about the effect that the constant c has on our curves? Does it affect their gradient at each x value? Why not? So how many original functions, in the form y = , could we get from $\frac{dy}{dx} = x$?

Ideal Response: All the constant does is to translate the curve up and down. The gradient at any given x point remains the same on all of the curves. All $\frac{dy}{dx} = x$ tells us is the gradient of the function at each x value. It tells us nothing about the position of the function, and hence we could draw an infinite number of functions having been told $\frac{dy}{dx} = x$.

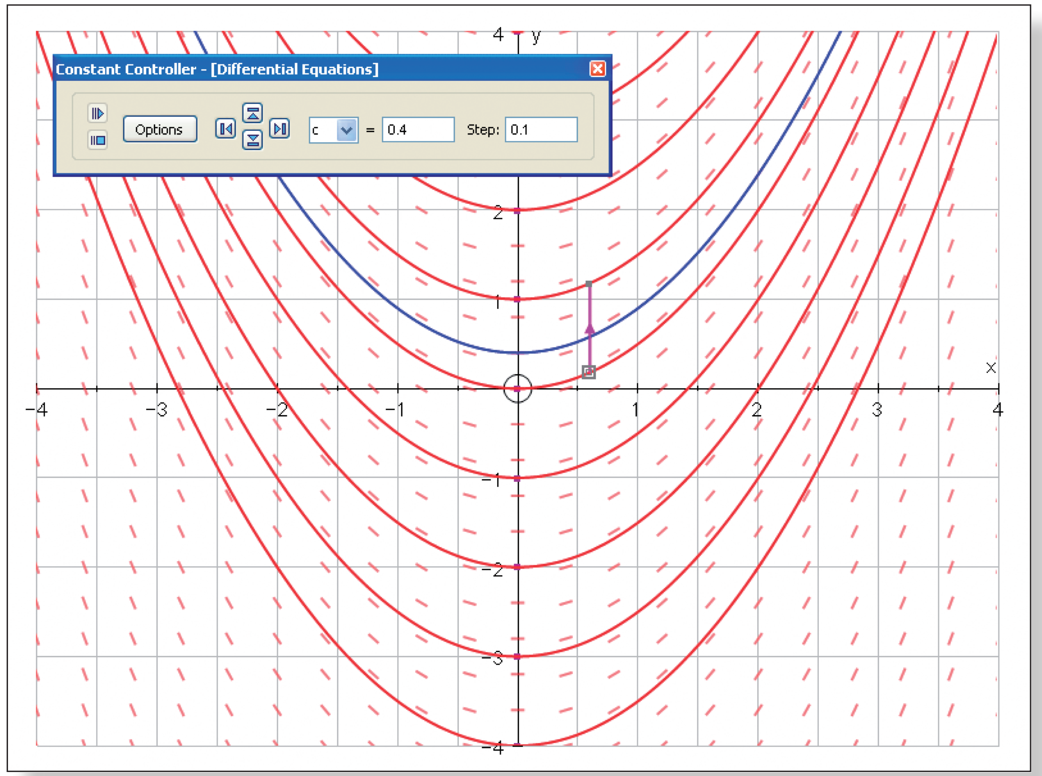
Teacher: Excellent. And we can show this even more clearly using vectors…

Click on Point Mode and place a point somewhere on the red curve which passes through origin.

Right-click and select Vector from the menu.

Enter the vector $\binom{0}{1}$ and click OK.

Your screen should look something like this:



Use the left-right buttons on the keyboard to move along the curve.
Use the up-down buttons to jump between curves.

Point out that since the gradient does not depend on y, lines which start a certain distance apart vertically, remain the same distance apart.

Teacher: Now, equations that contain terms like $\frac{dy}{dx}$ are called differential equations. And if you are given no more information, then you end up with a family of curves like we have here, all of which are in the form $y = \frac{1}{2}x^2 + c$. So, what we actually say is that the general solution to our differential equation is $y = \frac{1}{2}x^2 + c$.

Give the students a few moments to digest this, before moving on.

Activity 3: Particular Solutions

Teacher: Okay, let's make life a little more difficult. Imagine you were asked to find the particular solution to the differential equation $\frac{dy}{dx} = x$, given that when $x = 1.3$, $y = -1.6$. How would we do that?

Prompt: What is the general solution? How would that help us find the particular solution? Look at the family of curves if it helps you.

Ideal Response: General Solution: $y = \frac{1}{2}x^2 + c$
When $x = 1.3$, $y = -1.6$
So: $-1.6 = \frac{1}{2}(-1.3)^2 + c$
$c = -2.445$
So, the particular solution to this differential equation is: $y = \frac{1}{2}x^2 - 2.445$

Teacher: Sounds good, but let's just check…

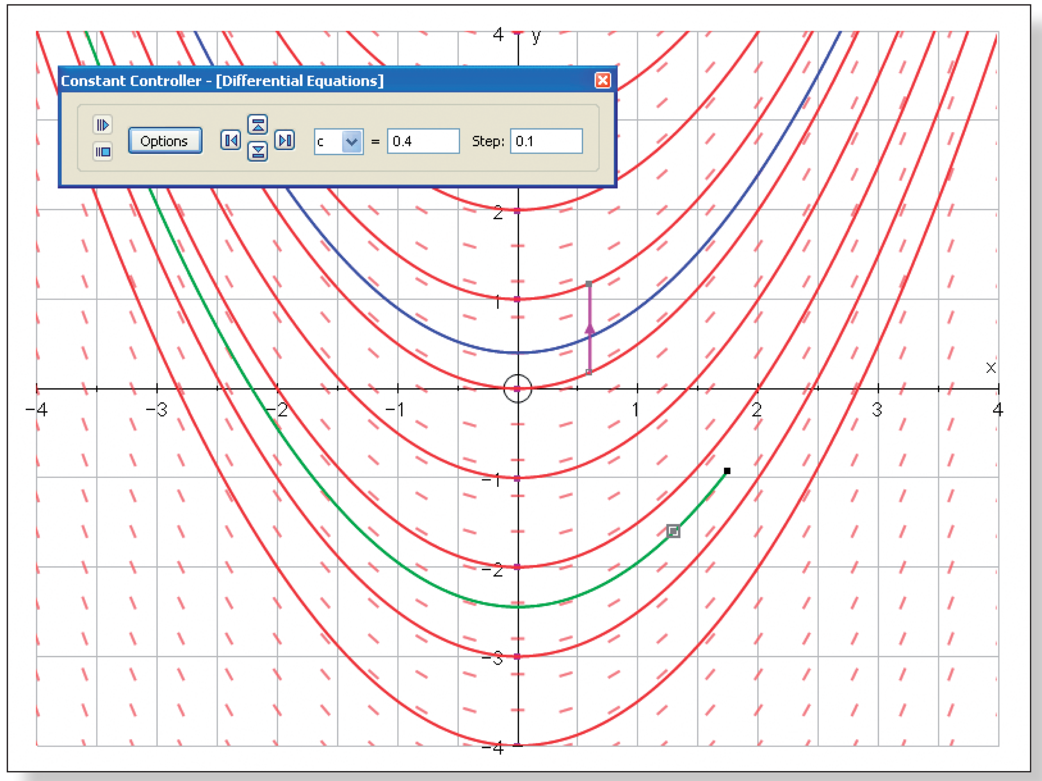
Enter the point (−1.5, 4.75).

Turn on **Slow Plot** mode.

Enter the equation: $y = \frac{1}{2} x^2 - 2.445$

The curve should appear on the screen passing through the point.

Your screen should look something like this:

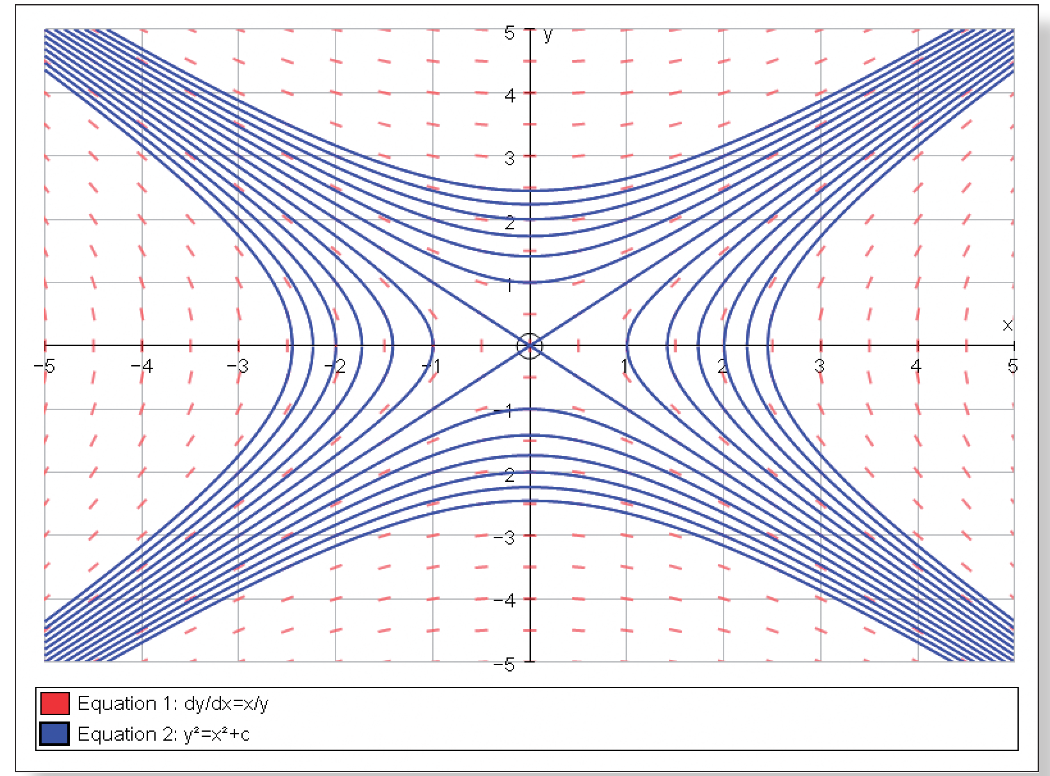


Ideas for Further Work

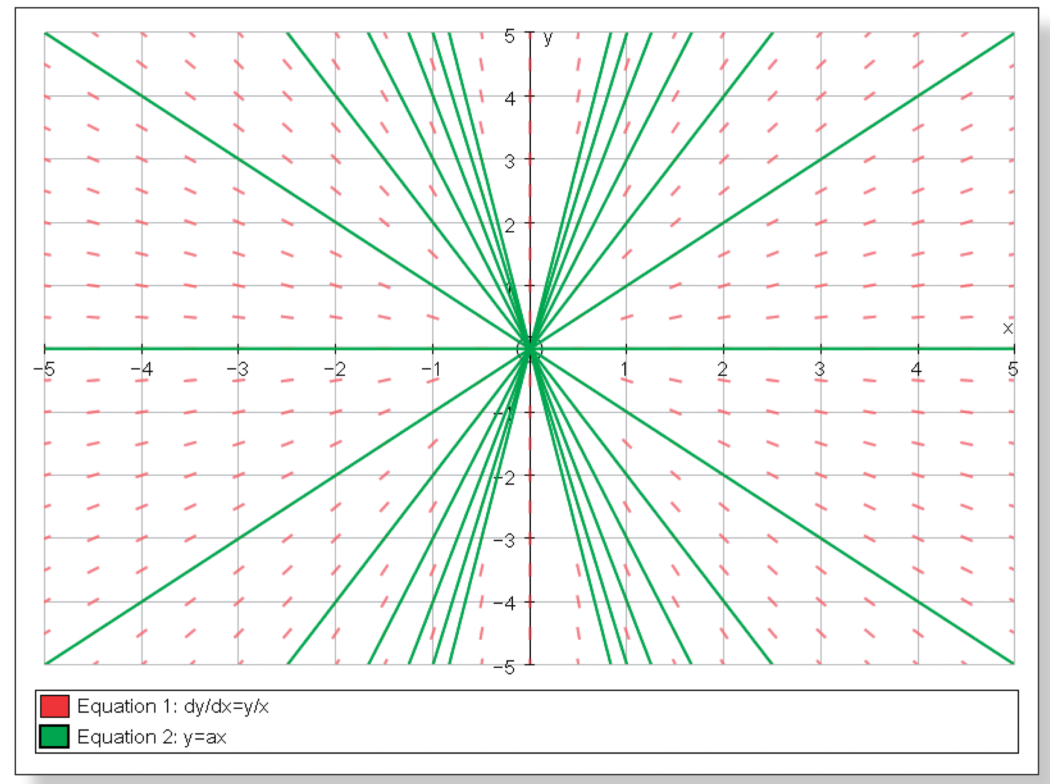
- Further practice finding the general solutions and particular solutions of other differential functions, both in the form: $\frac{dy}{dx} = g(x)$ and $f(y)\frac{dy}{dx} = g(x)$.
- Differential functions involving e and natural logs.
- Forming differential equations.
- Using differential equations to help solve exponential growth and decay problems.

Note: Once students' have acquired the necessary skills to solve them, it would be quite nice to use Autograph to show them the family of curves for other differential equations. The following three are particularly nice:

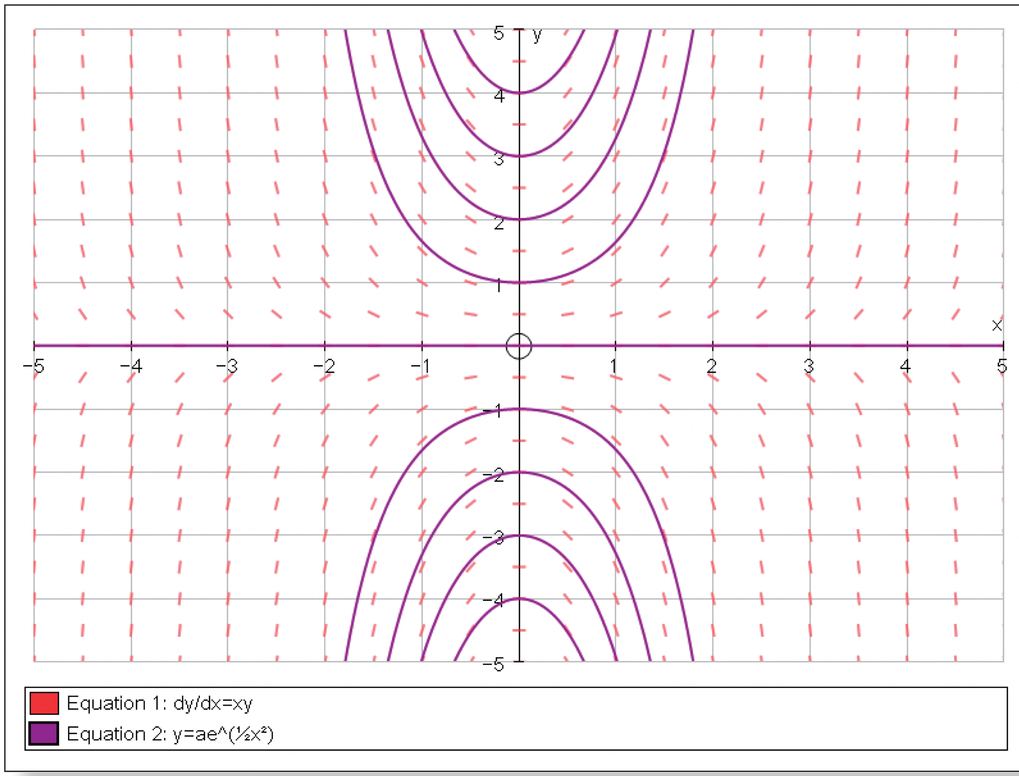
1. $\frac{dy}{dx} = \frac{x}{y} \to y^2 = x^2 + c$



2. $\frac{dy}{dx} = \frac{y}{x} \to y = ax$



3. $\frac{dy}{dx} = xy \to y = Ae^{\frac{1}{2}x^2}$



y
x
5
4
3
2
1
−1
−2
−3
−4
−5
Equation 1: dy/dx=xy
Equation 2: y=ae^(½x²)

Learning Objectives

- To obtain a graphical representation of the approximation to the Binomial Expansion: $(1 + x)^n$ for each of the three cases where n is positive, fractional, and negative.
- To be able to understand how and why accuracy is changed by expanding to higher powers of x.
- To understand why $|x| < 1$ if n is negative or a fraction, introducing the concept of divergence.
- To reinforce the features to look out for when attempting to visualise the shape of a function, recapping work on Transformations.
- To re-enforce and consolidate the skill of calculating binomial expansions algebraically.

Required Pre-Knowledge

- To know how to expand expressions in the form $(1 + x)^n$ up to a given power of x, where n can take any value, including negative and fractional.
- To be comfortable working with indices.

Pre-activity Set-up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

For this activity, you will need to set up three pages, each containing a set of axes, as follows:

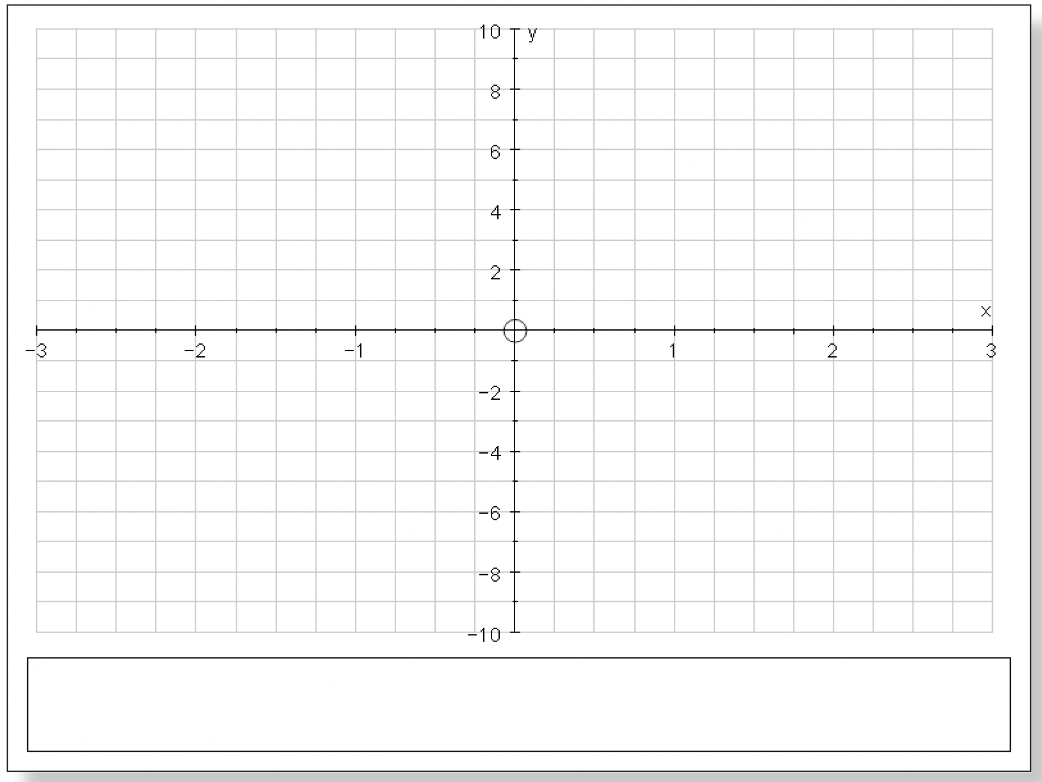
Page - 1:

Edit the axes as follows:

x:	Minimum: -3	Maximum: 3	Numbers: 1	Pips: 0.25
y:	Minimum: -10	Maximum: 10	Numbers: 2	Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Your screen should look something like this:



Page - 2:

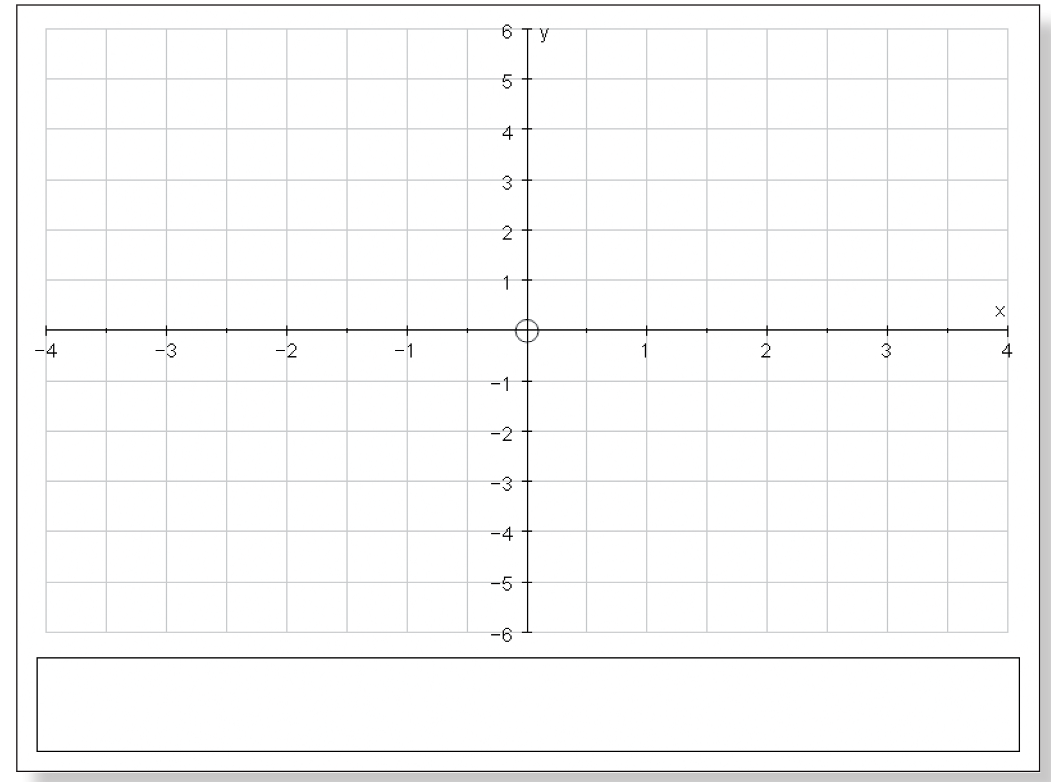
Open another blank **2D Graph Page**.

Edit the axes as follows:

x:	Minimum: −4	Maximum: 4	Numbers: 1	Pips: 0.5
y:	Minimum: −6	Maximum: 6	Numbers: 1	Pips: 1

Remove all of the green ticks underneath **Auto**.

Your screen should look something like this:



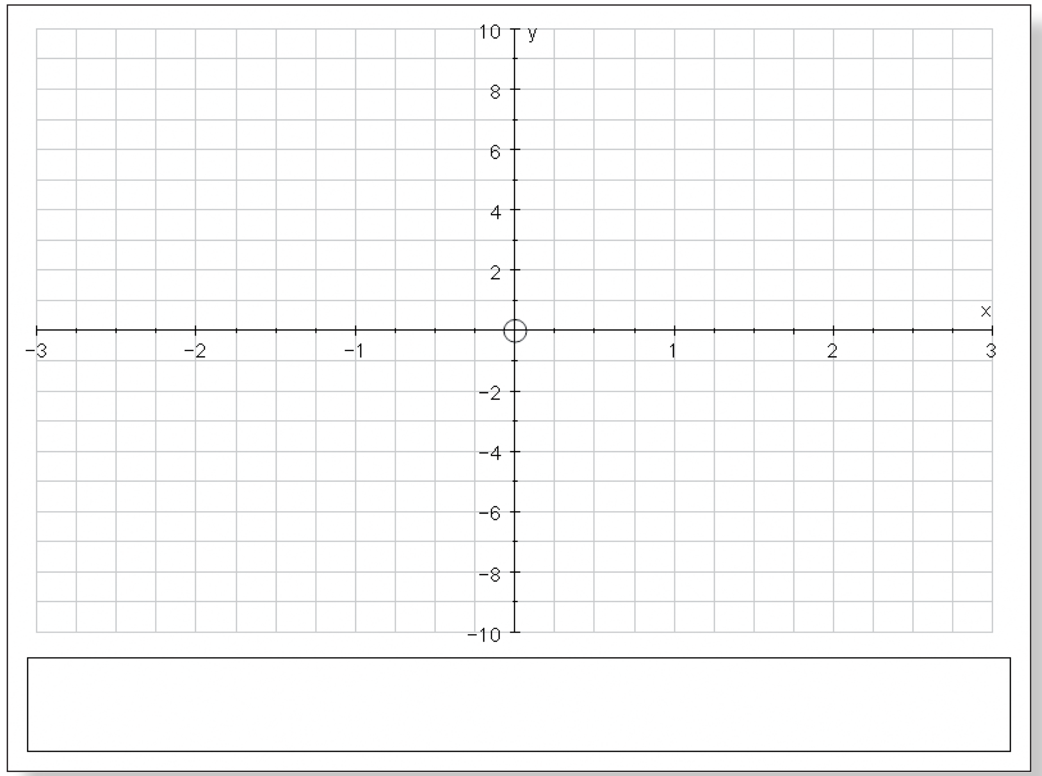
Page - 3:

Open another blank **2D Graph Page**

Edit the axes as follows:

x:	Minimum: −3	Maximum: 3	Numbers: 1	Pips: 0.25
y:	Minimum: −10	Maximum: 10	Numbers: 2	Pips: 1

Remove all of the green ticks underneath **Auto**.
Your screen should look something like this:



Your three pages are now available as Tabs on the top of the screen and can be accessed any time by simply **left-clicking** on them.

Step-by-Step Instructions

Activity 1: Positive Powers of n

Open Page 1 by **left-clicking** on the tab.

Teacher: Okay, I am going to draw the equation $y = (1 + x)^n$, and set the value of n to 1. What will the graph look like?

Ideal Response: When $n = 1$, the equation just becomes $y = x + 1$, which is a straight line, with gradient 1, passing through the point (0, 1).

Enter the equation: $y = (1 + x)^n$
Note: To enter n as a power, press "**alt n**" together, or type "**^n**"

The graph of $y = x + 1$ should now appear on the screen.

Left-click on the curve (it should turn black)

Click on the Thick Line tool, and choose a thickness of 2¼ points. This will help the line stand out more later on.

Click on Text Box.
Delete the words "Equation 1: ", so you are just left with the red text.
Tick the box next to Show Detailed Object Text.
Click OK.

The equation of the function, along with the current values of n should now be displayed.

Click on the Constant Controller.
The **up-down** buttons adjust the value of the constant.
The **left-right** buttons adjust the value of the step.
Adjust the value of the step to 1.

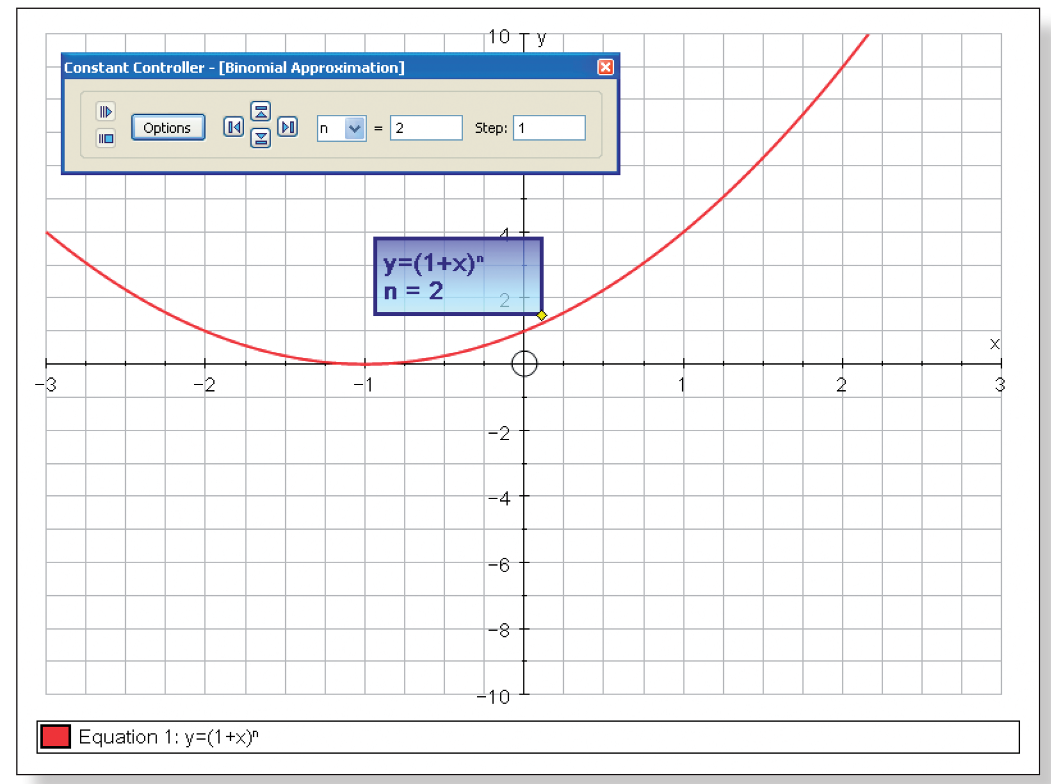
Teacher: If I increase the value of n from 1 to 2, what would my graph look like then?

Prompt: No need to expand the brackets. Think what the $y = (x + c)^2$ form of an equation tells you. Think back to our work on Transformations.

Ideal Response: The equation of the function is $y = (x + 1)^2$, which is just the graph of $y = x^2$, translated 1 unit to the left.

Click the **up** button of the Constant Controller to show the students they are correct.

Your screen should look something like this:



Teacher: And now if I was to increase the value of n from 2 to 3, what would my graph look like then?

Prompt: Again, think about Transformations. What points do you know definitely lie on the line?

Ideal Response: The equation of the function is $y = (x + 1)^3$, which is just the graph of $y = x^3$, translated 1 unit to the left.

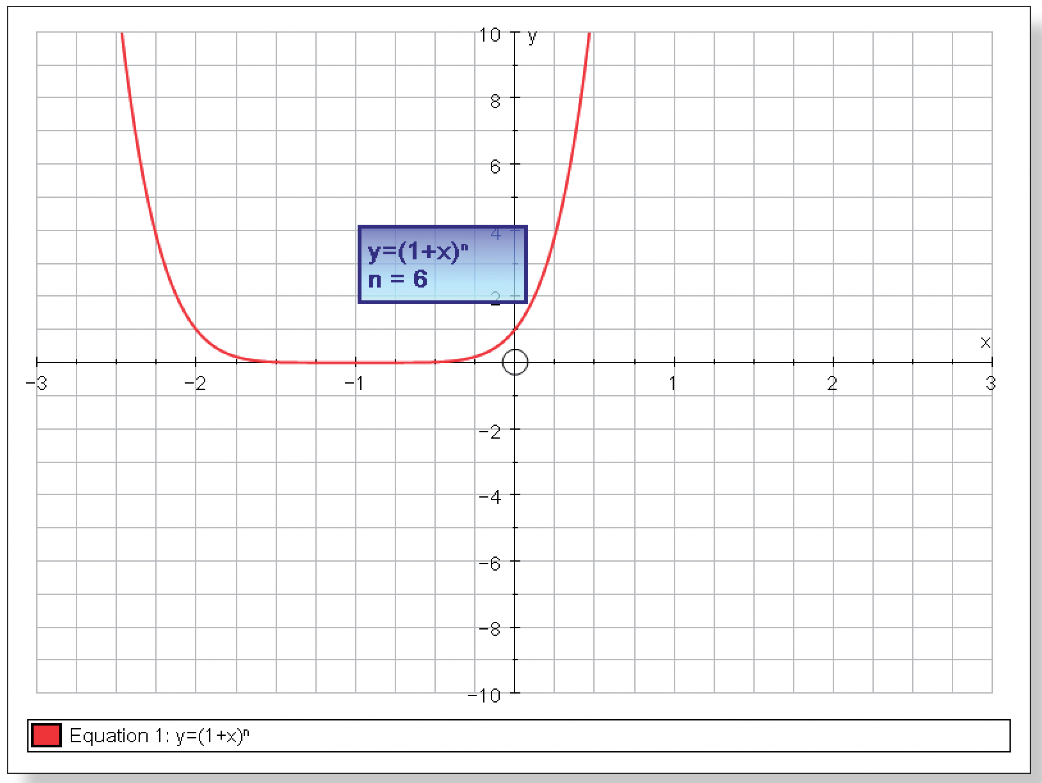
Click the **up** button of the Constant Controller to show the students they are correct.

Note: You can continue this process for the next three powers of x if you feel the class would benefit from it, or simply click through them quickly. When getting them to predict the shapes of the functions, have them think about what points they know lie on the curve (it is worth pointing out that every function passes through (0, 1) just as the non-translated functions all passed through the origin). Also, have the students think about whether really large values of x give positive or negative values of y.

Click the **up** button of the **Constant Controller** so that n = 6.

Close the **Constant Controller**.

Your page should look like this:



Teacher: Okay, first challenge. I would like you to completely expand the equation $y = (1 + x)^6$.

Prompt: How do we do the Binomial Expansion?

Note: If the Binomial Expansion has been taught using Pascal's Triangle co-efficients, then it might be a good idea to quickly draw Pascal's triangle somewhere on the board.

Alternative, if the Binomial Expansion has been taught by using the formula:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2\times 1} + \frac{n(n-1)(n-2)x^3}{3\times 2\times 1} + \dots$$

then this too could be prominently displayed to help students out.

Ideal Response: The compete expansion of $y = (1 + x)^6$ is: $y = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Teacher: Good. Now, often in exam questions, they only ask you to expand the expression for the first few terms. They usually say something like "expand in ascending powers of x, up to the term in x^3…". Because we are not using all the terms, these expansions are only approximate. Well, let's see just how accurate these approximate expansions actually are.

Teacher: Firstly, looking at our expansion, what would be a *linear approximation* to $y = (1 + x)^6$?

Prompt: What is the highest power of x in a linear transformation? So, how many terms do we take from our expansion?

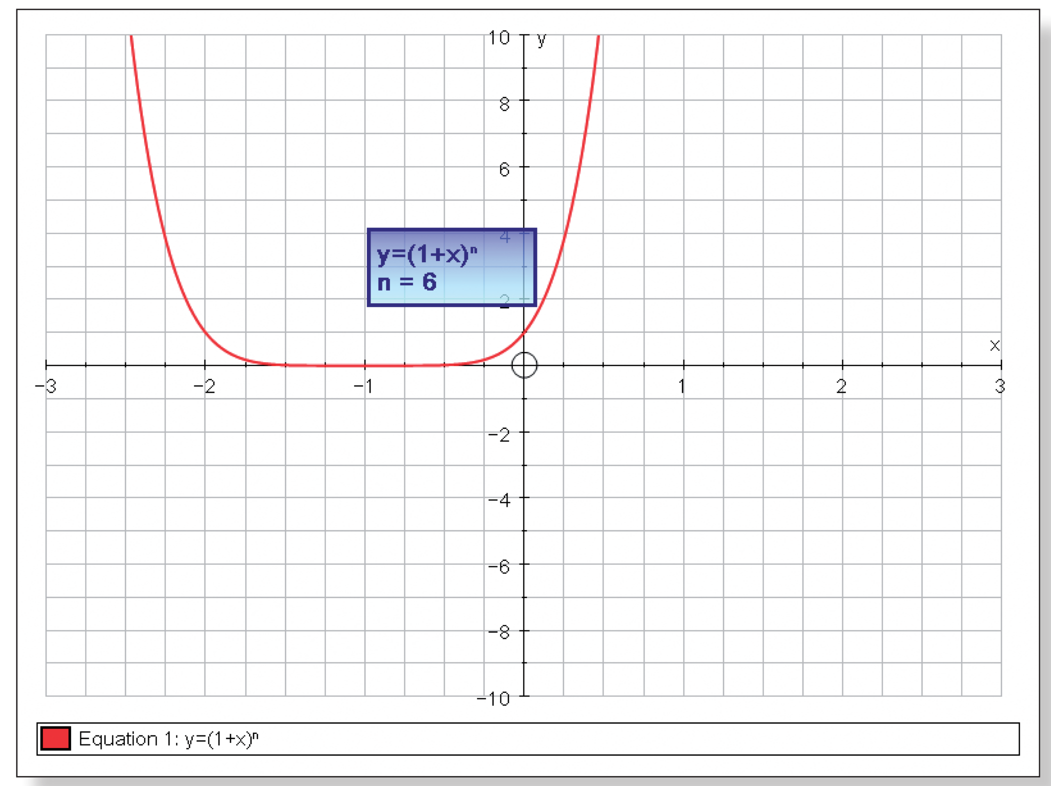
Ideal Response: A linear approximation would involve all terms with powers of x equal to one or less, and so it would be $y = 1 + 6x$.

Teacher: Good. Well, let's see how close an approximation that is to our curve:

Click on **Slow Plot** mode.

Enter the equation: $y = 1 + 6x$

Your screen should look something like this:



Teacher: So, how good an approximation is $y = 1 + 6x$?

Prompt: For what values of x does it seem to be a good approximation?

Use the **Drag** and **Zoom In** functions to get a closer look at what is going on. Notice how the *scale* automatically adjusts.

Return to the original view of the graph.

Note: Pressing the Undo button several times is often a quicker way of doing this.

Ideal Response: It seems like a good approximation for x values between −0.1 and 0.1, but a poor approximation for everything else.

Teacher: Okay, let's try and improve things. What would be a *quadratic approximation* to $y = (1 + x)^6$?

Ideal Response: A quadratic approximation would involve all terms with powers of x equal to two or less, and so it would be $y = 1 + 6x + 15x^2$.

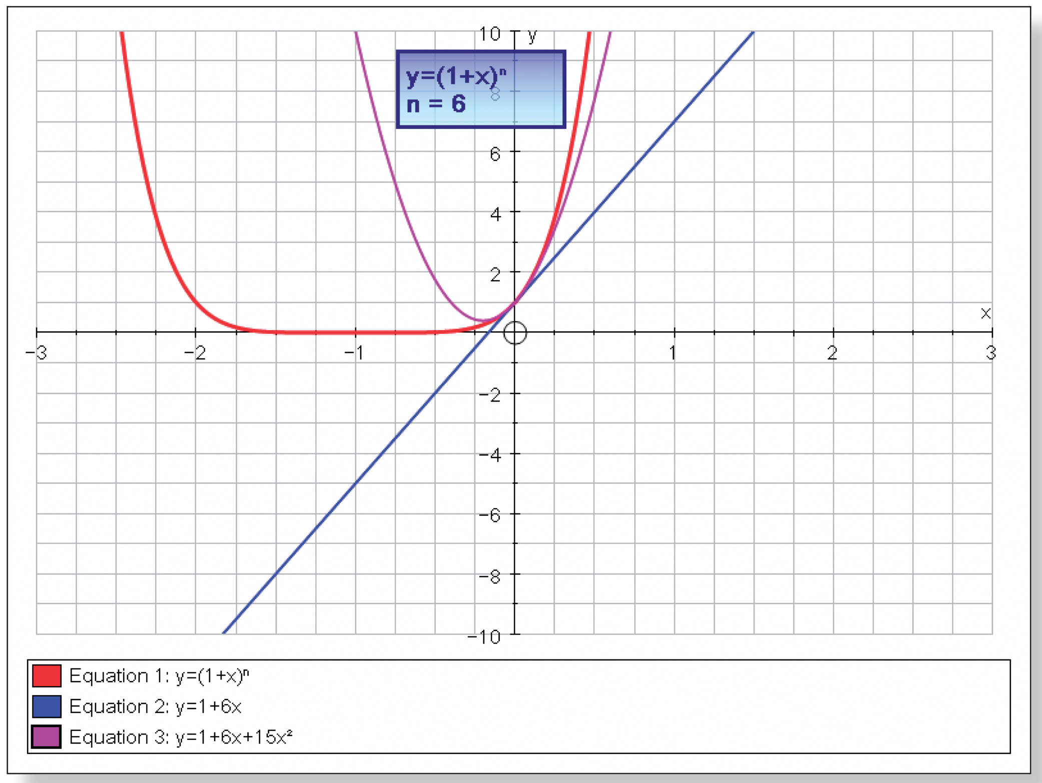
Make sure Slow Plot mode is still on.

Enter the equation: $y = 1 + 6x + 15x^2$

Note: To enter the squared term, either use the little 2, press "alt 2" together, or type "xx".

Press Pause Plotting both to stop the process and to resume it to focus on the key features of the graph.
Note: The Spacebar can also be used to serve this function.

Your screen should look something like this:



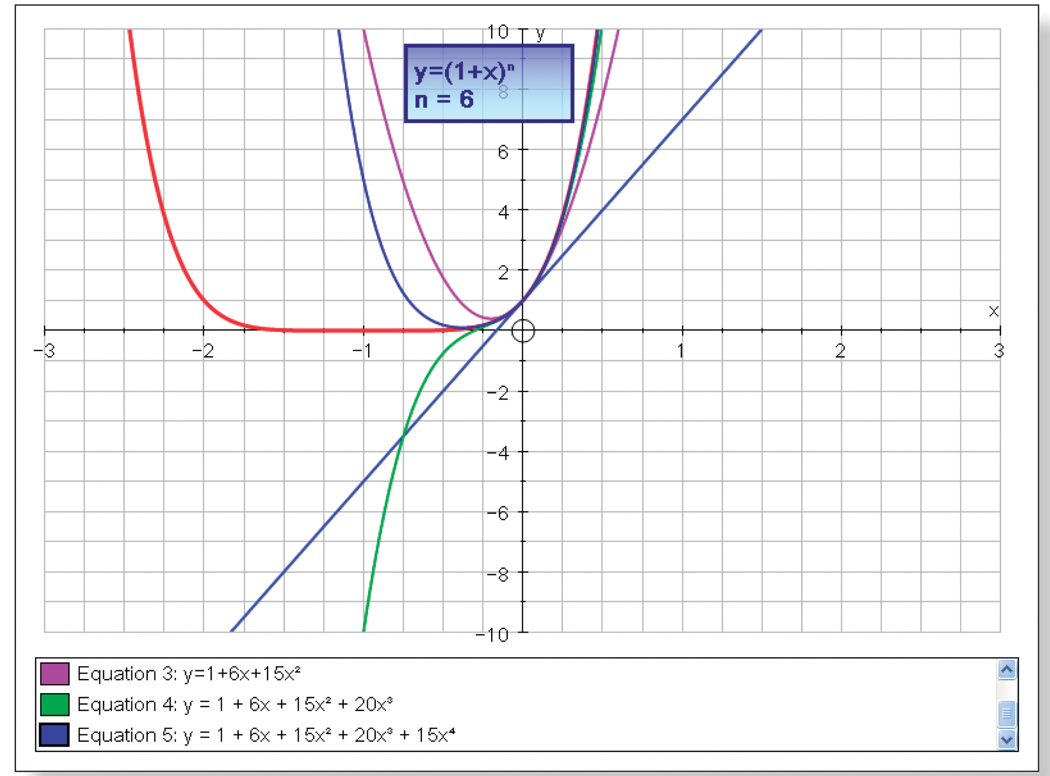
Teacher: Does this seem like a better approximation? What values of x does this seem a good approximation?

Again make use of the Zoom and Drag functions, always remembering to return to your original view of the graph using Undo.

Enter the cubic approximation: $y = 1 + 6x + 15x^2 + 20x^3$

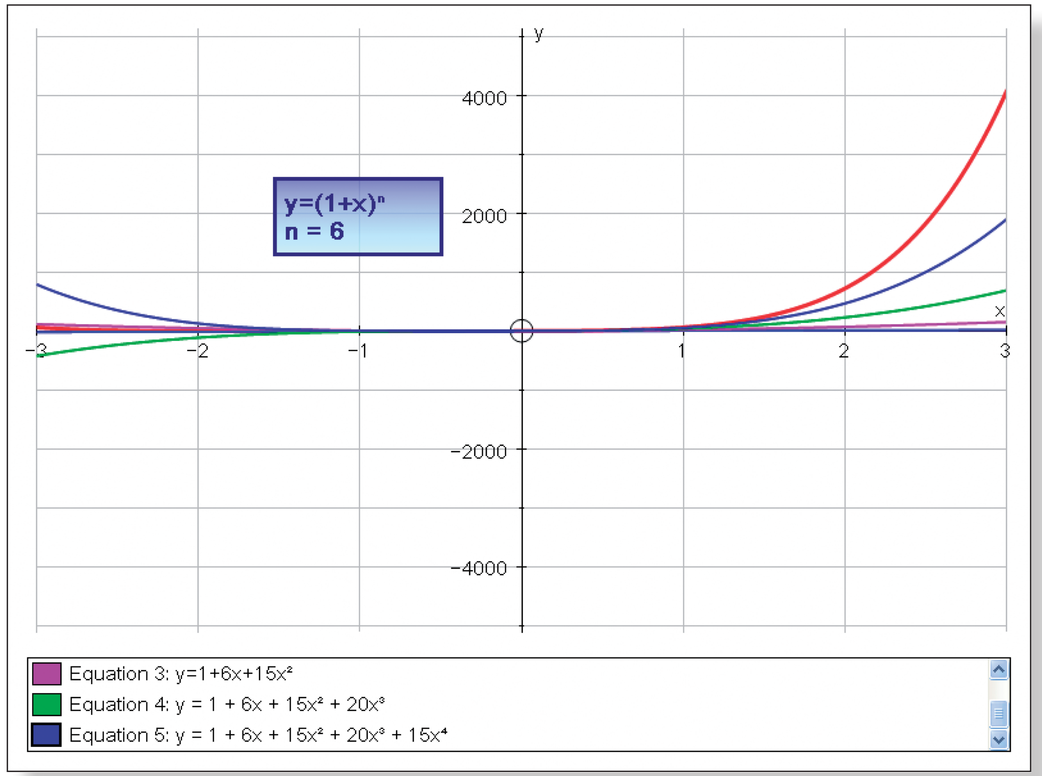
And then enter the quadratic approximation: $y = 1 + 6x + 15x^2 + 20x^3 + 15x^4$

Your screen should look something like this:



Use the Zoom Out y function, left-clicking close to the origin to get a better view of the graph. Draw the students' attention to how the scale is automatically adjusting.

Your screen should look something like this:



Use the **Drag** tool to have a look at both sides of the graph. Follow the graph for large (positive and negative) values of x. **Zoom Out** even further if it helps. Ask the students which part of the graph they want to have a look at, or even get them up to the front to use the **Drag** tool themselves.

Teacher: Does anybody have any comments about the graphs? Why do you think the graph appears to give a poor approximation for large and small values of x?

Prompts for discussion: Look at the graphs at different values of x. Which terms of the original function are omitted when using the binomial approximation? Why are these more significant for larger and smaller values of x? Point out that usually, when using the binomial we are asked to use very small values of x, and so in terms of an approximation, many of these graphs actually seem quite good. Introduce the ideas of *convergence* and *divergence*. Notice how the cubic and linear graphs seem to be diverging for negative values of x. Encourage the students to think why this might be the case. Point out that whilst the other approximations appear a long way away from the original graph of $y = (1 + x)^6$, at least they are "heading in the right direction". This will be very important for the next two examples.

Activity 2: Fractional Powers of n

Open Page 2 by clicking on the tab.

Ensure you are working in **Select Mode**.

Teacher: Okay, still working with the $y = (1 + x)^n$ graph, we are now going to look at changing the value of n, to see what effect, if any, that has on the accuracy of our approximations. We have seen the shapes of the graph for various positive values of n, so now let's try some fractions! Does anybody have any idea what the graph would look like if I changed the value of n to $\frac{1}{3}$? Come up and sketch it if you are feeling brave!

Prompts: What does "to the power of a third" actually mean? Can anybody think of a couple of points which we know must lie on the line? Where does the graph cross the axes? What happens to the graph for really large values of x? Does the x exist for negative values of x?

Encourage students to come to the front to sketch the curve or mark on their points using the **Scribble Tool**.

Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on **Edit** > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph itself and select **Delete Objects** from the menu).

Teacher: Let's have a look…

Click on **Slow Plot** mode.

Enter the equation: $y = (1 + x)^{\frac{1}{3}}$
Note: To enter "to the power of one third", type: "^ (1/3)"

Click **OK**, and the curve should begin to plot, hopefully resembling the curves drawn by the students.

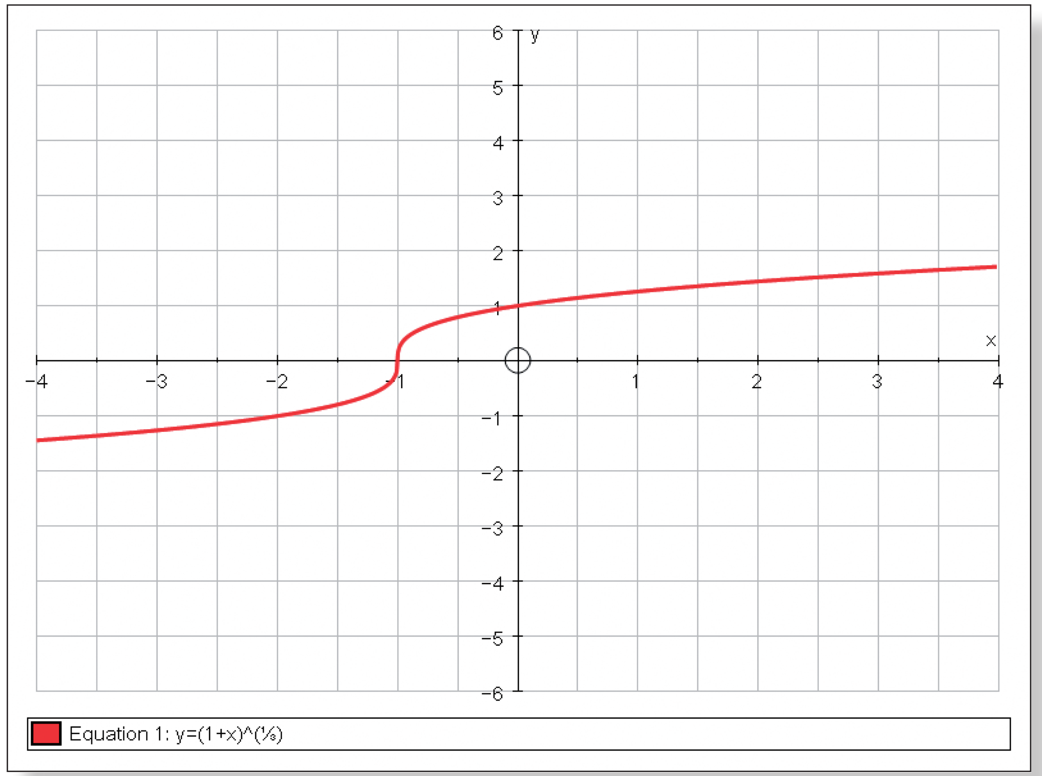
Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph. Discuss the shape and the points where the graph crosses the axes.

When the curve has finished plotting, *delete all scribbles* as described above.

Left-click on the curve (it should turn black).

Click on the **Thick Line** tool, and choose a thickness of 2¼ points.

Your screen should look something like this:



Teacher: Okay, your turn again. I would like you to expand the equation $y = (1 + x)^{\frac{1}{3}}$ as far as the term in x^3. And remember to simplify any fractions as much as possible.

Prompt: Again, a reminder of the formula might be handy, reminding the students to keep an eye on all those negative signs!

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2\times1} + \frac{n(n-1)(n-2)x^3}{3\times2\times1} + \ldots$$

Ideal Response: The expansion of $y = (1 + x)^{\frac{1}{3}}$ as far as the term in x^3 is $y = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$.

Teacher: Good. Now, we are going to do the same thing as before. I will begin by plotting the linear approximation, then the quadratic, and then the cubic, and see how good an approximation to the original line these are…

Ensure **Slow Plot** mode is still on.

Enter the equation: $y = 1 + \frac{1}{3}x$

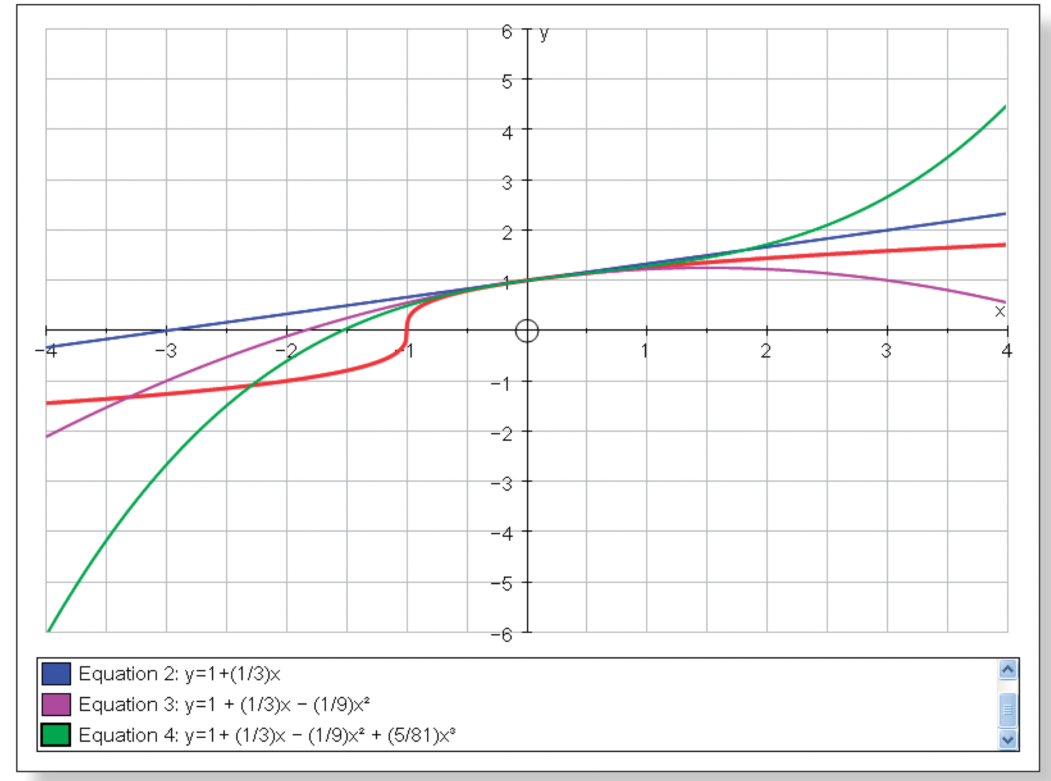
Enter the equation: $y = 1 + \frac{1}{3}x + \frac{1}{9}x^2$

Enter the equation: $y = 1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{5}{81}x^3$

Note: To enter fractions, use the forward slash / key, and remember the brackets. For example, to type $\frac{5}{81}x^3$, you could write "(5/81) **xxx**".

Each time an equation is being plotted, press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph. Give the students time to comment and discuss.

In the end your screen should look something like this:

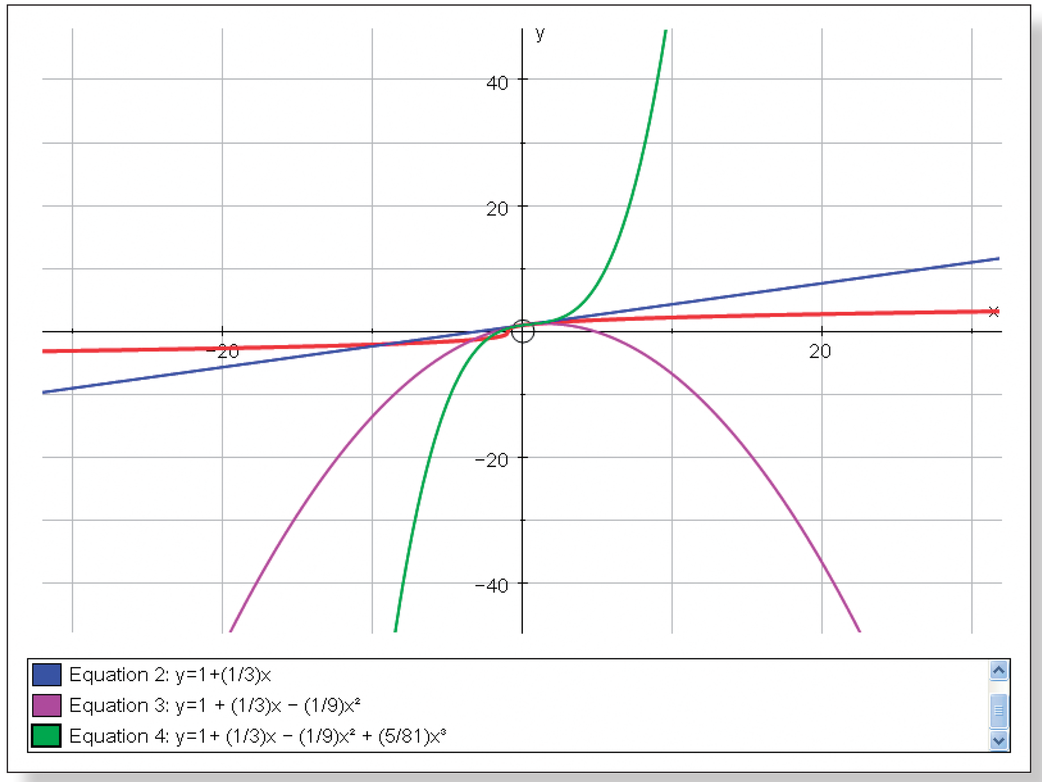


Teacher: Does anybody have any comments about the approximations this time.

Prompts for discussion: Look at the graphs at different values of x. Again, point out that usually, when using the binomial we are asked to use very small values of x, and so in terms of an approximation, many of these graphs actually seem quite good. However, also focus on large (positive and negative) values of x…

Use **Zoom Out** tool centred on the origin to help the students get a good grasp of what is going on.

Your page should look something like this:



Teacher: Does anybody have any further comments about the approximations?

Prompts for discussion: How would you describe what is happening to the approximations for really big or really small values of x? Emphasis the divergence, and contrast this with what was happening in the previous example on Page 1. Use the tabs at the top of the screen to quickly flick between pages.

Use the **Drag** tool to have a look at both sides of the graph. Follow the graph for large (positive and negative) values of x. **Zoom Out** even further if it helps.

Press the **Undo** button several times to return to the original view of the graph.

Teacher: So, looking at our graph, if n is a fraction, what values of x should we limit our approximations to?

Ideal Response: Values of x between −1 and 1.

Activity 3: Negative Powers of n

Open Page 3 by clicking on the tab.

Ensure you are working in **Select Mode**.

Teacher: Okay, still working with the $y = (1 + x)^n$ graph, we are now going to try negative values of n. Now, the value in particular that I want us to look at is when $n = -5$. Does anybody have any idea what on earth the graph of $y = (1 + x)^{-5}$ would look like? Come up and sketch it if you are feeling brave!

Prompt: What does "to the power minus five" actually mean? How else could we write it? Are there any values of x for which the curve is undefined? Can anybody think of a couple of points which we know must lie on the line? What happens to the graph for really large and really small values of x? Does the graph ever touch the x-axis?

Encourage students to come to the front to sketch the curve or mark on their points.

Use the **Erase** tool to rub out any mistakes.

If you want to get rid of all scribbles, click on Edit > **Select All Scribbles**, and press **delete** on the keyboard (or **Right-click** on the graph itself and select **Delete Objects** from the menu).

Teacher: Let's have a look…

Ensure **Slow Plot** mode is still on.

Enter the equation: $y = (1 + x)^{-5}$

Note: To enter "to the power of minus 5", type: "^(−5)"

Click **OK**, and the curve should begin to plot, hopefully resembling the curves drawn by the students.

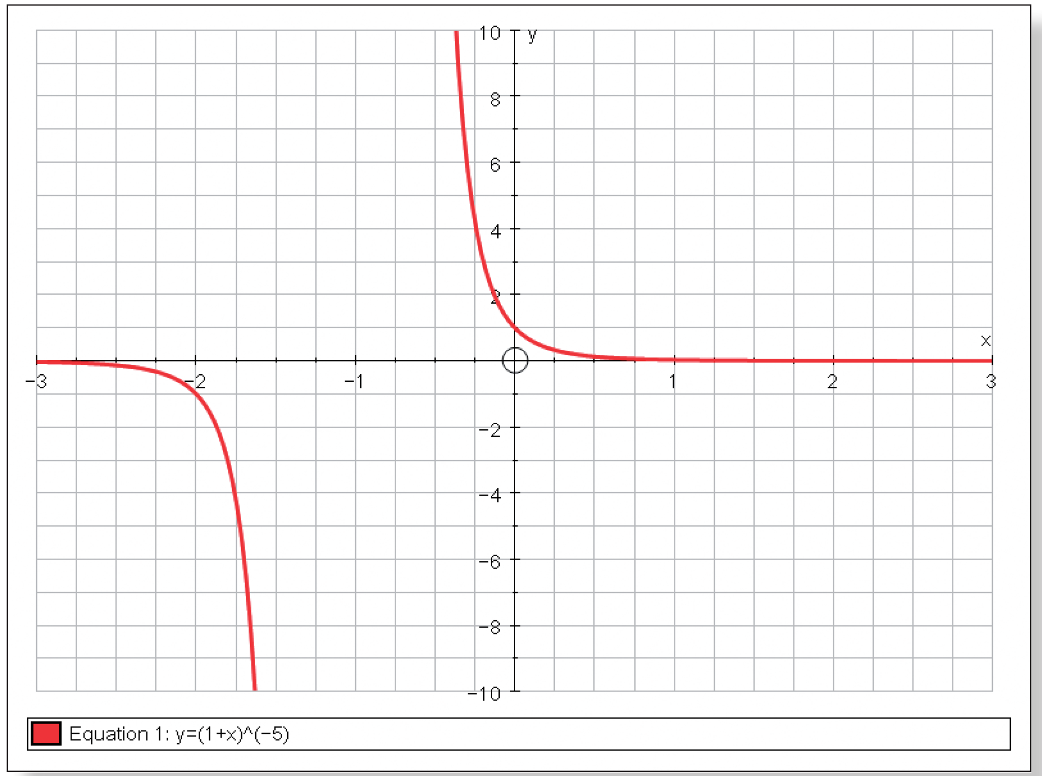
Press **Pause Plotting** (or **Spacebar**) both to stop the process and to resume it to focus on the key features of the graph. Focus on the asymptote at $x = -1$, and the fact that the graph never crosses the x-axis.

When the curve has finished plotting, *delete all scribbles* as described above.

Left-click on the curve (it should turn black).

Click on the **Thick Line** tool, and choose a thickness of 2¼ points.

Your screen should look something like this:



Teacher: Okay, your turn again. I would like you to expand the equation $y = (1 + x)^{-5}$ as far as the term in x^4. And remember to simplify any co-efficients as much as possible.

Prompt: Again, a reminder of the formula might be handy, reminding the students to keep an eye on all those negative signs!

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2\times1} + \frac{n(n-1)(n-2)x^3}{3\times2\times1} + \ldots$$

Ideal Response: The expansion of $y = (1 + x)^{-5}$ as far as the term in x^4 is:

$$y = 1 - 5x + 15x^2 - 35x^3 + 70x^4$$

Teacher: Good. Now, we are going to do the same thing as before. I will begin by plotting the linear approximation, then the quadratic, then the cubic, then the quartic, and see how good an approximation to the original line these are…

Ensure Slow Plot mode is still on.

Enter the equation: $y = 1 - 5x$

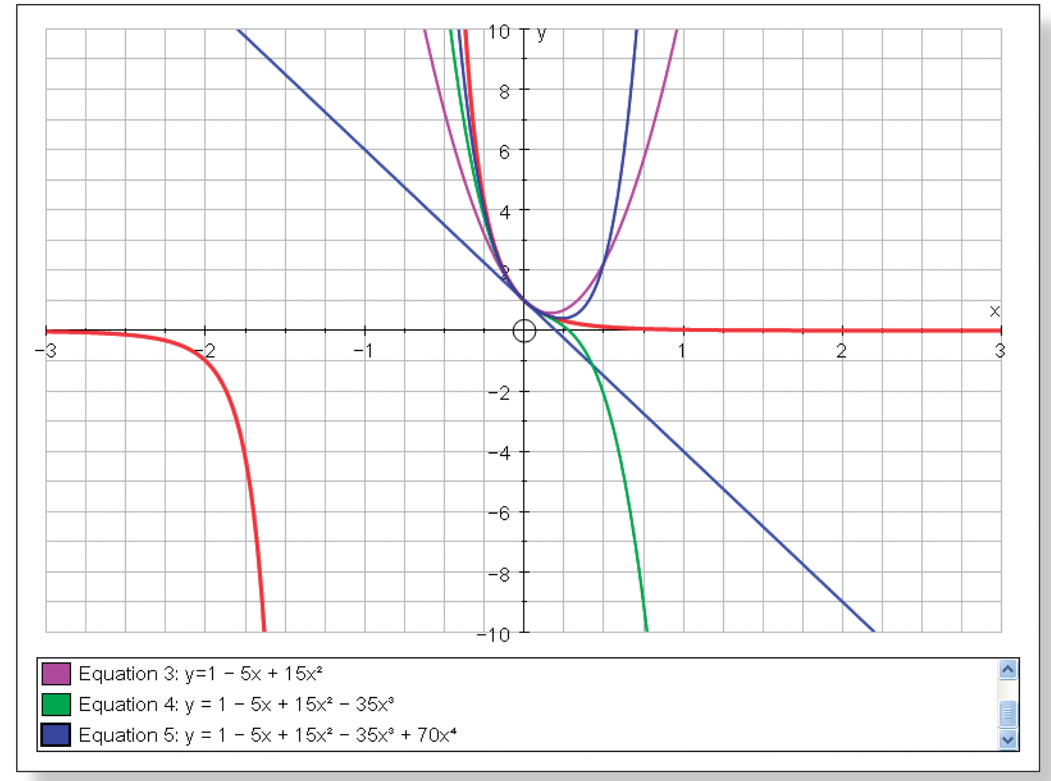
Enter the equation: $y = 1 - 5x + 15x^2$

Enter the equation: $y = 1 - 5x + 15x^2 - 35x^3$

Enter the equation: $y = 1 - 5x + 15x^2 - 35x^3 + 70x^4$

Each time an equation is being plotted, press Pause Plotting (or the Spacebar) both to stop the process and to resume it to focus on the key features of the graph. Give the students time to comment and discuss.

In the end your screen should look something like this:

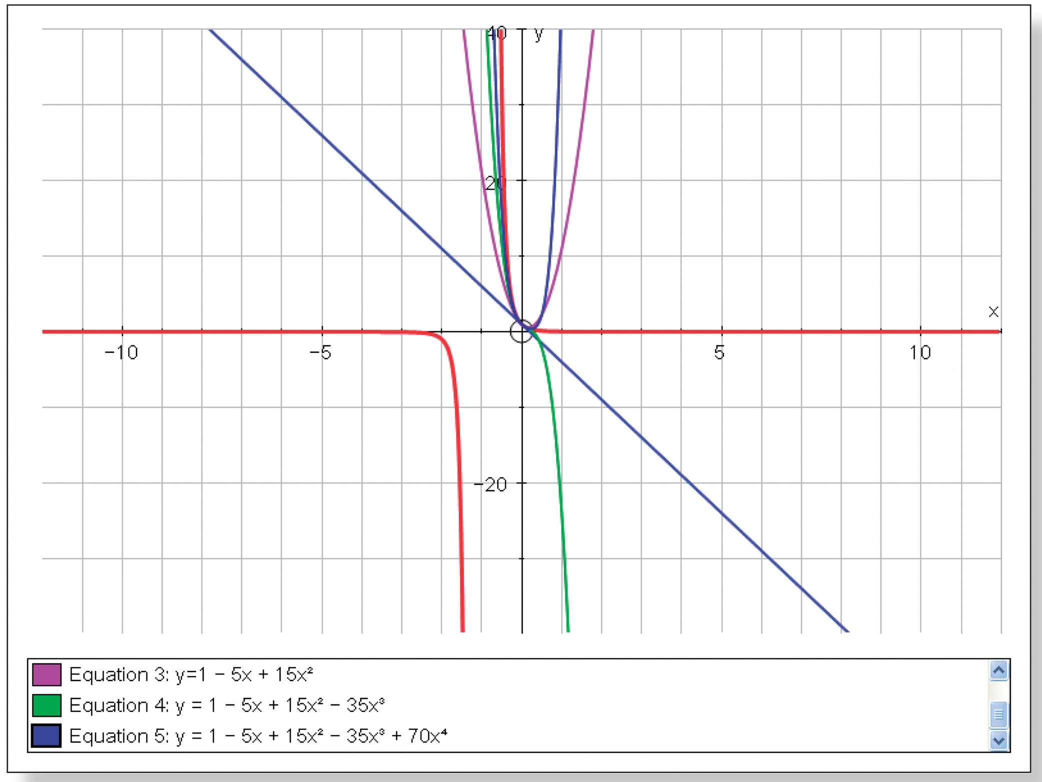


Teacher: Does anybody have any comments about the approximations this time?

Prompts for discussion: Again, it should be clear that for small values of x, the graphs give good approximations. However, there is a clear problem on the negative side. Again, focus on large (positive and negative) values of x…

Use Zoom Out tool centred on the origin to help the students get a good grasp of what is going on.

Your page should look something like this:



Teacher: Does anybody have any further comments about the approximations?

Prompts for discussion: Again, ensure the discussion focuses on divergence, and the values of x for which the approximation is a reasonable one, and the values for which it is not.

Use the tabs to flick between Page 1, Page 2 and Page 3.

Teacher: Can anybody briefly summarise what the graphs of the binomial approximation have shown us?

Prompts: Think of the three cases in turn, when n is positive, fractional, and negative. What values of x make for good approximations? What values of x do not make for good approximations? How does this relate to the type of questions we are often asked to tackle using the binomial approximation?

Ideal Response: The case where n is positive appears to have the widest range of values of x for which we get a reasonable approximation. However, even in this case it seems to be that the larger the value of x, the weaker the approximation. This is because the larger the value of x, the more significant terms involving higher powers of x on our expansion will be, and these terms are often omitted from the approximations. For the cases when n is fractional and negative, large and small values of x give extremely poor approximations. In fact, in many cases the graphs of the approximations appear to diverge quite considerably from the graph of the original function. Hence, it is only appropriate to use values of x between -1 and 1 for these values of n, and even then (especially in the case of negative values of n) we still may not get accurate results. However, generally we are asked to use values of x extremely close to zero, so our graphs have shown us that those approximations should be reasonably accurate.

Ideas for Further Work

- Examples of using the binomial expansion to find approximate answers for things like $\sqrt{1.03}$ and $\sqrt{26}$ would follow on nicely from this demonstration.
- If it has not already been covered, look at how the binomial can be used to expand expressions such as $(5 + 4x)^{1/2}$.

Learning Objectives

- To derive the value of e.
- To better understand the relationship between functions and their gradient function, including polynomials.
- To understand that the gradient function of e is the same as the function itself.

Required Pre-Knowledge

- To understand the concept of the gradient function and of tangents.
- To be able to derive the gradient functions of quadratic and cubic functions using differentiation.
- To be comfortable with index notation.

Pre-Activity Set-Up

Open up Autograph in Advanced Mode and ensure you have a blank 2D Graph Page.

At the top of the screen go to Page > Edit Settings, and adjust the number of significant figures up to 8. This will increase the accuracy of our calculations.

Select Whiteboard Mode.

Edit the axes as follows:

x:	Minimum: -4	Maximum: 4	Numbers: 1	Pips: 0.5
y:	Minimum: -16	Maximum: 16	Numbers: 2	Pips: 1

Remove all of the green ticks underneath Auto.
Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Enter the equation: $y = x^2$

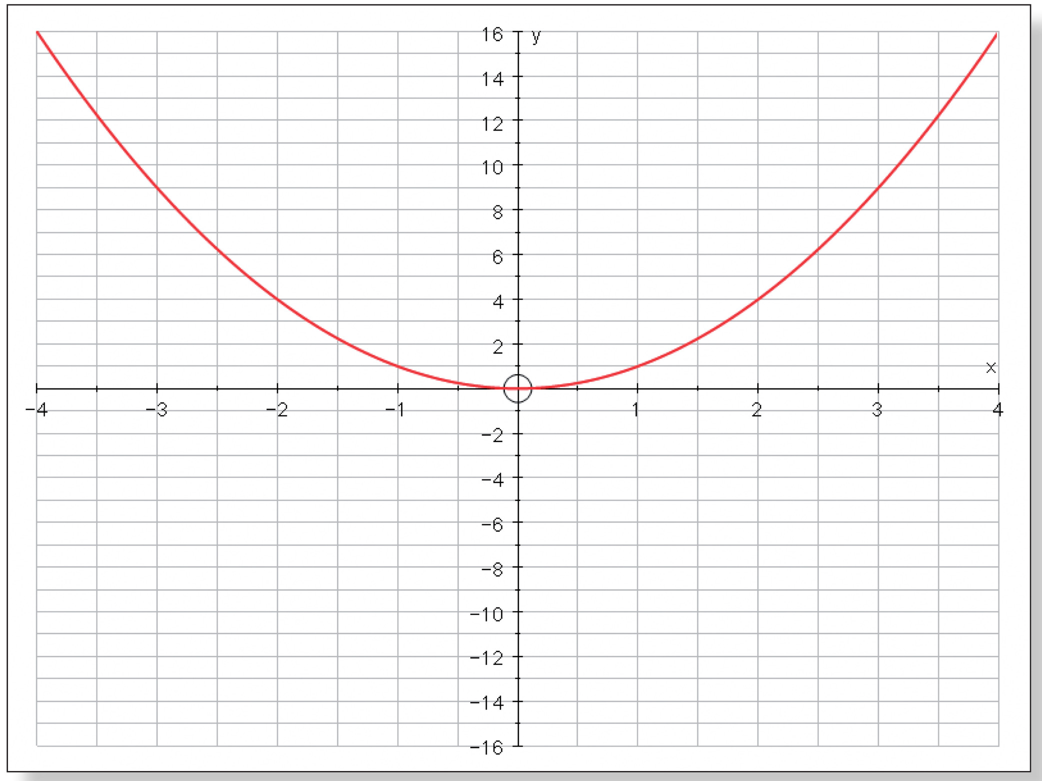
Note: To enter x^2, either use the little 2 button, or press "alt 2" together, or press "xx".

At the top of the screen go to Axes > Show Key.

This should make the key at the bottom of the screen disappear.

Note: This can also be done by right-clicking on the Key towards the bottom of the screen where it says "Equation 1: y = 8", and from the menu left-click on Show Key.

Your screen should look something like this:



Step-by-Step Instructions

Activity 1: Polynomials

Teacher: Just to warm you up, what is the equation of the *red* line on the screen?

Prompt: Think about the various points which lie on the line.

Note: If the students are struggling, it might be a nice idea to add a point to the curve as follows:

Left-click on the curve (it should turn black).

Add a point onto the curve with an x value of 0.

Select Text Box from the top toolbar and click OK.

The co-ordinates of the point should now be labelled.

You can then use the left-right arrow keys on the keyboard to move the point along the curve and draw the students' attention to the changing co-ordinates. This should help them better see the link between x and y.

When you are finished, ensure that only the point is selected, right-click and select Delete from the menu, or simply press delete on the keyboard to clear both the point and the Text Box.

Ideal Response: $y = x^2$

Teacher: And what is the equation of the *gradient function* of the line?

Prompt: What does the gradient function mean? What technique do we use to find the gradient function?

Ideal Response: $y = 2x$

Teacher: Good, now let's watch as Autograph plots the gradient function of the curve by calculating the gradient of the tangent at many different points on the curve.

Left-click to select the curve (it should turn black).

Click on **Slow Plot** mode.

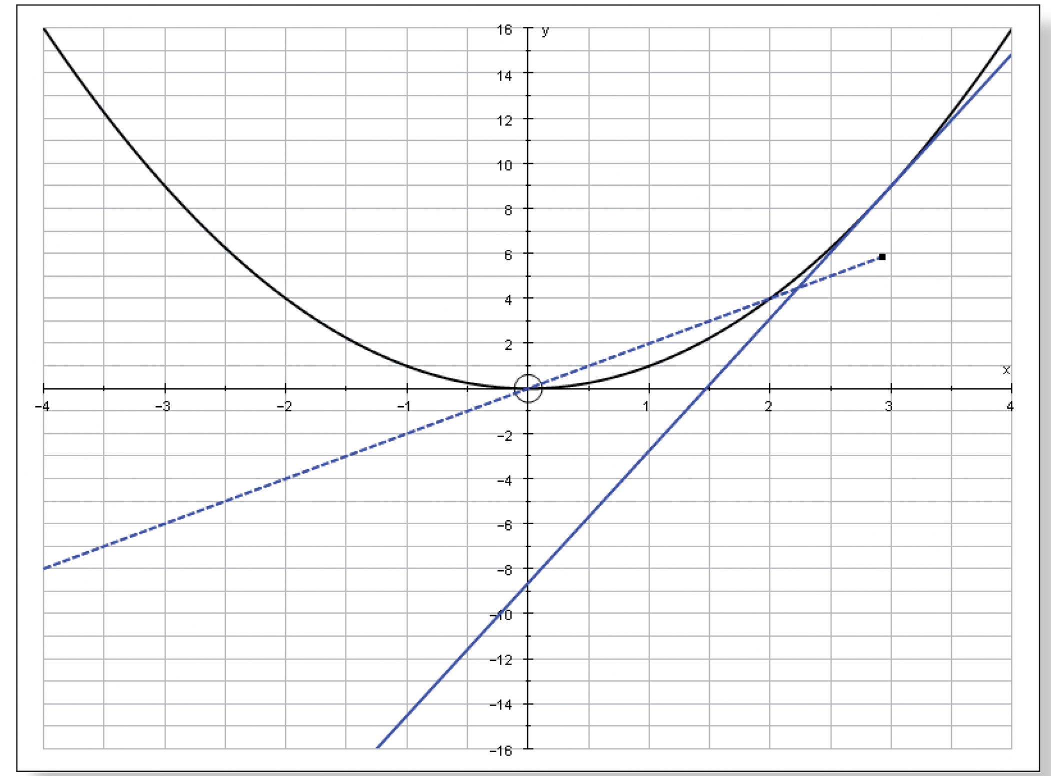
Click on **Gradient Function**, and click **OK**.

This should now show how the gradient function is derived by plotting the gradient of tangents to the curve.

Press **Pause Plotting** both to stop the process, and to resume it (the **Spacebar** on the keyboard can also be used for this).

Note: It is worth emphasising at this stage that the gradient of the curve goes from negative to positive, and that this is shown in the y values of the gradient function. This is important later.

Your screen should look something like this:



Teacher: Here's a question for you. Will there ever be a quadratic function whose gradient function is *identical* to the function itself?

Prompt: Think about what all the gradient functions of quadratic functions have in common. Think about their shape.

Ideal Response: No, because the gradient function of any quadratic function will always be linear.

Left-click to ensure the curve is still selected (it should still be black).

Right-click and select **Delete Object** from the menu, or simply press delete on the keyboard.
Note: Don't worry about the warning message. This is just to inform you that the gradient function will also be deleted as it is defined with respect to the curve itself.

You should now be left with just your axes.

Teacher: Would anybody like to predict what the graph of the equation $y = x^3 + 4x^2 - 3x - 8$ would look like?

Prompt: Which direction will it slope? Where will it cross the y-axis?

Ensure **Slow Plot** mode is still turned on.

Enter the equation: $y = x^3 + 4x^2 - 3x - 8$
Note: To enter x^3, either use the little 3 button, or press "**alt 3**" together, or press "**xxx**".

With the curve selected, choose **Text Box** from the top toolbar and click **OK**. The equation of the curve should now be labelled.

Teacher: Can you work out the gradient function of this curve? And for an extra challenge, can you tell me where the gradient function would cross the x-axis?

Prompt: Remember, we differentiate each term one at a time. Multiply the whole term by the power, and subtract one to get your new power. What must a function be equal to when it crosses the x-axis?

Ideal Response: The gradient function is: $3x^2 + 8x - 3$

Factorised this becomes: $(3x - 1)(x + 3)$

This crosses the x-axis when the function is equal to 0, which occurs when x is $\frac{1}{3}$ and -3.

Ensure the curve is selected (it should be black).

Ensure **Slow Plot** mode is still on.

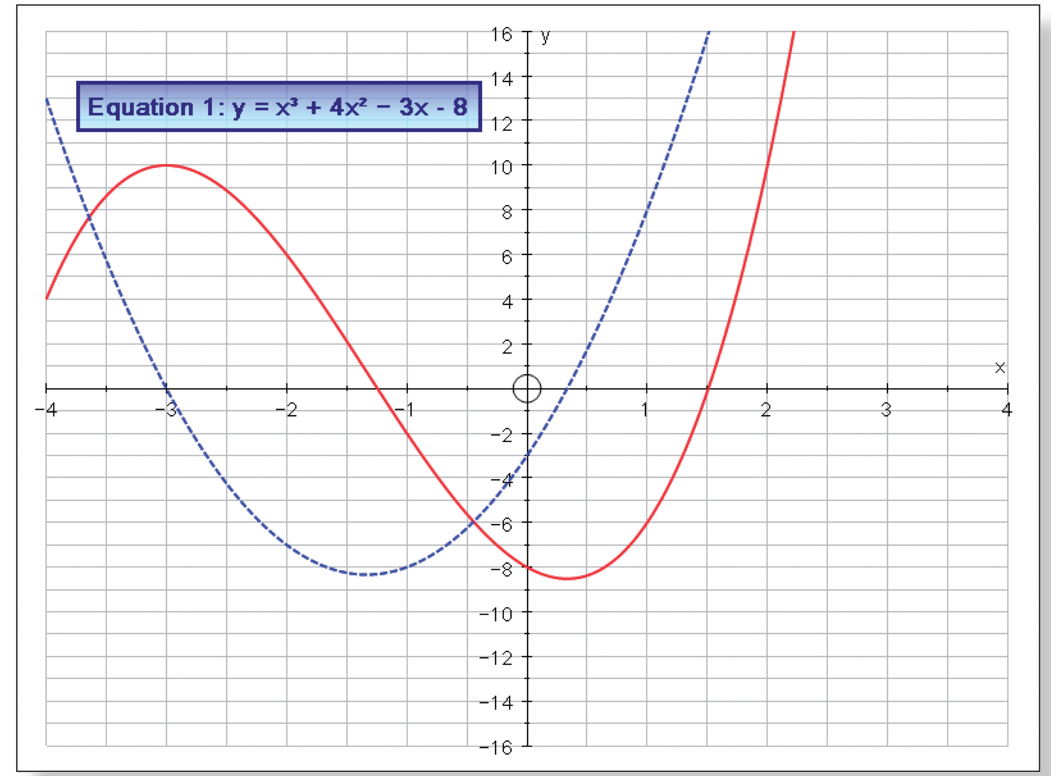
Click on **Gradient Function**, and click **OK**.

This should now show how the gradient function is derived by plotting the gradient of tangents to the curve.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process, and to resume it.

Note: Again, emphasise how the gradient of the curve goes *from positive to negative to positive again*, and how this is shown in the gradient function.

When the gradient function has finished plotting, your screen should look something like this:



Teacher: Okay, so will there ever be a cubic function whose gradient function is identical to the function itself?

Prompt: Think about what all the gradient functions of cubic functions have in common. Think about their shape.

Ideal Response: No, because the gradient function of any cubic function will always be quadratic.

Teacher: Good, and can anybody use this to explain why there can never be a *polynomial* whose gradient function is exactly the same as the function itself?

Prompt: Use your knowledge of how to differentiate polynomials. Think about the powers.

Ideal Response: The gradient function of any polynomial will always be of an order *one less* than the function itself, and so the two functions can never be the same as each other.

Teacher: Good, but there are a group of functions whose gradient functions are actually very similar to the functions themselves…

ACTIVITY 2: $y = 2^x$

Left-click to ensure the curve is still selected (it should still be black).

Right-click and select **Delete** from the menu, or simply press delete on the keyboard.

You should now be left with just a set of axes.

Teacher: Can anybody tell me what the curve $y = 2^x$ would look like?

Prompt: Tell me some points which are definitely on the curve. Where does the curve cross the y-axis? Think about what happens to y as x gets bigger. How about when x gets smaller? How about when x is negative? Will the curve ever go below the x-axis?

Note: If the students do come up with points which lie on the line, you could have them mark those points on the board them using the **Scribble Tool**, or enter them as points using the **Enter Co-ordinates** tool.

Ideal Response: A curve which starts just above the x-axis for negative values of x, remains relatively flat/horizontal, passes through the point (0, 1), and then continues to get steeper and steeper as the value of x increases.

Ensure **Slow Plot** mode is still turned on.

Enter the equation: $y = 2^x$
Note: To enter x as a power, either use the little x button, or press "**alt x**" together.

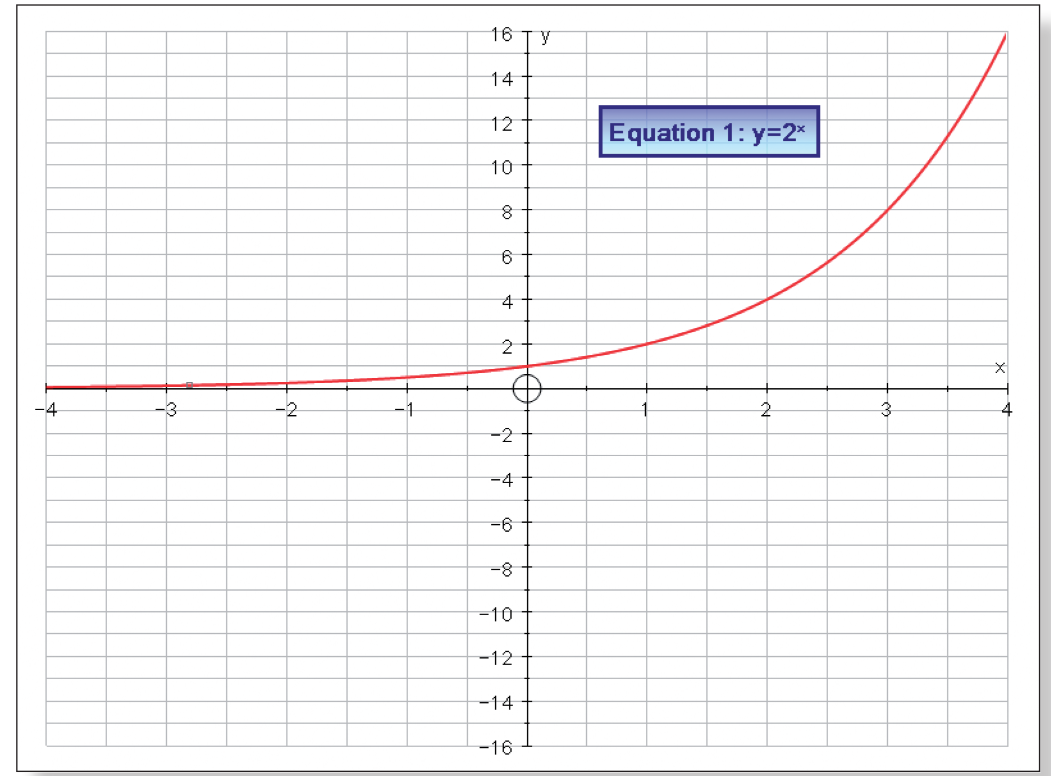
The curve should now start to plot.

Press **Pause Plotting** (or the **Spacebar**) both the stop the process, or to resume it.

Left-click on the curve when the plotting has finished to ensure the curve is selected (it should be black).

Select **Text Box** from the top toolbar and click **OK**.
The equation of the curve should now be labelled.

Your screen should look something like this:



Use the **Drag** and **Zoom In** functions to emphasise that although the curve gets closer and closer to the x-axis for negative values of x, it never actually touches.

Use the **Drag** and **Zoom Out** functions to return to the original view of the graph.

Note: Pressing the **Undo** button several times is often a quicker way of doing this.

Teacher: Can anybody describe the gradient function of this curve?

Prompt: Is the gradient ever negative? Think about the gradient when x is negative. What is happening to the gradient as x increases?

A good way to illustrate these points is to have a moving tangent as follows:

Add a point on the left hand side of the curve, somewhere between $x = -4$ and $x = -3$.
Note: The cross will turn into an arrow sign when the pointer is over the curve.

Right-click and select **Tangent** from the menu.

This should display the tangent to the curve at that given point.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click to select the tangent (it should turn black).

Select **Text Box** from the top toolbar and click **OK**.

The equation of the tangent should now be labelled.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the point you placed on the curve (it should have a square around it).

You can now use the **left-right arrows** on the keyboard to change the position of the point.

Draw the students' attention to the effect it is having on the *gradient of the tangent*.

Get them to imagine the gradient function as the point moves from left to right along the curve.

Ideal Response: The gradient function is pretty similar to the function itself. It is a curve which starts just above the x-axis for negative values of x, remains relatively flat/horizontal, and then continues to get steeper and steeper as the value of x increases.

Teacher: Sounds good. Let's have a look…

Left-click on an unoccupied part of the graph area to *de-select everything*.

Left-click on the curve (it should turn black).

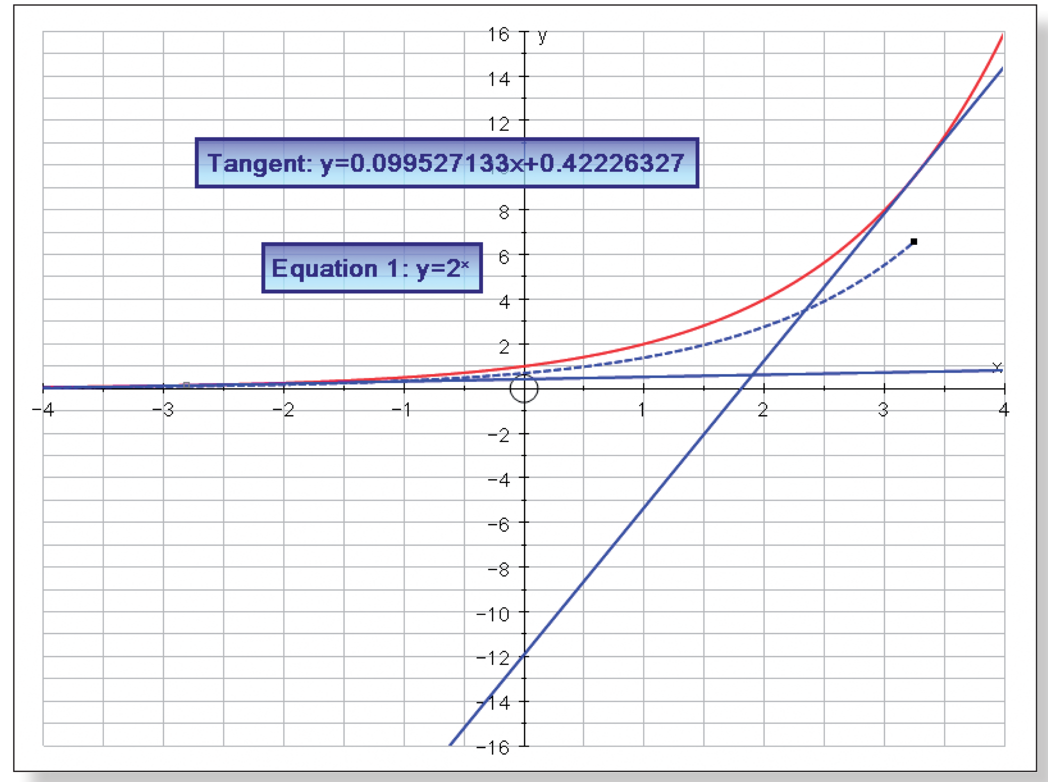
Make sure **Slow Plot** mode is still turned on.

Click on **Gradient Function**, and click **OK**.

This should now show how the gradient function is derived by plotting the gradient of tangents to the curve.

Press **Pause Plotting** (or the **Spacebar**) both the stop the process, or to resume it.

Your screen should look something like this:



Teacher: Now, we can see that the gradient function is pretty similar to the function itself. But it is not identical. I wonder if it is possible to find a "special function" in the same form as $y = 2^x$, but whose gradient function is identical to the function itself…

Activity 3: In Search of that Special Function

Go to **Edit** in the top toolbar, then **Select All**, and then press delete on the keyboard.

All lines and text boxes should now have disappeared, leaving you with the set of axes again.

Make sure **Slow Plot** mode is turned off.

Enter the equation: $y = a^x$
Still on the **Add Equation** screen, click on **Edit Constants**.
Set the value of a to 2 and click **OK** twice.
This ensures you still have the same graph on the screen as before.

Click on **Gradient Function**, and click **OK**.

Teacher: Okay, so here we have the graph of $y = a^x$ where $a = 2$ as before. As you can see, we got pretty close to finding our "special function", so now let's try some other values of a…

Click on the **Constant Controller**.
You can now adjust the value of a.
The **up-down** buttons adjust the value of a.

The **left-right** buttons adjust the size of the step.
The current value of a is recorded in the **Constant Controller**.

Teacher: What do you think would happen to the graph of the function and the graph of the gradient function if I decreased the value of a?

Prompt: Try some different values of x and a in your head. What happens to the function?

Ideal Response: The function itself would become flatter/less steep as the y values decrease, and so the gradient function would also become flatter.

Keeping the value of the step at 0.1, slowly decrease the value of a, focussing on the slope of the curve. Stop when a = 1.5.

Teacher: What do you notice about the function and the gradient function?

Ideal Response: They are getting further away from each other.

Teacher: Incidentally, can anybody tell me what happens to the curve when a = 1, and what the gradient function would look like?

Prompt: We get the graph of $y = 1^x$. Try some different values of x. What do you notice? What does this mean for the shape of the graph? How about for the shape of the gradient function?

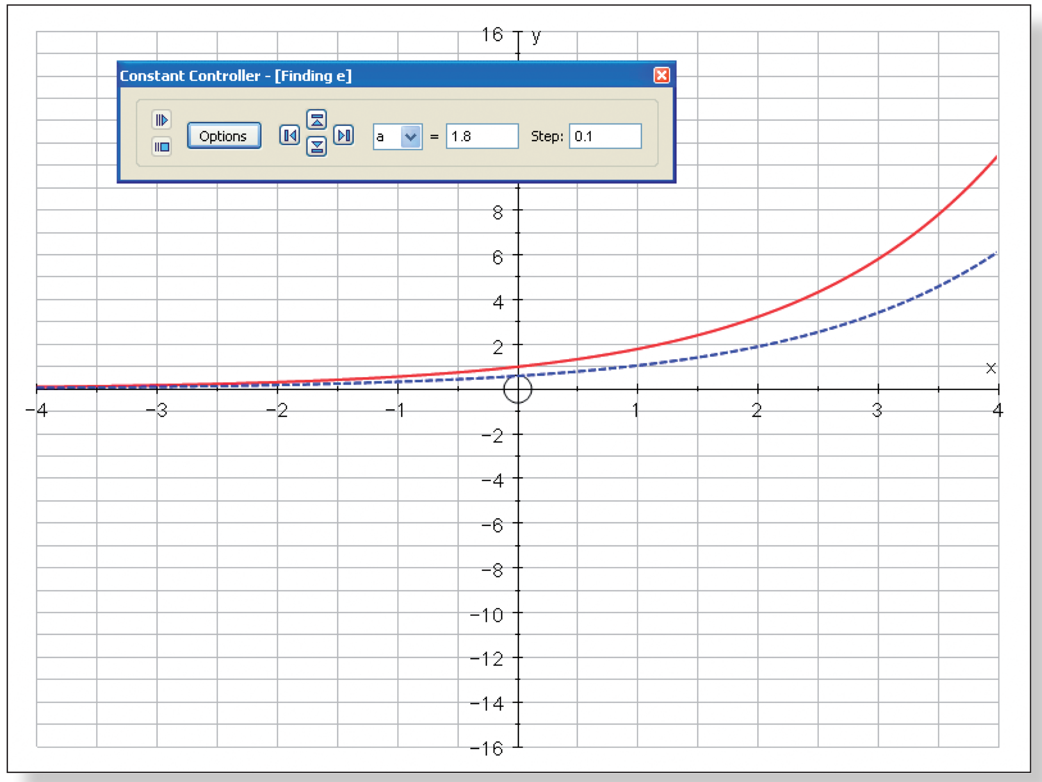
Ideal Response: We end up with a horizontal line, passing through the point (0, 1) or a line with the equation y = 1, because 1 to the power of anything is always 1. The gradient of this line would always be 0, so that gradient function would have the equation y = 0, which is the equation of the x-axis!

Use the **Constant Controller** to reduce the value of a to 1 to confirm this to the students.

Teacher: Okay, so it looks like we need to increase the value of a to find our special function…

Keeping the value of the step at 0.1, slowly increase the value of a, drawing the students' attention to the slope of the two curves.

Your screen should look something like this:



Teacher: Here's a question for you. Why do all these curves seem to be passing through the same point on the y-axis?

Prompt: What is the value of x at this point?

Ideal Response: x is equal to zero, and anything to the power of zero is equal to 1, so a^x will always be 1 at this point.

Draw the students' attention to the fact that the curve and the gradient function are getting closer together as the value of a increases.

Keep increasing the value of a by the same step (0.1) until you get to 2.7.

Teacher: Well, this looks pretty close, but let's zoom in and take a closer look.

Use the hand and zoom functions to look closer at a portion of the curve, somewhere between the values of x = 2 and x = 2.5.

Teacher: The curves are certainly close, but as we can see, they are not touching.

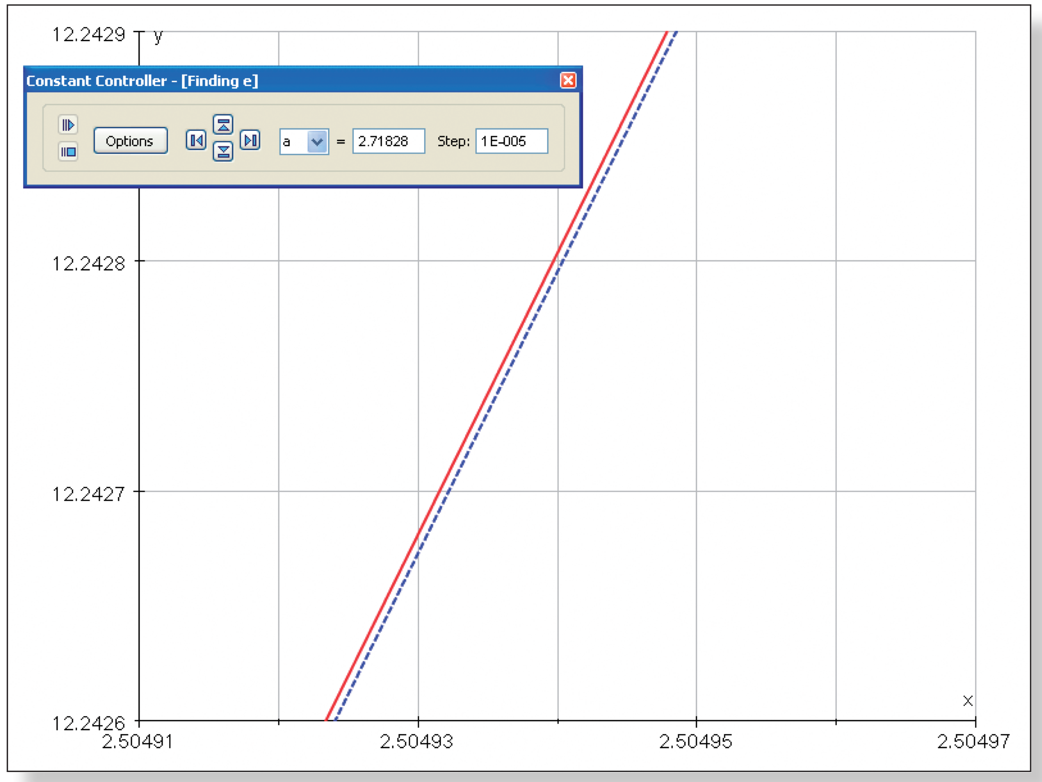
Adjust the size of the step to 0.01 and increase the value of a some more.

Each time the curves appear to be touching, zoom in some more and adjust the step.

Draw students attention to the scale on the axes, emphasising just how small the numbers are getting.

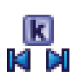
Use **Undo** if you ever lose the graph!

Your screen should look something like this:



Keep zooming in and adjusting the step as much as you feel necessary.

Teacher: Now, we could keep going and going trying to find the exact value of a which would make the function and the gradient function perfectly coincide, but we would be going forever. In fact, if we did keep zooming in, eventually the computer would fail as no amount of decimal places would ever describe the number perfectly. That is because the “magic number” itself is actually *irrational*. We have found a pretty good approximation to it here, and thankfully the number itself has a special, and far easier to remember, name than 2.71828… That number is called e.



In the **Constant Controller**, change the current value of a to “e”.

The two curves should now lie perfectly on top of each other.

Use the hand and the **Zoom Out** tools to return to the original view of the graph.

Teacher: Yes, the gradient function of $y = e^x$ is exactly the same as the function itself, making functions involving e incredibly important in mathematics.

Teacher: So, can anybody differentiate $y = e^x$ with respect to x…

Ideas for Further Work

- The students should now be in a strong position to start differentiating functions involving e.
- This activity could also be linked into work on exponential growth, and is also a good introduction to the logarithmic function - see Teacher Demonstration 14: The Natural Log Function.

Discovering the Natural Log Function

Learning Objectives

- To understand the importance of the natural log function.
- To build up a picture of the natural log function using the graph of $y = \frac{1}{x}$, and hence deduce its role in integrating $y = \frac{1}{x}$.
- To reinforce the importance of constants when integrating functions.
- To appreciate the important link between e and the natural log function.
- To look at the functions $y = \ln(-x)$ and $y = \ln(|x|)$.
- To begin to look at integrating functions using natural logs.

Required Pre-Knowledge

- To know how to integrate functions involving both positive and negative powers of x.
- To understand the role of the constant in integration.
- To be able to use integration to find the area underneath a curve.
- To have encountered the value e before – see Teacher Demonstration 13: Discovering e.

Pre-Activity Set-Up

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

On the top toolbar click on **Page** > **Edit Settings**.
Change the number of significant figures to 8 to improve accuracy.

Edit the axes as follows:

x: Minimum: −6 Maximum: 6 Numbers: 1 Pips: 0.5
y: Minimum: −12 Maximum: 12 Numbers: 2 Pips: 1

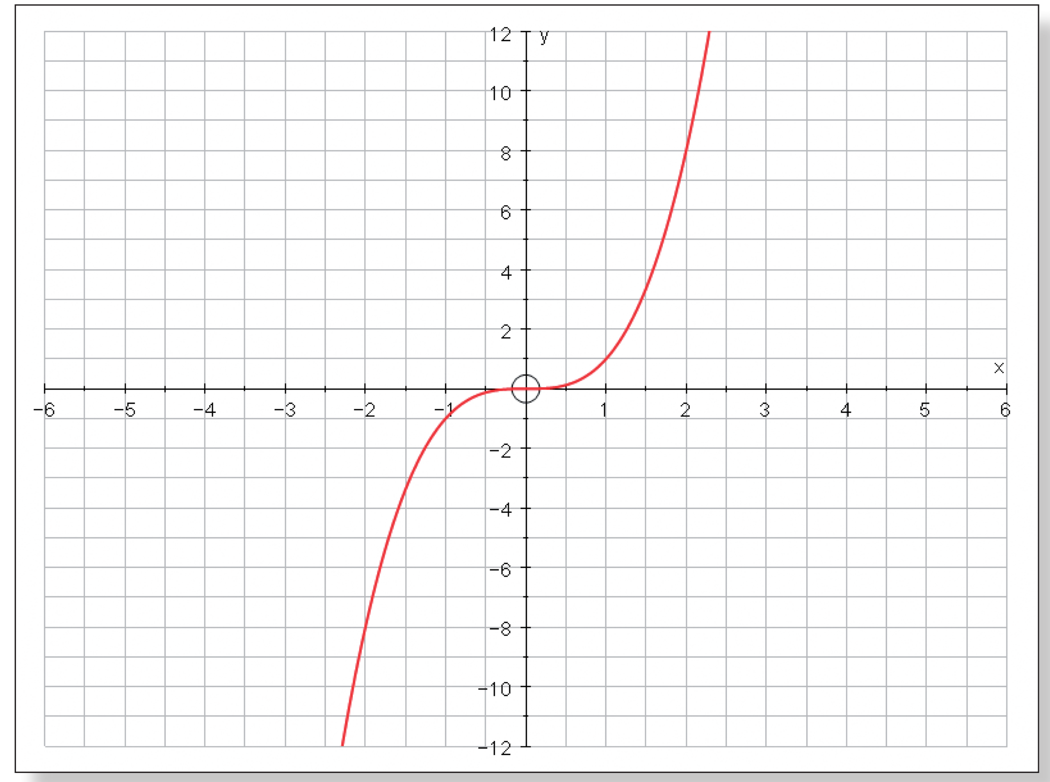
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Enter the equation: $y = x^3$
Note: To enter the cubed either use little 3 button, press "**alt 3**" together, or type "**xxx**".

On the top toolbar click on **Axes**, and then **Show Key**.

This should remove the key from the bottom of the screen.

Your screen should now look something like this:



In a prominent place at the front of the room (possibly on another whiteboard, or on a large piece of paper) make a copy of the following table:

Gradient Function	Function

Step-by-Step Instructions

Activity 1: Finding the Functions

Teacher: Okay, let's start with something nice and easy. What is the name of the function drawn on the board?

Prompt: Think about the shape. Think about which points it passes through.

Note: If the students are struggling, it might be a nice idea to add a point to the curve as follows:

Left-click on the curve (it should turn black).

Add a point onto the curve with an x value of 0.

Select **Text Box** from the top toolbar and click **OK**.

The co-ordinates of the point should now be labelled.

You can then use the **left-right** arrow keys on the keyboard to move the point along the curve and draw the students' attention to the changing co-ordinates. This should help them better see the link between x and y.

When you are finished, ensure that only the point is selected, **right-click** and select **Delete** from the menu, or simply press delete on the keyboard to clear both the point and the **Text Box**.

Ideal Response: $y = x^3$

Teacher: Good. Right, something a little more difficult. If $y = x^3$ was the gradient function of a certain function, what might that function be?

Prompt: What does gradient function mean? How do we work out gradient functions? So, if we already know what the gradient function is, how might we work out the original function? Might integration help?

Possible Response: $y = \frac{1}{4}x^4$

Teacher: And is that the only possibility? Are there any other functions which might give us a gradient function of $y = x^3$? Think about what we must also include when integrating?

Ideal Response: $y = \frac{1}{4}x^4 + c$

Teacher: Good. Now, let's just take a look at that, and in particular let's make sure we are happy with the role that the constant plays…

Click on **Slow Plot** mode.

Click on the **Integral Function** tool, and click **OK**.

Nothing should appear on the screen, as Autograph is waiting for you to select the initial values of x and y for each solution.

Left-click on the point (0, 0).

This should give you the graph of $y = \frac{1}{4}x^4$.

Teacher: Thinking about what we have just said, what is the equation of this function?

Prompt: Think about what points the graph goes through. What is the significance of the fact that it passes through the origin?

Ideal Response: $y = \frac{1}{4}x^4$

Teacher: Good. Now, how about this one?

Left-click on the point (0, 2).

This should give you the graph of $y = \frac{1}{4}x^4 + 2$.

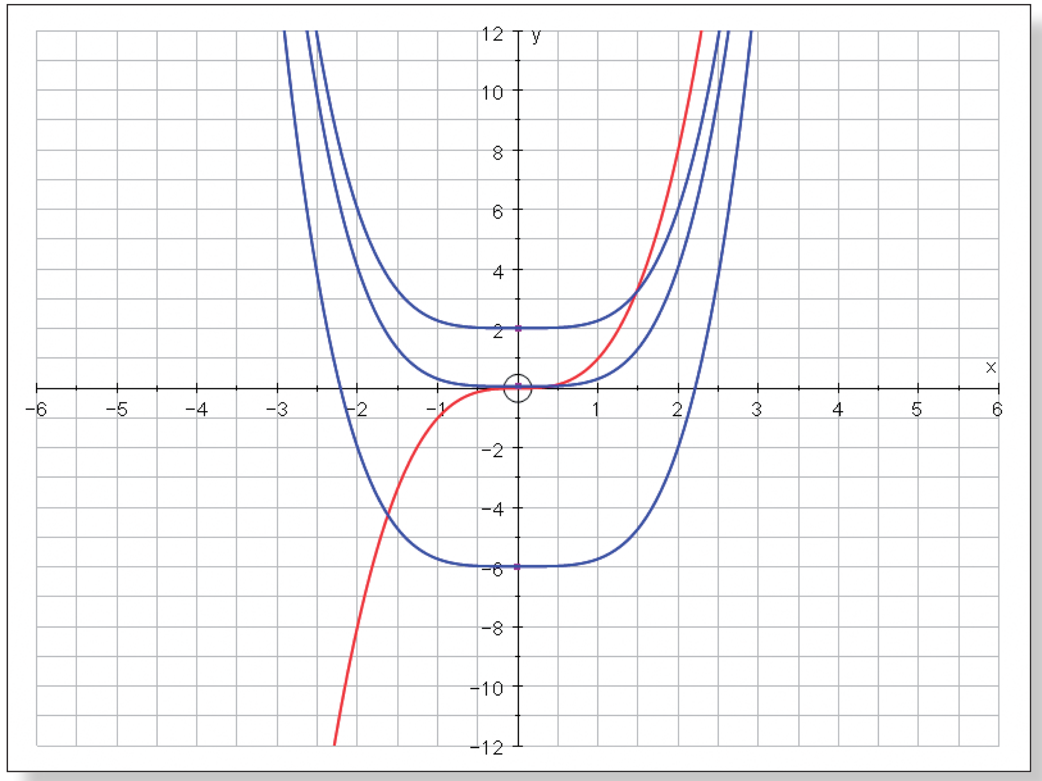
Ideal Response: $y = \frac{1}{4}x^4 + 2$

Teacher: And this one?

Left-click on the point (0, −6).

This should give you the graph of $y = \frac{1}{4}x^4 - 6$.

Your screen should look something like this:



Teacher: Now, think about the equations of those three functions. Do you agree that if you differentiated each of them, you would always arrive at the same gradient function, namely: $y = x^3$?

Ideal Response: Yes!

Teacher: And can somebody just remind us why that is the case, graphically?

Prompt: Look at the family of graphs. Think about the gradient of the curves for any given x value.

Ideal Response: The effect of the constant is simply to translate the curves up and down. The gradient at any given x value remains the same, and so the gradient function remains the same.

Teacher: Good. Now, the other thing I want to draw your attention to is the fact that we could have predicted the shape of the original function even if we didn't know how to integrate. Thinking about the gradient function ($y = x^3$) can anybody tell us how?

Prompt: What does the shape of the gradient function ($y = x^3$) tell us about the shape of the original function? Think about some of the points on the gradient function, in particular the y values, and what they mean for the graph of the original function.

Ideal Response: The gradient function ($y = x^3$) starts off negative between $x = -2$ and $x = 0$. This tells us that the function itself must start by sloping downwards, which it does. Similarly, the gradient function has a y value close to zero around the origin, which tells us that when x is close to zero, the function itself must have a gradient of zero, and hence be relatively flat. Finally, when x is positive, the gradient function is also positive, which tells us that the function itself must slope upwards.

Teacher: Good. Now, let's just make a quick note of our findings in this table:

Gradient Function	Function
$y = x^3$	$y = \frac{1}{4}x^4 + c$

Ensure you are in **Select Mode**.

Go to **Edit** on the top toolbar, then **Select All**, and then press **delete** on the keyboard (or alternatively **right-click** on any point on the graph area and select delete from the menu).

This should leave your screen clear apart from the set of axes.

Enter the equation: $y = x^2$

Note: To enter the squared either use little 2 button, press "**alt 2**" together, or type "**xx**".

Teacher: Right, no prizes for telling me that this is the graph of $y = x^2$. What I want to know is that if this was in fact the graph of a gradient function of a certain function, what might that function be?

Ideal Response: $y = \frac{1}{3}x^3 + c$

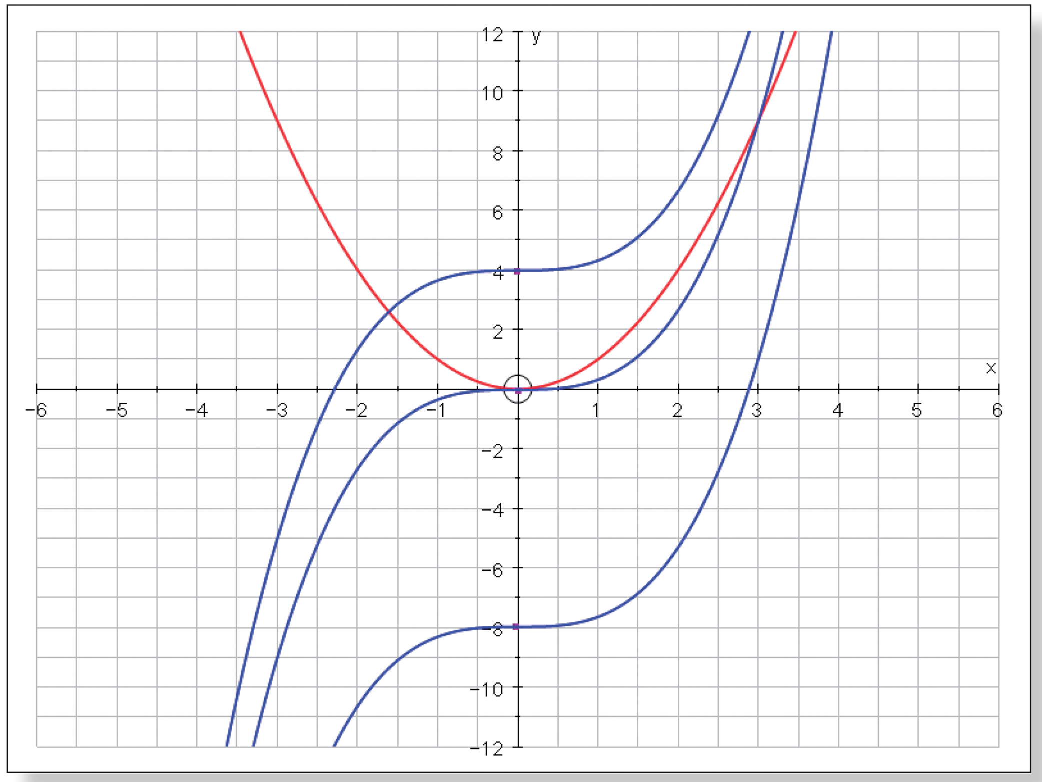
Proceed in a similar manner as described above:

Make sure **Slow Plot** mode is on.

Click on the **Integral Function** tool, and click on points on the y-axis to plot a family of functions which all share the same gradient function.

Draw the students' attention to the relationship between the functions themselves and the gradient function ($y = x^2$), namely how it is possible to determine the shape of the function by looking carefully at the gradient function. This will be important later on.

When complete, your page should look something like this:



Fill in the table:

Gradient Function	Function
$y = x^3$	$y = \frac{1}{4}x^4 + c$
$y = x^2$	$y = \frac{1}{3}x^3 + c$

Teacher: Okay, so now all I want you to do is to try to fill in the rest of the table yourself using the following gradient functions on the left:

Gradient Function	Function
$y = x^3$	$y = \frac{1}{4}x^4 + c$
$y = x^2$	$y = \frac{1}{3}x^3 + c$
$y = x$	
$y = x^0$	
$y = x^{-1}$	
$y = x^{-2}$	
$y = x^{-3}$	

Allow the students sufficient time to complete this task.

If any functions are proving particularly difficult (apart from $y = x^{-1}$ of course!), then it might be useful to repeat the above method using Autograph to illustrate the original functions and their relation to the gradient function to the students.

Spending a couple of minutes discussing the case of $y = x^0$ might also prove useful.

The completed table should look like this:

Gradient Function	Function
$y = x^3$	$y = \frac{1}{4}x^4 + c$
$y = x^2$	$y = \frac{1}{3}x^3 + c$
$y = x$	$y = \frac{1}{2}x^2 + c$
$y = x^0$	$y = x + c$
$y = x^{-1}$	?
$y = x^{-2}$	$y = -x^{-1} + c$
$y = x^{-3}$	$y = -\frac{1}{2}x^{-2} + c$

Activity 2: The Mystery Function

Teacher: Okay, so we seem to have a problem with $y = x^{-1}$. Can somebody just quickly explain what the problem seems to be?

Ideal Response: When we integrate, we usually add one on to the power, and then divide by the new power. But that would mean we were dividing by zero!

Teacher: Okay, let's have a look at this function more closely to see if we can figure out what is going on. Can anybody describe what the graph of $y = x^{-1}$ looks like?

Prompt: Think about how else we could write the function. Are there any values of x for which the function is undefined? What happens to the function for really big (both positive and negative) values of x? What happens when x is close to zero?

Ideal Response: The graph is undefined when $x = 0$, meaning $x = 0$ is an asymptote. When x is really big (both positive and negative), y is really small, and when x is close to zero, y is really big.

Teacher: Good. So let's look at the graph…

Ensure you are in **Select Mode**.

Go to **Edit** on the top toolbar, then **Select All**, and then press **delete** on the keyboard (or alternatively **right-click** on any point on the graph area and select **Delete** from the menu).

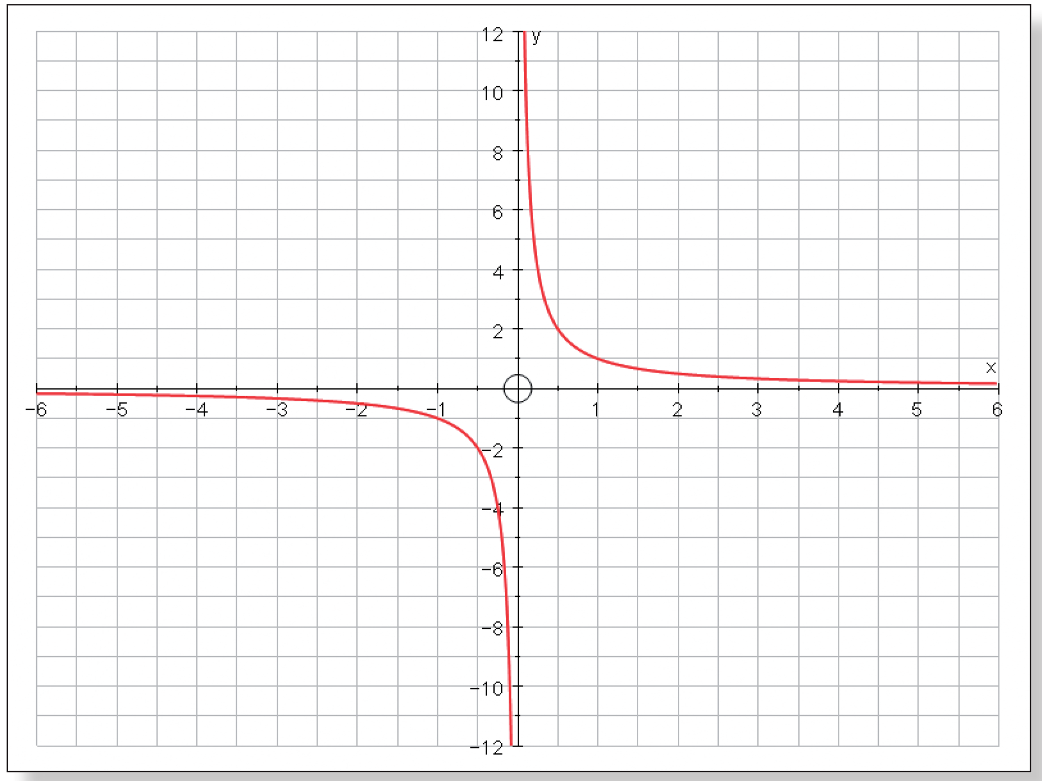
This should leave your screen clear apart from the set of axes.

Make sure **Slow Plot** mode is still turned on.

Enter the equation: $y = x^{-1}$.
Note: To enter the minus one either use little −1 button, or type "**^(−1)**"

Press **Pause Plotting** both to stop the process and to resume it (the **Spacebar** on the keyboard can also be used for this).

Your screen should look like this:



Teacher: Okay, so it seems we cannot use our normal method of integration to work out what function $y = x^{-1}$ is the gradient function for. But can we at least use the graph of $y = x^{-1}$ to describe what our actual function will look like? And to make matters easier, let's focus on just the positive values of x.

Prompt: Think about the y values of the gradient function for given x values. What do they tell you about the shape of our function? Think about the asymptote of the gradient function? What does that tell you about the function itself?

Ideal Response: For positive values of x, the gradient function is always positive, and so the function itself must always slope upwards. However, the y values of our gradient function are falling as x increases, which suggests that the function itself will start off very steep around $x = 0$, and then gradually begin to flatten out. Also, because our gradient function is not defined for $x = 0$, the function itself must not be defined for $x = 0$ either.

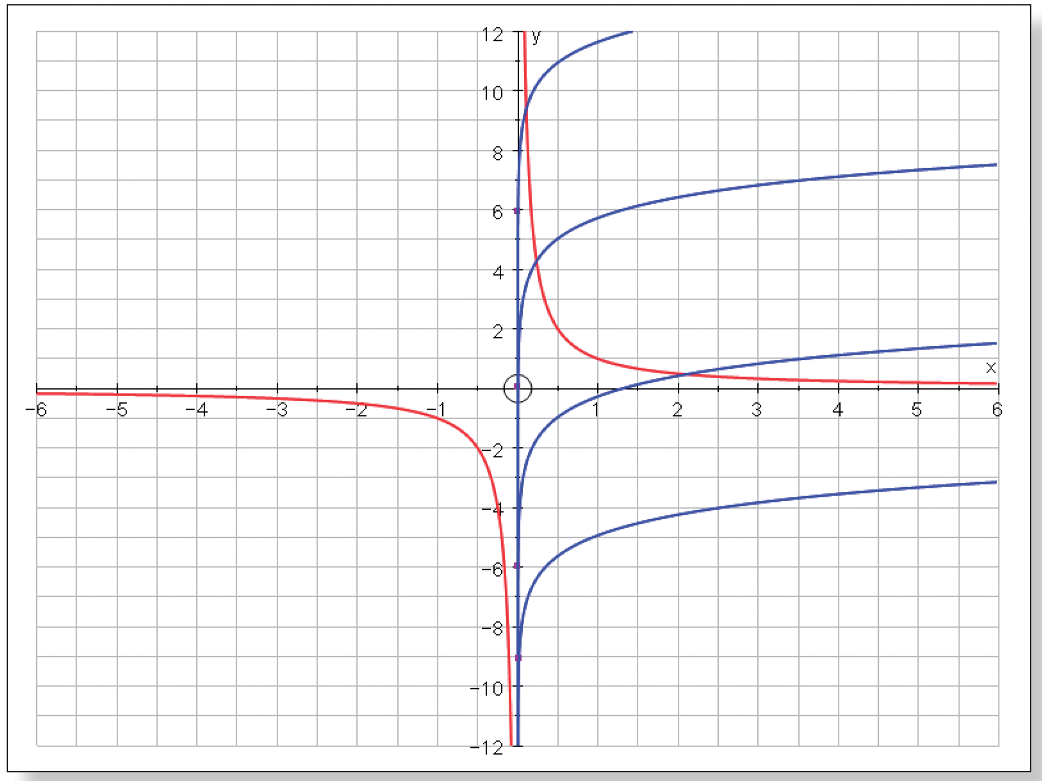
Teacher: Excellent. Well, let's use Autograph to take a look at the family of functions…

Click on the **Integral Function** tool, and click **OK**.

Again, select three or four initial values of x and y along the x-axis.

As the functions begin to appear, discuss their shape and relate this back to the shape of the gradient function.

Your screen should look something like this:



Teacher: Now, this family of functions are incredibly important in mathematics, and they have a special name. They are called the natural logarithms, and they are all in the form of $y = \ln(x) + c$. Let's clear away the family of curves, and look at $y = \ln(x)$ in particular.

ACTIVITY 3: $y = \ln(x)$

Turn off **Slow Plot** mode.

Click on **Manage Equation List**.

Select **Integral Function** from the list and click the red cross in the corner.

The equation should be removed form the list and when you click okay, you should just be left with $y = x^{-1}$.

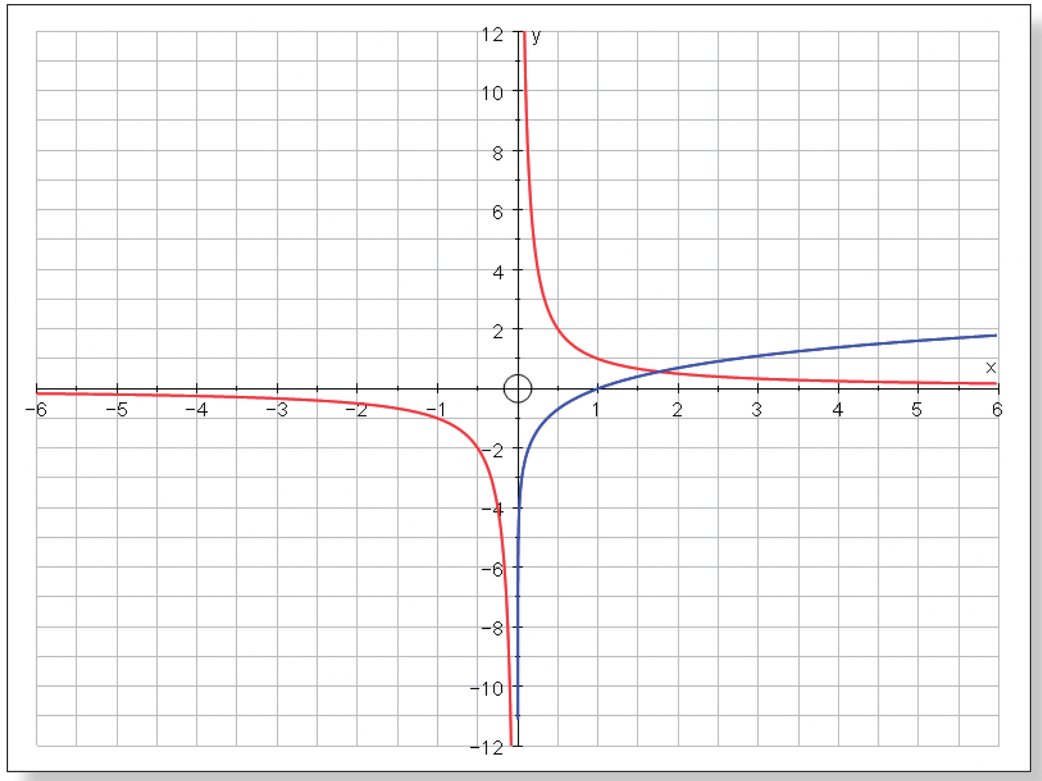
Turn **Slow Plot** mode back on.

Enter the equation: $y = \ln(x)$

The curve should begin to plot.

Press **Pause Plotting** (or the **Spacebar**) both to stop the process and to resume it to focus on the key properties of the graph.

Your screen should now look like this:



Teacher: Now, this function has some very important features. Firstly, what is the value of $\ln(x)$ when x is 1?

Ideal Response: 0

Teacher: And for what values is $y = \ln(x)$ negative?

Ideal Response: When x is between 0 and 1.

Teacher: And how about when $x = 0$?

Ideal Response: The curve is undefined.

Teacher: And last but not least, what value of x makes ln(x) equal 1?

Ideal Response: Somewhere between 2.5 and 3.

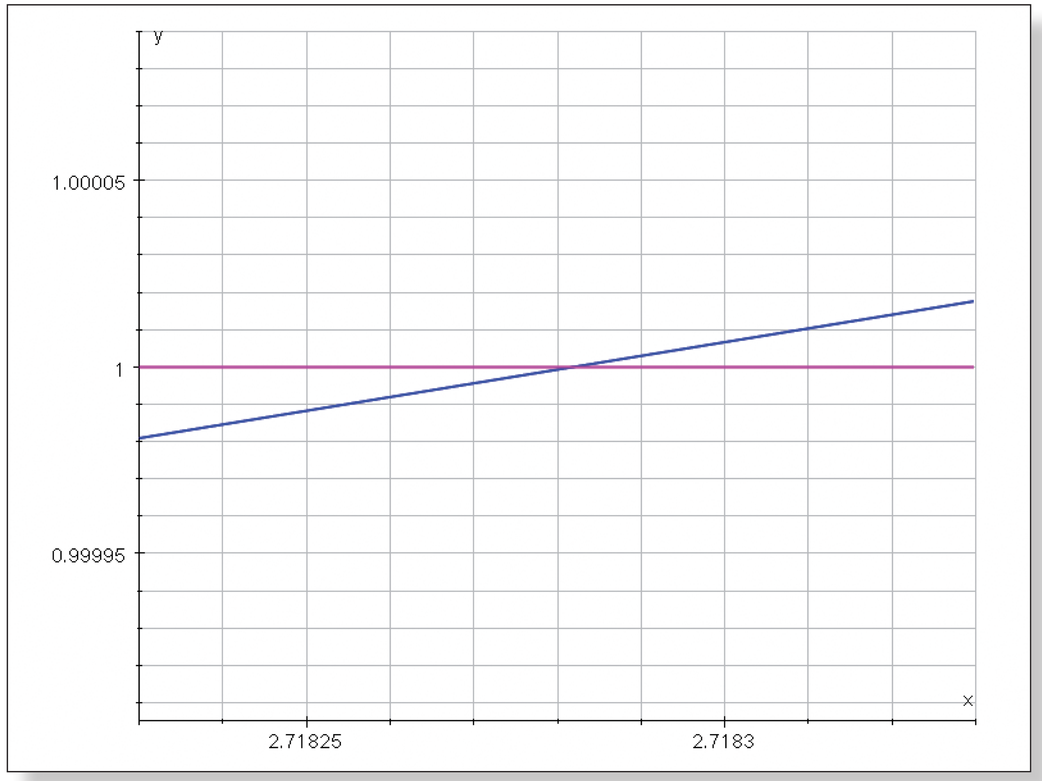
Teacher: Well, let's just take a closer look at that point as it is very important.

Enter the equation: $y = 1$

Use the **Zoom In Box** tool to focus on the point of intersection of $y = 1$ and $y = \ln(x)$.

Do this several times, getting closer and closer, pointing out to the students how the scale is automatically adjusting.

Your screen should look something like this:



Teacher: Does anybody recognise the value we are honing in on?

Ideal Response: e.

Teacher: Let's have a look…

Ensure you are in **Select Mode**.

Left-click on the curve $y = \ln(x)$ (it should turn black).

Enter the point with x co-ordinate: e

A point should appear at the intersection.

Use the **Zoom In** tool to zoom in even further to convince the students that this is the point of intersection.

Teacher: So, what does ln(e) equal?

Ideal Response: 1

Teacher: So, to summarise: $\int \frac{1}{x}\,dx = ?$

Ideal Response: $\ln(x) + c$

Teacher: And: $\frac{d}{dx}(\ln(x)) = ?$

Ideal Response: $\frac{1}{x}$

Teacher: Brilliant. But that just leaves on question: what about the left hand part of $y = x^{-1}$?

ACTIVITY 4: $y = \ln(-x)$ AND $y = \ln(|x|)$

Turn **Slow Plot** off.

Hit **Undo** as much as needed to return to the original view of the axes, and just the functions $y = x^{-1}$ and $y = \ln(x)$.

Teacher: If the left hand portion of this graph is the gradient function, can anybody guess what the function itself might be?

Prompt: It has something to do with natural logs.

Possible Response: $y = -\ln(x)$

Teacher: Think back to our work on Transformations. What effect does the transformation $-f(x)$ have on the graph?

Ideal Response: $y = \ln(-x)$

Teacher: Well, let's check…

Turn **Slow Plot** mode on.

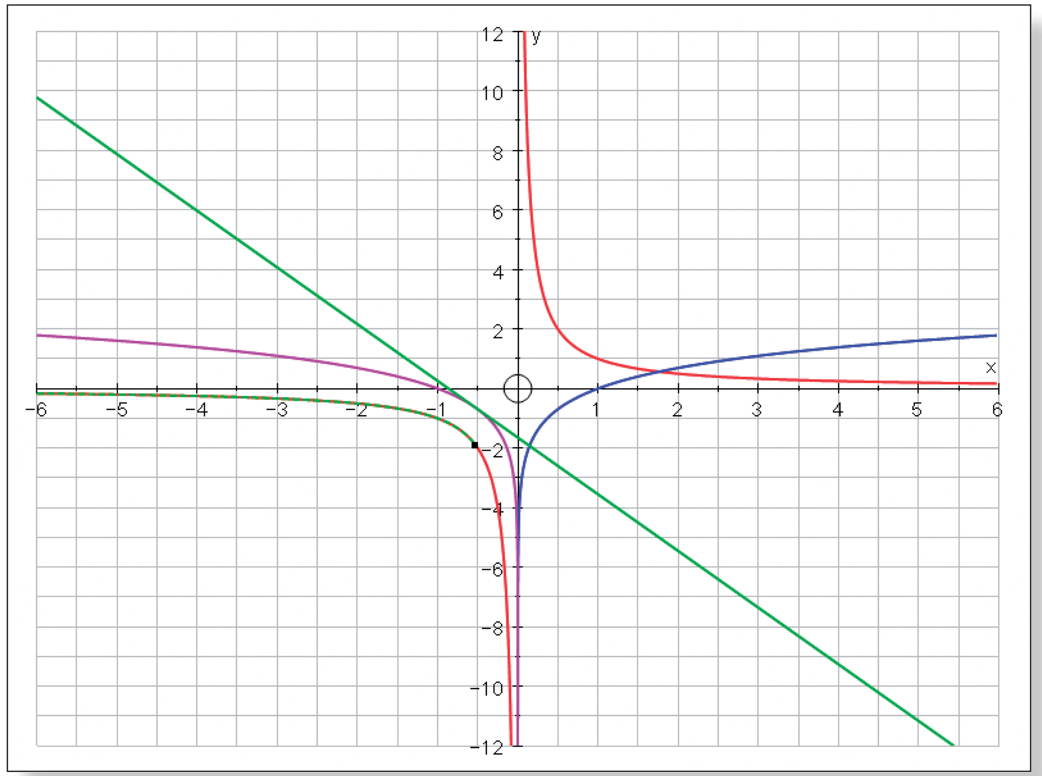
Enter the equation: $y = \ln(-x)$

Teacher: Now, if we are correct, then the gradient function of this graph should be the left hand side of the $y = x^{-1}$ graph…

Click on **Gradient Function** and press **OK**.

The gradient function will now be plotted exactly over the left hand portion of the graph of $y = x^{-1}$.

Your page should look something like this:



Teacher: And last but not least, can anybody think of a *single function*, which when we plot the gradient function will give us the entire graph of $y = x^{-1}$?

Prompt: Think about how to counter the positive/negative problem.

Ideal Response: $y = \ln(|x|)$

Teacher: Well, let's check…

Enter the equation: $y = \ln(|x|)$

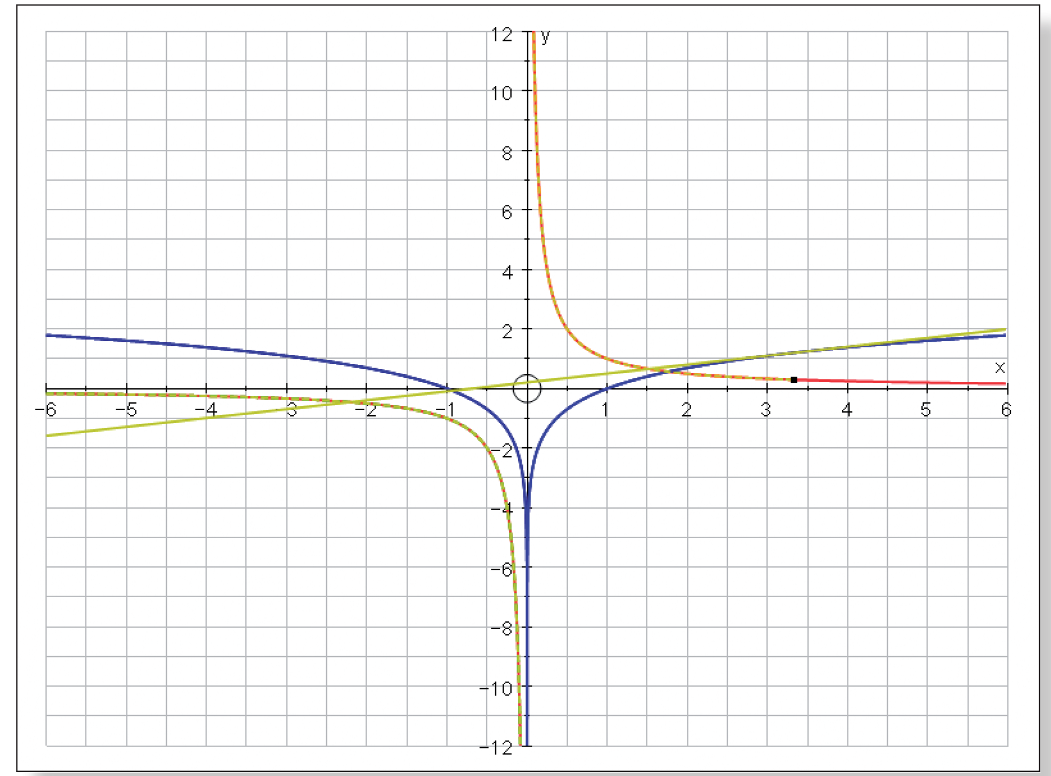
Note: To enter the modulus signs, just click the button with the vertical lines.

When then function has finished plotting…

Click on **Gradient Function** and press **OK**.

The gradient function will now be plotted exactly over both portions of the graph of $y = x^{-1}$.

Your page should look something like this:



Ideas for Further Work

- Looking at integrating functions like $\int_1^3 \frac{1}{x}\,dx$, $\int_2^6 \frac{5}{x}\,dx$, $\int_{-5}^{-2} \frac{1}{x}\,dx$, $\int_0^5 \frac{1}{x-4}\,dx$, $\int_2^3 \frac{2}{4-3x}\,dx$.
- Differentiating functions involving natural logs.
- Looking at log functions with different bases.
- Inverse functions.
- Laws of logs.
- Solving equations by taking logs.

Teacher Demonstration T15

Laws of Logs : A Graphical Approach

Learning Objectives

- To introduce the log function
- To use an interactive method to derive the following Laws of Logs:

1. $\log_c(ab) = \log_c(a) + \log_c(b)$
2. $\log_c(\frac{a}{b}) = \log_c(a) - \log_c(b)$
3. $\log_c(a^n) = n\log_c(a)$

Required Pre-Knowledge

- It may help if students have been introduced to the concept of a logarithm, and understand the following definition:

$a^b = c \Leftrightarrow \log_a(c) = b$

However, this demonstration could serve as an introduction to the concept of logs, and work on definitions and bases could follow.

Pre-Activity Set-Up

Ensure you have access to the Autograph File named "Laws of Logs".

Ensure the following table is displayed prominently at the front of the classroom:

x	log(x)	x	log(x)
1	0	11	
2	0.30102…	12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	
10		20	

Step-by-Step Instructions

Activity 1: Introducing the Concept

Teacher: Today we are going to look at a new function on your calculator – the log function. To start off with, can you make a quick copy of the following table, use you calculator to confirm you agree with the first two values, and then fill in the rest!

Prompt: Again, the usual problem of students having several different models of calculators could become apparent here, so just be on hand to steer students in the right direction.

Ideal Response:

x	log(x)	x	log(x)
1	0	11	1.04139…
2	0.30102…	12	1.07918…
3	0.47712…	13	1.11394…
4	0.60205…	14	1.14612…
5	0.69897…	15	1.17609…
6	0.77815…	16	1.20411…
7	0.84509…	17	1.23044…
8	0.90308…	18	1.25527…
9	0.95424…	19	1.27875…
10	1	20	1.30102…

Teacher: Excellent. Now, what I want you to imagine is a number line, stretching between 0 and 1.5, with all the values from log(1) to log(20) marked on it. If we were to mark each of the values in our table on the number line, what would the marks look like?

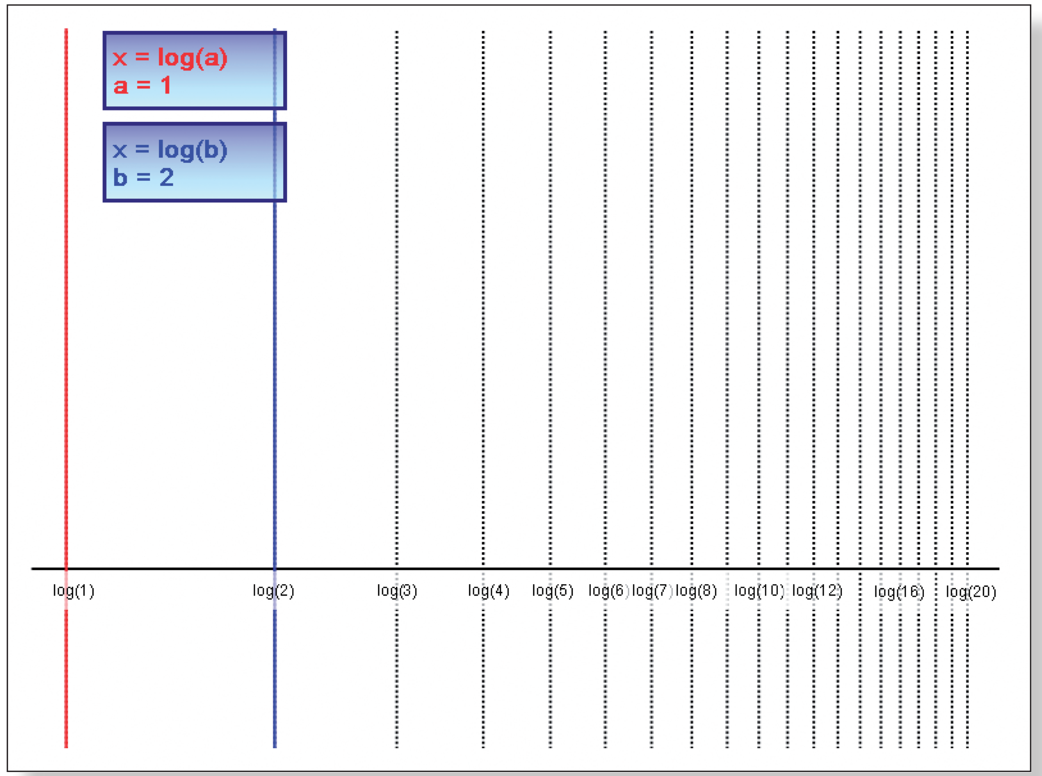
Prompt: Look carefully at the numbers in the table. Look at the gaps between them. Would the marks be evenly spaced? Would they all be clumped together at one end?

Ideal Response: The interval between each value is decreasing, so the gaps between each mark would get smaller and smaller, meaning the marks would be clustered together at the right hand side of the number line.

Teacher: Good. Now, it just so happens that I have such a number line here…

Open up the Autograph file named "Law of Logs".

Your screen should look like this:



Give the students a few moments to relate the values in the table to the number line before moving on.

Activity 2: log(a) + log(b)

Teacher: Now, you will see that on this number line I have also marked on the values of log(a) and log(b), with the values of a and b set to 1 and 2 respectively, as shown in the textbox. Now, the question is, where do you think the line of log(a) + log(b) would go?

Prompt: It might be an idea to not give too much away at this stage, as the element of surprise is quite powerful!

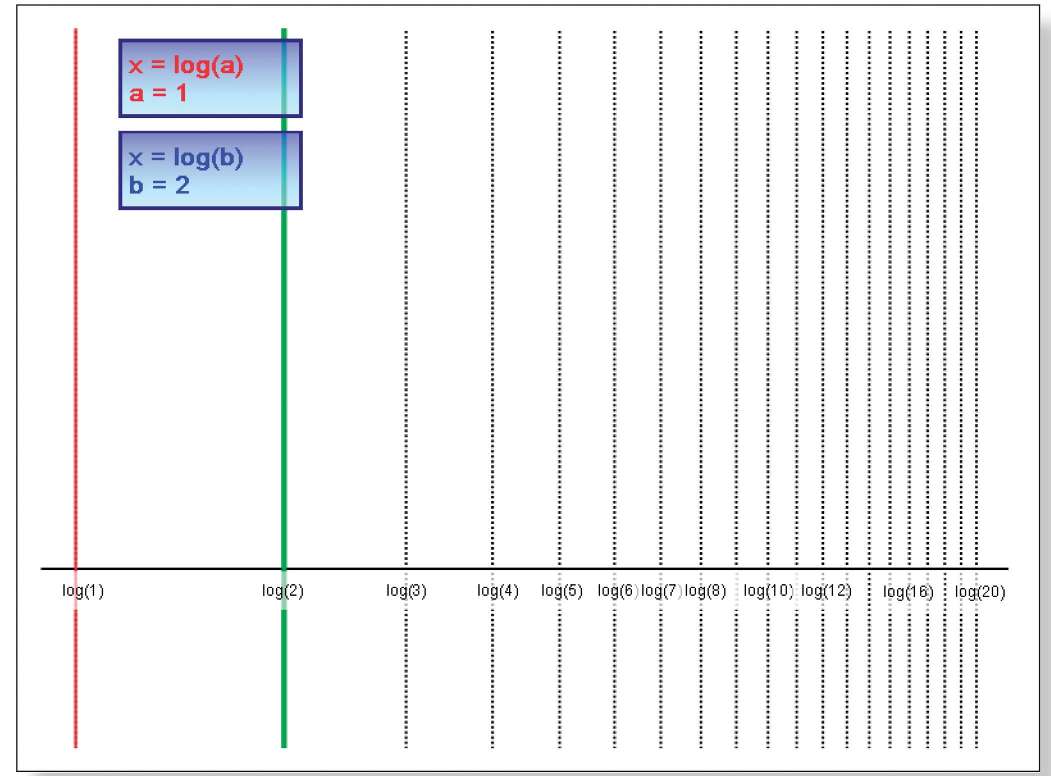
Expected Response: log(3)

Teacher: Well, that sounds logical enough. Let's just check…

Ensure Slow Plot mode is turned off.

Enter the equation: x = log(a) + log(b)
Note: The brackets are not necessary, and can be excluded if you like.
Still on the Enter Equation screen, click on Draw Options, change the colour to green, and the line thickness to 3 pts.
Note: A dotted line can also look quite nice, but that is entirely up to you!
Click OK twice.

Your screen should look something like this:



Teacher: Well, what on earth has happened there? How can log(1) + log(2) = log(2)?

Prompt: For any two numbers, p and q, if p + q = q, what must be true about p? Have a look at the values in your table to see if they help?

Ideal Response: log(1) = 0.

Teacher: That makes sense. Zeros are always causing us trouble in mathematics. Let's try another one. Say I change the value of a to 3. What do you think log(3) + log(2) would equal?

Expected Response: log(5)

Teacher: Sounds good, but we had better just check…

Click on the Constant Controller.
The drop-down menu allows you to select each constant.
The up-down buttons adjust the value of the constant.
The left-right buttons adjust the value of the step.

Move the Constant Controller to a convenient position on the page.

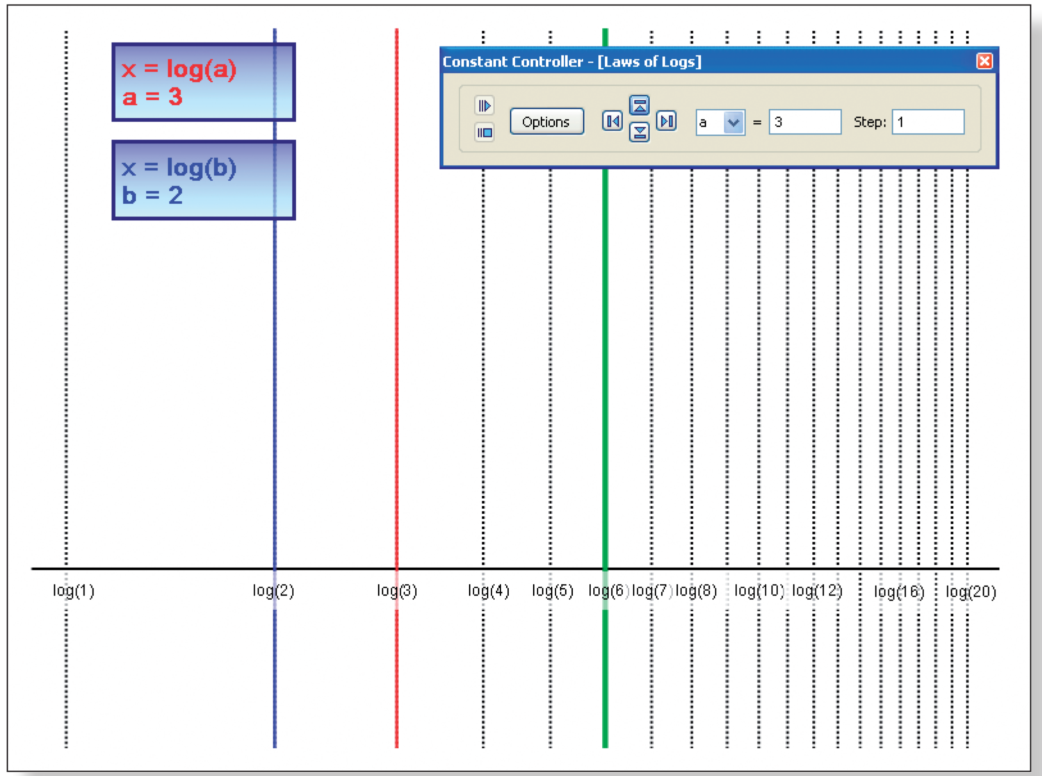
Select constant a.

Change the value of the step to 1.

Use the right button to increase the value to of a to 3.

The line representing log(a) + log(b) should move at the same time.

Your screen should look something like this:



Teacher: Okay, again something funny is going on. It looks like log(3) + log(2) = log(6)! How can that be? Is that supported by our table of values?

Ideal Response: According to the table, log(3) = 0.47712, log(2) = 0.30102, and log(6) = 0.77815, so yes it seems to be the case that log(3) + log(2) = log(6).

Teacher: Okay, let's see if we can get to the bottom of this. Use your table to predict what answer we will get to: log(3) + log(5).

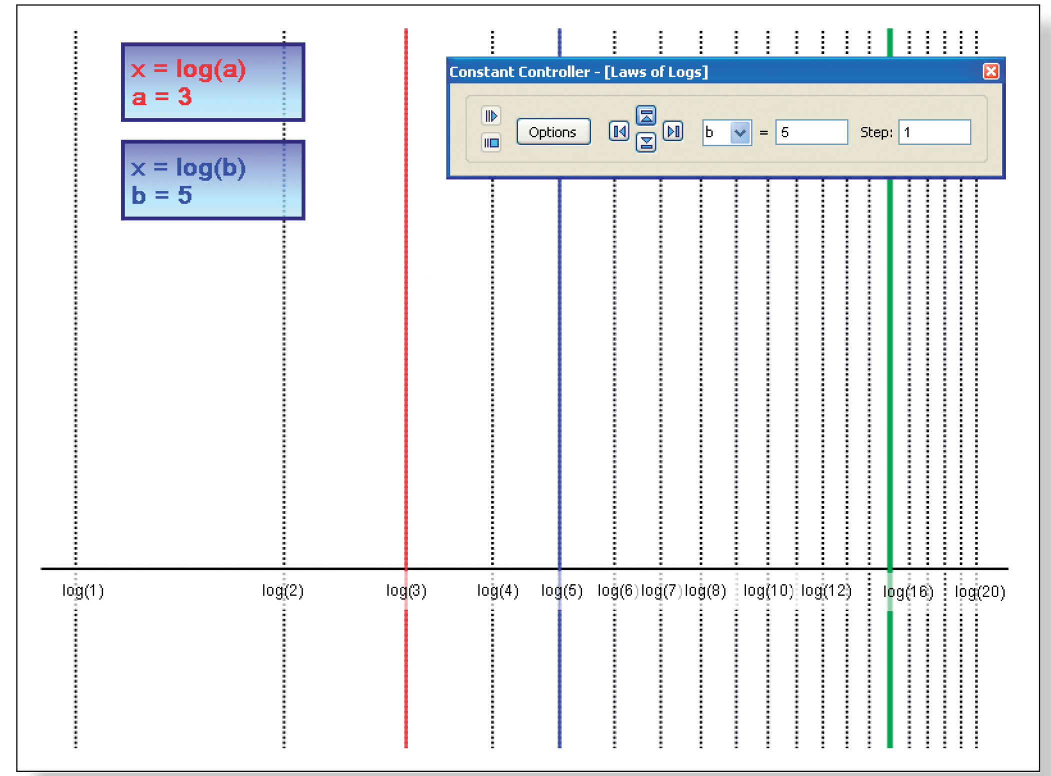
Prompt: Add the two values in the tables together and see if it matches any other value.

Ideal Response: According to the table, log(3) + log(5) = log(15).

Teacher: Let's check that:

Select constant b.
Change the value of the step to 1.
Use the **right button** to increase the value to of b to 5.

Your screen should look like this:



Teacher: Okay, so what is going on here? What is the rule for adding two logarithms together?

Prompt: Can you express it in words? Can you generalise? log(a) + log(b) = ?

Ideal Response: When adding logarithms, you must multiply the two numbers together. So, log(a) + log(b) = log(ab).

Teacher: Sounds good. Let's check that with a few examples.

Encourage the students to make predictions and then use the values in the table, and finally the **Constant Controller**, to check their answers. Ask questions like:

"What is log(4) + log(5)?"

"The sum of which two logarithms would give us log(7.5)… are there any other logarithms that would give us this answer?"

When you are ready…

ACTIVITY 3: log(a) - log(b)

Click on **Manage List**.

Click on x = log(a) + log(b) and delete it using the cross in the corner.

Click **OK**.

Your should now once again be left with just the lines representing log(a) and log(b).

Teacher: Now, the question that follows on naturally from this, is what happens when we subtract logarithms? For example, what do you think we would get if we did log(8) − log(2)?

Prompt: What would make sense following what we have just done? Use your table of values to help you.

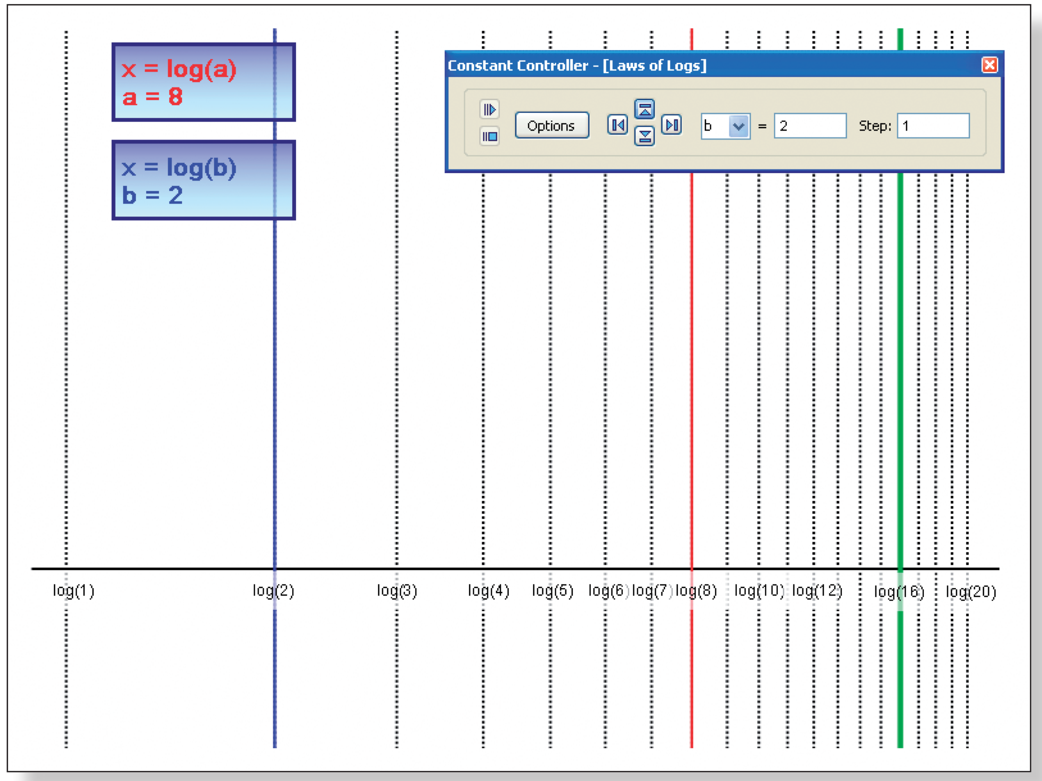
Ideal Response: Well, when adding logarithms, we had to multiply, so it would make sense that if we were subtracting logarithms, we would have to divide. This is supported by the numbers in our table. So, I predict that log(8) − log(2) = log(4).

Teacher: Sounds good, but let's check…

Set the value of a to 8, and the value of b to 2.

Enter the equation: x = log(a) − log(b)
Still on the **Enter Equation** screen, click on **Draw Options**, change the colour to purple, and the line thickness to 3 pts.
Click **OK** twice.

Your screen should look something like this:



Teacher: So what is the rule for subtracting two logarithms? Can you generalise this?

Ideal Response: When subtracting logarithms, you must divide the two numbers together. So, log(a) − log(b) = log(a/b).

Teacher: Sounds good. Let's check that with a few examples.

Again, encourage the students to make predictions and then use the values in the table, and finally the constant controller, to check their answers. Ask questions like:

"What is log(20) − log(5)"

"When two logarithms are subtracted, the answer is log(1.5). Give me three subtractions that could have given us this answer"

When you are ready…

Activity 4: nlog(a)

Click on **Manage List**.

Click on x = log(a) − log(b) and **delete** it using the cross in the corner.
Click on x = log(b) and delete it using the cross in the corner.
Click **OK**.

You should now be left with just the line representing log(a).

Set the value of a to 2.

Teacher: Okay, now there is just one more thing we need to look at, and it's all to do with the functions in the from nlog(a). What value of n would mean that our line stays in the same place?

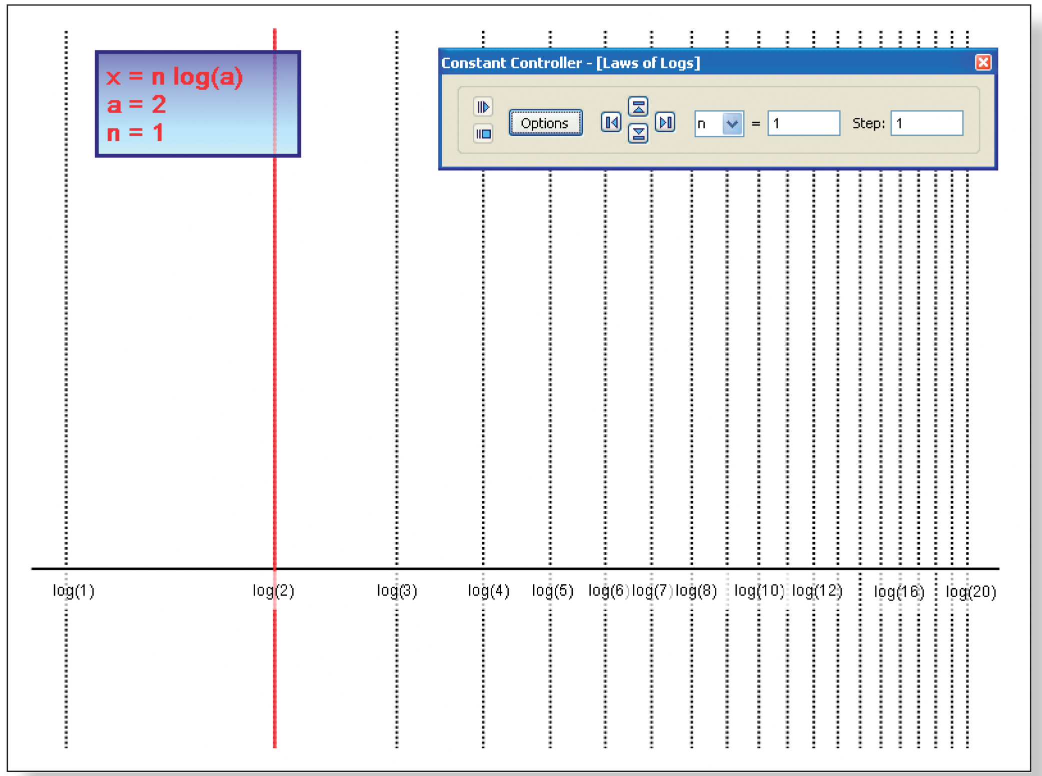
Ideal Response: n = 1

Click on **Manage List**.

Double-click on x = log(a) so the **Edit Equation** box comes up.

Enter the equation x = nlog(a).
Click on **Edit Constants** and just make sure the value of n is set to 1.
Click **OK** twice.

Your screen should look something like this:



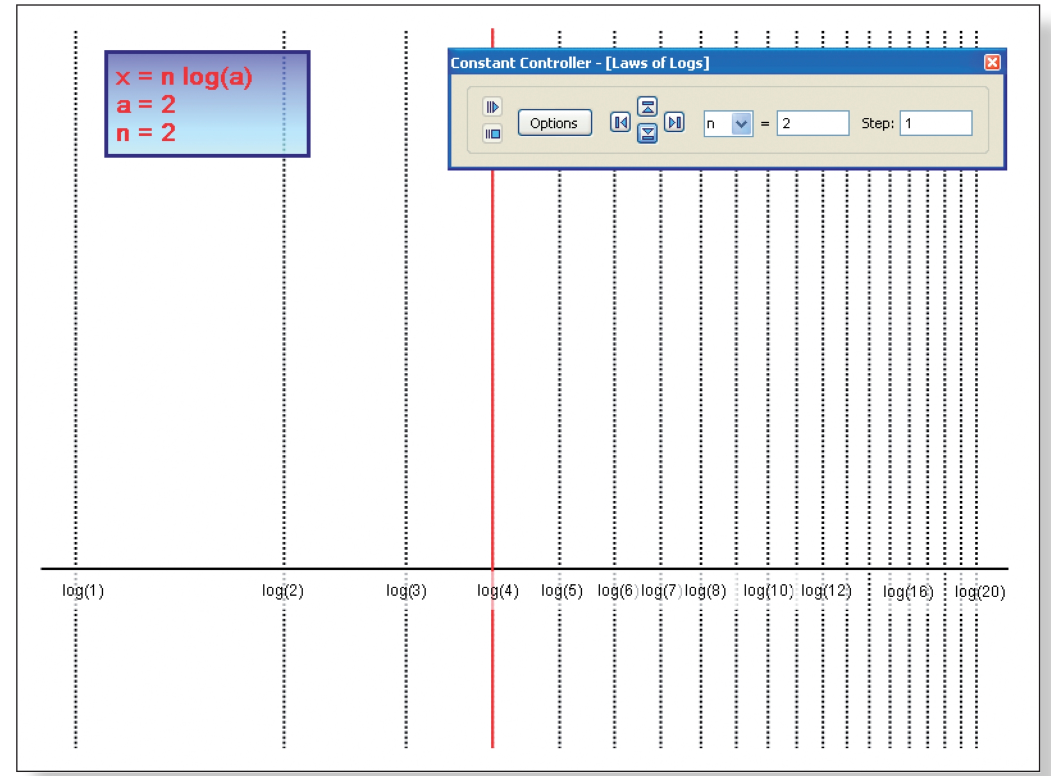
Teacher: Now, the question is: what is going to happen when we increase the value of n? Where do you predict the line will go when we increase n to 2, giving us 2 log(2)?

Expected Response: log(4)

Teacher: Well, that sounds logical enough. Let's just check…

Select constant n.
Change the value of the step to 1.
Use the **right button** to increase the value to of n to 2.

Your screen should look something like this:



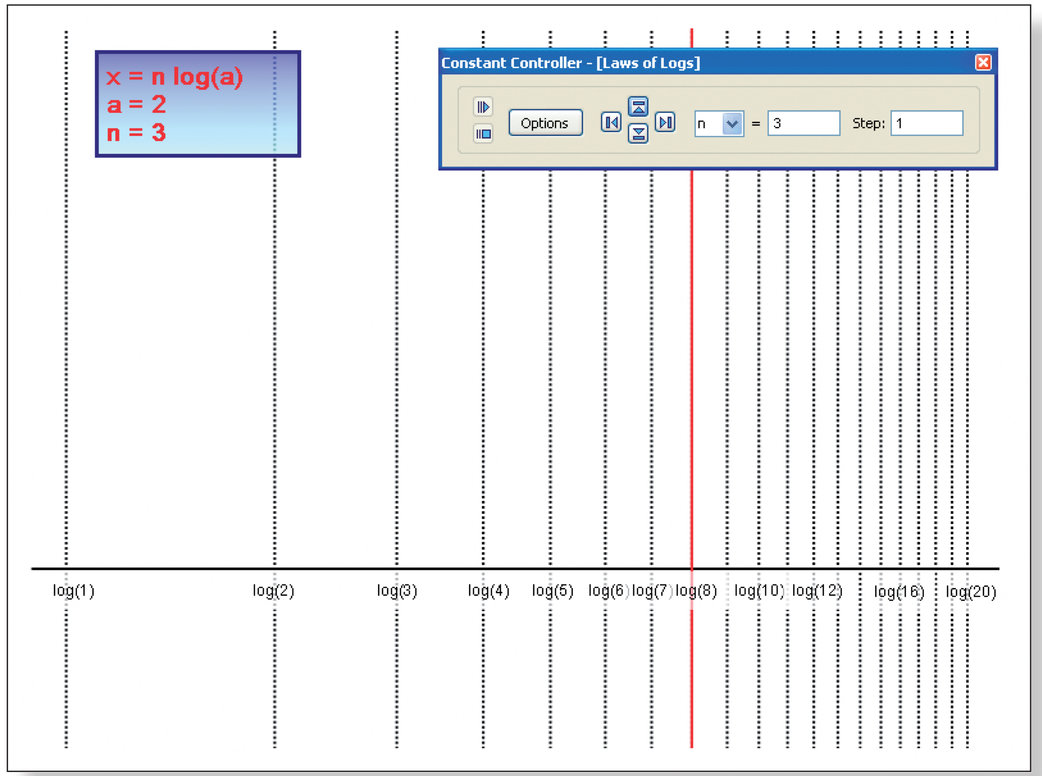
Teacher: Finally, something that seems to go as expected. So, if we increase n to 3, giving us 3log(2), what do you predict we would get?

Expected Response: log(6)

Teacher: Let's check…

Use the **right button** to increase the value to of n to 3.

Your screen should look something like this:



Teacher: Okay, again something funny is going on again! It looks like 3log(2) = log(8)! How can that be? Is that supported by our table of values?

Ideal Response: According to the table, log(2) = 0.30102, so 3 × log(2) = 0.90308, which is equal to log(8).

Teacher: Okay, let's see if we can get to the bottom of this. Use your table to predict what answer we will get to: 4log(2)

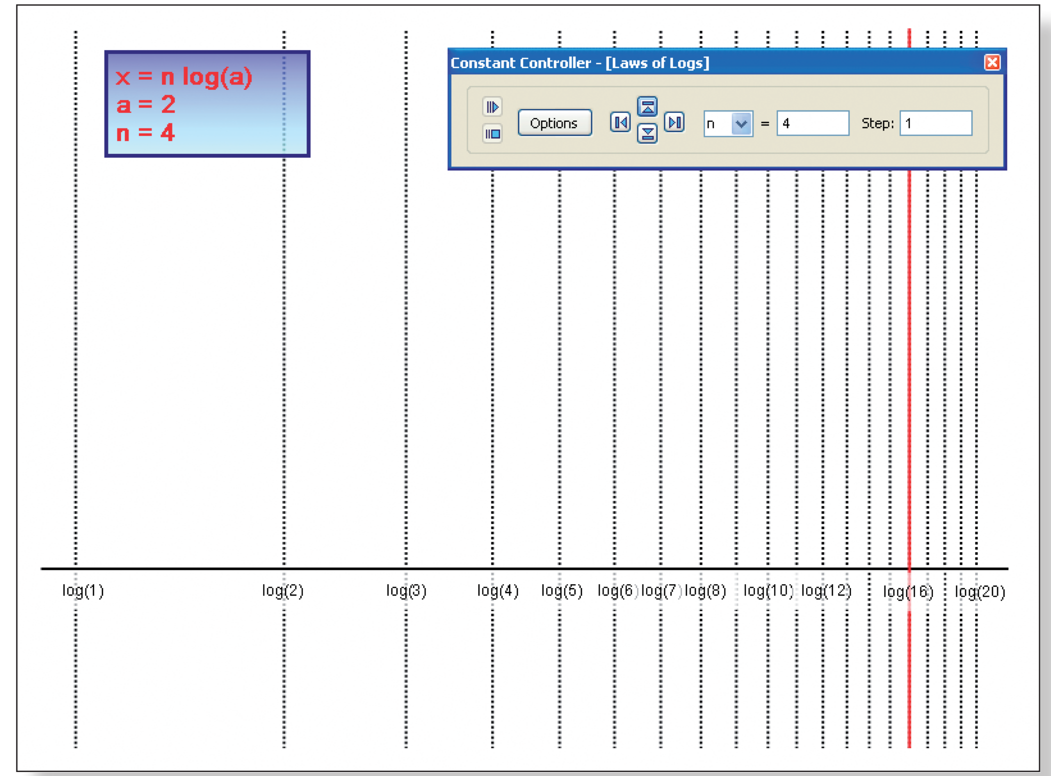
Prompt: Work out what 4 × log(2) is and see if it matches any of the values in your table.

Ideal Response: According to the table, 4log(2) = log(16).

Teacher: Let's check that:

Use the right button to increase the value to of b to 4.

Your screen should look something like this:



Teacher: Okay, so what is going on here? What is the rule for multiplying a logarithm by a constant?

Prompt: Can you express it in words? Can you generalise? nlog(a) = ?

Ideal Response: When multiplying a logarithm by a constant, you get the same answer as if you took the original logarithm and raised the number to the power of that constant. So, nlog(a) = log(a^n).

Teacher: Sounds good. Let's check that with a few examples.

Encourage the students to make predictions and use the Constant Controller to check. Ask questions like:

"What is 2log(3)?"

"How about 0.5log(16)?"

"What about −2log (4)?"

"−0.5log(9)?"

Ideas for Further Work

- If it has not been covered already, students could be taught about the bases of logarithms, and introduced to the definition: $a^b = c \Leftrightarrow \log_a(c) = b$.
- Using the laws of logs to combine and simplify expressions involving logs.
- The relationship between exponentials and logarithms.

- Graphs of log functions.
- The natural log function – see **Teacher Demonstration 14: The Natural Log Function**.