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STUDENT INVESTIGATIONS

In this investigation students are introduced to each of the 6 types of transformations that can be applied to a function. The use of Autograph's constant controller allows students to test their predictions and deepen their level of understanding.
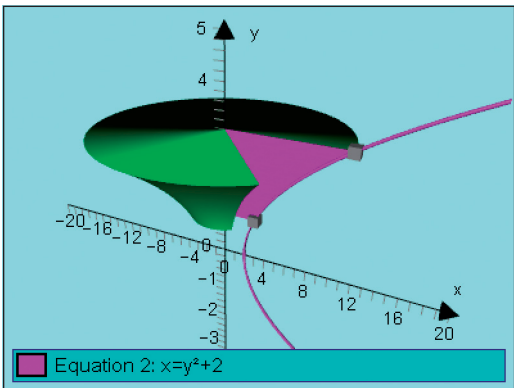
Equation 1: y=2x³+3x²−2x+1
Equation 2: y=−2x³−3x²+2x−1

During this dynamic and interactive investigation, students are able to use Autograph's unique 3D interface to visualise the solids formed by rotating areas around axes and lines. This should strengthen the link between the algebra and the geometry of what can otherwise be a conceptually difficult topic.

Equation 2: x=y²+2
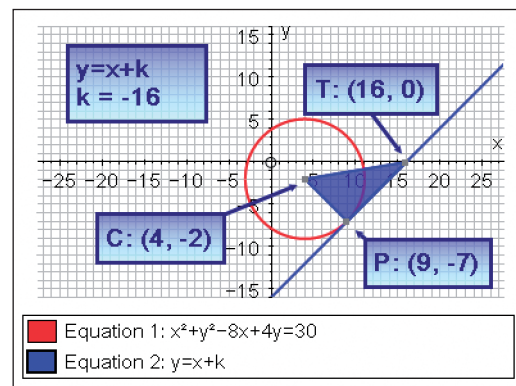
This interactive investigation tests students' knowledge and understanding of all the key skills involved in co-ordinate geometry. There is plenty of scope for prediction and investigation to help consolidate knowledge and deepen understanding.

An investigation into the unusual and interesting relationships between the roots of cubic equations, which tests students' knowledge of many important areas of co-ordinate geometry and mathematics in general.
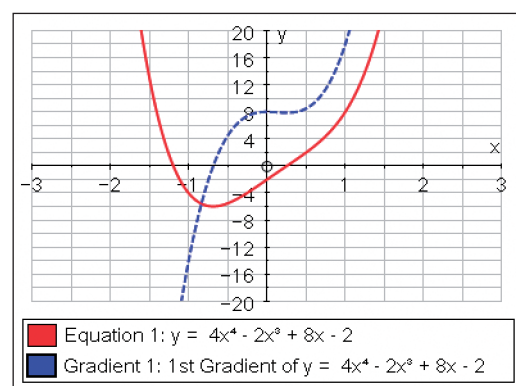
During this investigation, students discover the rule for finding the gradient function of quadratics, cubics, and functions involving higher powers of x. They are encouraged to predict and then use Autograph to test out their predictions in a highly visual and dynamic manner, thus deepening their levels of understanding.
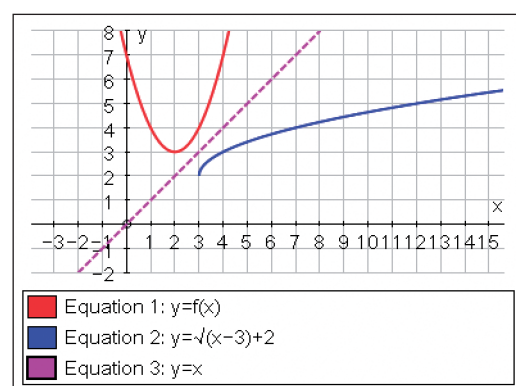
This interactive investigation is designed to introduce students to the main concepts involved in the study of functions, including different types of functions, domain and range, and inverse functions.
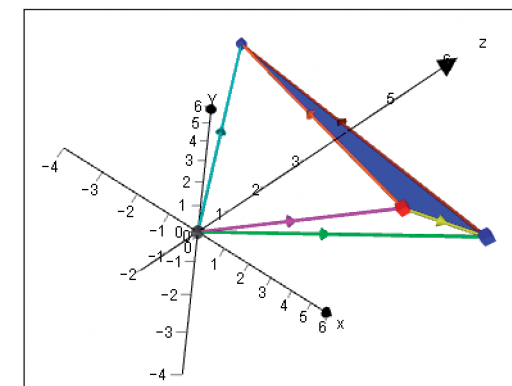
In this interactive investigation, students are able to consolidate, deepen and enrich their knowledge and understanding of vectors, both in 2D and 3D, using Autograph's dynamic tools and unique 3D interface.
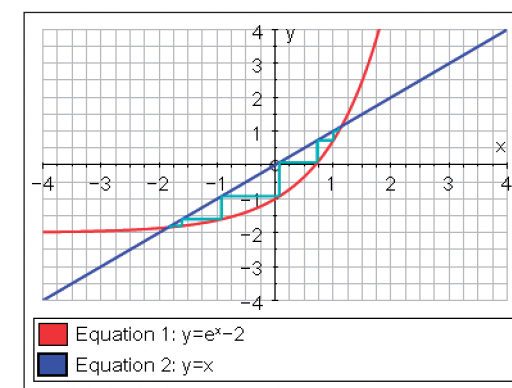
Students can make full use of one of Autograph's iteration functions in this dynamic and interactive investigation which conveys the link between the algebraic and graphical solution to iterative equations.
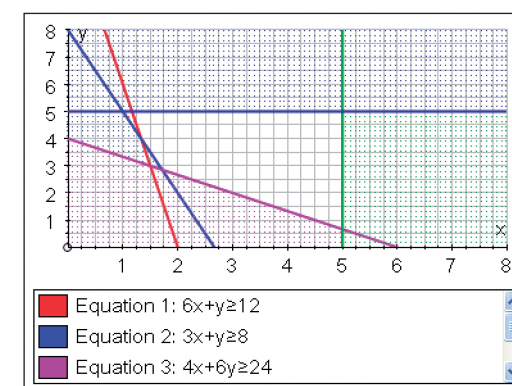
Three investigations into Linear Programming which make use of Autograph's dynamic and interactive controls to enable students to visualise the graphical solutions to such problems, and hence deepen understanding.
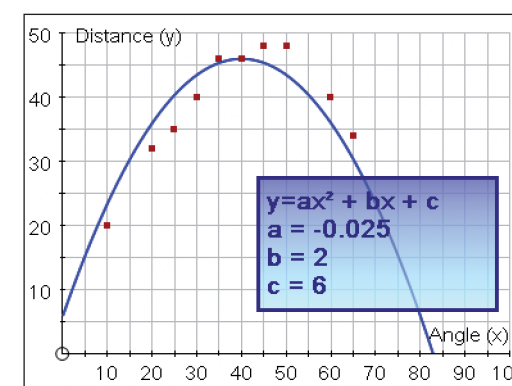
A fun investigation into a real life application of mathematics. Students must devise a model which enables Gonzo to fly successfully from a cannon into a bucket of water. Many important mathematical concepts are incorporated along the way!

I remember when I was studying for my Mathematics A Levels. It was unusual enough for boards to be white, let alone interactive. Cutting edge technology was a watch with an inbuilt calculator, and a dynamic geometry lesson would be one in which the teacher suddenly switched from white chalk to red in the process of drawing a wonky circle. The late 1990s were a dark time indeed.

However, the technological revolution that has swept the world over the course of the last few decades has also landed in our classrooms. Overhead projectors have gone digital, scientific calculators are now graphical, static whiteboards have given way to their interactive cousins, computers are faster, mountains of dedicated mathematical software are widely available, and the internet has opened up possibilities far beyond what Pythagorus, Euler and the greatest minds in the history of the world could ever have imagined.

Alas, in many instances, this technology is not being used to its full potential. It either sits rotting away in storeroom cupboards or on the digital dumping ground of a computer's desktop screen, or it is used in an extremely limited and ineffective way, which can often cause frustration on the part of both the teacher and the students, thus causing more harm than good.

This sorry state of affairs could be, as many have argued, because a large proportion of teachers are reluctant to change; unwilling to try new things and break the habits and traditions that have served them perfectly well over the course of their long and distinguished careers. But I disagree. The teachers that I have talked to are ready to embrace this influx of technology, they are ready to welcome it with open arms into their classrooms, but they are just a little scared to do so.

Now, this fear stems from many different sources, and much of it is well founded. It may be fear that their pupils will know far more about each piece of technology than they ever will, thus making the teacher feel vulnerable, exposed and uncomfortable, no longer the all-knowing figure of confidence and authority that they feel they should be. Or it could be because teachers fear that by the time they have got to grips with a certain piece of technology, the world will have once more moved on, making their newly discovered knowledge both redundant and useless.

Or – and this I feel is the most prevalent reason – it is because teachers fear that they simply do not have enough time. With curriculums changing every couple of years, new strategies and initiatives bounding in from every direction, lessons to plan, books to mark, students to teach, and (in the precious few seconds of each day that remain) lives to be led, where is the time to sign up to a course, or read a book, or sit at a computer hoping the ability to use whatever technology it may be might just seep through the pixels on the screen to the pores on the skin?

And so what often happens is this: through no fault of their own, eager teach-

ers try the new technology without having adequately planned for its use, and disaster inevitably follows. During computer-based lessons, students sneak onto other internet sites, to play games, check email, or poke friends. Whilst the teacher is happily clicking through their immensely detailed, perfectly animated PowerPoint presentation, students appear bored, lack focus and seem unmotivated. Using Autograph to draw the line y = 3x + 4 does not magically make the students understand straight line graphs any better than had the teacher simply drawn it by hand, and now the teacher cannot remember how to return to the interactive whiteboard's inbuilt flip-chart. The Excel spreadsheet that worked like a dream at home is now threatening to blow up the school's computer system because of the existence of a mysterious thing called a macro. The ideal lesson is suddenly scuppered by the fact that the school network is down, or YouTube videos are blocked, or some delightful little devil has stolen the batteries from the mouse. Having spent four times as long to plan than normal, the lesson is declared a failure by all, and the teacher vows to never use that stupid piece of technology again.

It's a dark place where everyone has been one time or another, and where no-one wants to return.

For a while now I have been a firm believer in what I like to call the *effective use of technology*. Simply turning on an interactive white board does not automatically make the lesson interactive, neither does clicking through a PowerPoint presentation, nor simply using a software package such as Autograph to draw a series of straight lines. The technology should be the facilitator in the lesson, not the driving force. Effective use of technology is when that piece of technology, whatever it may be, genuinely enhances the learning experience of the students and, just as importantly, improves the teacher's experience and enjoyment of the lesson.

What I hope to show both in this book, and in its sister publication *Autograph Activities: Teacher Demonstrations*, is how to use one such piece of technology, Autograph, effectively. In my experience Autograph is one of the most underused pieces of software in schools. Many have used Autograph to do things like draw lines, curves and circles, but few seem to have fully exploited its true potential, and that is a great pity.

Once more I feel the major reason behind this is the time factor. However, what I hope to show in this book is that Autograph is incredibly simple and logical to use. Once you have spent about one hour having a go at the included interactive *Autograph Tutorial*, you will be in a position to help guide your students through the ten *Student Investigations* contained in this book. Better still, it will not take long before you start using the software to develop your own activities based on your own ideas. Once again, technology is effective only when it is the facilitator of teaching and learning.

The *Student Investigations* provide the perfect vehicle for independent, dynamic learning. Autograph is an excellent tool for investigation, and mathematics is at its strongest and most appealing when students can embark upon such journeys of self-discovery. Students use Autograph and the accompanying

worksheet to discover and examine concepts that would not be viable in the normal classroom setting. They are encouraged to derive things for themselves, to predict, to observe, to trial out hypotheses to enrich their learning experience and deepen their levels of understanding. Topics covered include: examining the transformation of functions; understanding vectors in two and three dimensions; and looking at numerical methods to solve equations.

What I hope this book will achieve is to offer ways of effectively using a fantastic piece of software without you having to put in hours and hours of your precious time. I hope it will encourage both you and your students to use the software to plan activities and investigate concepts not covered in this book. I hope it will further strengthen your students' enjoyment of learning mathematics, exciting them, enriching their learning experience, further opening their eyes to what a wonderful and fascinating subject mathematics is. Most importantly of all, I hope the book will further help you enjoy teaching mathematics, inspiring your students, embracing technology, and using it in the way it was intended – as a facilitator of excellent teaching and learning, and an incredibly effective one at that.

Also available in this series:

Autograph Activities: Teacher Demonstrations

The Teacher Demonstrations will allow you to dynamically introduce, review, extend or illustrate important topics or concepts in ways not previously possible. They are intended for use on an Interactive Whiteboard or by means of a digital projector. There is countless opportunity for student interaction, asking and answering probing questions, testing hypothesise and predictions. The topics covered include: introducing the natural logarithm function; deriving trigonometric identities; understanding the binomial approximation; introducing the concept of differentiation; and using Autograph's unique 3D interface to examine volumes of revolution.

This aim of this book is to help fully utilise Autograph's potential as a tool for independent investigation. Below are details about the Student Investigations contained in this book, including advice regarding timings, set-up and delivery.

Note: Before attempting the Student Investigations, it is strongly advised that both students and teachers work through the *Autograph Tutorial* on page 12 of this textbook. This interactive tutorial will help familiarise the user with the controls of Autograph, which will help boost confidence and allow far greater progress throughout the activities.

THE STUDENT INVESTIGATIONS

These investigations are designed to allow students to interactively utilise Autograph's power to explore and understand concepts at a depth which the normal classroom setting would not allow. Ideally the activities should take place in a computer suite with one student at each workstation, although groups of two or three students around each computer would also work well.

Each investigation comprises:

1. Teacher Notes – these contain:

   - Learning Objectives, broken down into Core Content and Extension Material

   - Required Prior Knowledge

   - A brief Description of the Activity

   - Details of any Pre-Activity Set-Ups that must be carried out before the lesson starts

   - A Suggested Introduction

   - Suggested Prompts if students get stuck during the activity, as well as advice about potential stumbling blocks

   - A Suggested Plenary

   - Ideas for Further Work to build upon the knowledge gained

2. Student Worksheets – these can be photocopied and each student given their own individual copy. The worksheets contain step-by-step instructions, as well as questions to answer and tasks to complete. All answers can be written on the accompanying worksheet, and at the end of each investigation there is a section where students are encouraged to summarise what they have learnt today.

   Throughout the investigations, students are also regularly encouraged to:

Predict: What do they think will happen before they press a certain button or perform an action.

Observe: Make the most of Autograph's interactive features to have a look around and experiment with the objects on the screen.

Many of the activities include several pages of worksheets so it might be a good idea for each student to store their worksheets in a separate folder or ring binder. This would prevent them getting lost or mixed up, and would provide an excellent revision aid at the end of their course.

3. Answers – a complete set of answers, including screen shots, for all of the questions and tasks from the student worksheet.

The Role of the Teacher

The role of the teacher is extremely important throughout these student investigations, as it is in any computer-based lesson. Ideally the teacher should have read through the accompanying Teacher Notes prior to the lesson, and perhaps tried the activity out themselves to ensure they are comfortable with all the tasks and agree with the answers given. This will also help the teacher predict where certain members of the class might get stuck and have an appropriate answer or prompt on hand.

During the activity itself the teacher should be on hand to answer any questions and to ensure the students are "on task" as much as possible. If possible, it would be extremely useful if the teacher's computer screen could be seen by all (either by means of a digital projector or an interactive whiteboard), and then the teacher could clearly demonstrate how to do a certain element of the activity if a few members of the class are getting stuck.

Lesson Timings

The timings of the student investigations will depend very much on the length of the school lesson and the ability of the group. Each investigation contains *Core Content* and *Extension Material*, as detailed in the Learning Objectives. The investigations have been designed so that the majority of students can cover the majority of the Core Content in **one hour**. The investigations have been differentiated so that the easier material is covered at the start, thus giving all students access to the investigations and allowing them to continue at their own pace.

If you have the time in your course for a second lesson, then the Extension Material allows students to delve deeper into each topic, strengthening their understanding and truly enriching their learning experience. Some of this material may be beyond the score of their course, but it has been designed to really get the students thinking and to allow them to make full use of Autograph's dynamic and interactive features to answer any questions they may have. Furthermore, you may well see potential to further extend the investigation beyond the material contained in the student worksheets. This is to be encouraged, and if this is the case, then the investigation may run the course of two or more les-

sons.

As with any computer based lesson, the first five or ten minutes will usually be spent logging onto computers, making sure everyone has the software loaded, and then setting up the activity. Likewise, the final ten minutes should be set aside to allow students to fill in the box on their worksheet explaining what they have learnt today, to read out the answers to the questions and the tasks, and for a whole class plenary. Whatever time remains in the middle can be spent working through the activity itself.

Saving Work

Students should be encouraged to save their work at regular intervals as they progress through the activity to prevent any moments of horror. Whilst it is not strictly necessary for each student to save the Autograph files once the investigation is completed as they will have a full record of their work on their worksheet, it might be a good idea. The students' completed Autograph files provide an interactive resource which they can use if they are ever stuck on a homework or exam question on this particular topic. If you are able to do quite a few of these activities through the year, revisiting the students' Autograph files would also provide a nice revision lesson.

TEACHER NOTES

OVERVIEW

The aim of this tutorial is to enable both teachers and students to become familiar with how Autograph works. The tutorial gives an overview of some of Autograph's major 2D graphing features[†] which will enable the user to cope better with the activities throughout this textbook. It is very much hands-on, challenging and interactive, allowing the user to really get to grips with how the software works and what it can do. This should increase the user's confidence with the software, hopefully allowing teachers to come up with activities of their own, and students to use the software to aid them with their studies.

Note: There is a separate *Autograph Additional Teacher Tutorial,* specifically designed to help teachers make the most of Autograph on an **Interactive Whiteboard** or a **Graphics Tablet**. The tutorial should only take around ten minutes to complete, and it highlights the two of Autograph's best features:

1. Whiteboard Mode
   There are a few subtle differences to make Autograph smoother and easier to operate with an interactive whiteboard or a graphics tablet. Therefore, it might be a good idea to have a quick run through this additional tutorial before attempting any demonstrations using Autograph.

   Please Note: A series of 15 step-by-step interactive Autograph demonstrations are available in the book *Autograph Activities: Teacher Demonstrations.*

2. The On Screen Keyboard
   Autograph comes complete with its own onscreen keyboard, which means you don't have to be tied to the keyboard attached to your classroom's computer. The **onscreen keyboard** makes Autograph fully functional using the pen of an interactive whiteboard, or frees you up to walk around the classroom using a graphics tablet. The onscreen keyboard is even more useful as it can be used with any other application, not just Autograph, and the maths symbols in the Extra panel will also work in other applications. Only a couple of characters are exclusive to Autograph font.

TIMING

The tutorial is designed to last between forty-five minutes to an hour, based on someone who has never used the software before. This will of course differ depending on the user's past experience and overall computer competency. Once again, the most important elements are covered earlier on in the tutorial, so if not all students reach the end, they will still be in a strong position to tackle the

rest of the activities in the book.

LOCATION

For students, this tutorial should ideally take place in a computer suite with one student allocated to each workstation. Paired work is also acceptable, but it is important that each student has the same hands-on experience. If your school has access to Autograph at home, either because your school has an Extended Site Licence, or your students have individual Student Licences, then this tutorial might make a nice homework assignment.

REQUIRED PRE KNOWLEDGE

Whilst the tutorial touches on concepts such as differentiation, integration and radians, specific knowledge of these is not required. This tutorial is designed for a student about to embark upon a post-16 maths course, and hopefully the little taster they will get of these concepts will whet their appetite for what is to come later.

PRE-ACTIVITY SET-UP

None required.

† As this textbook is aimed at Post-16 students, I have avoided including elements such as the transformations of shapes and basic data handling work to the tutorial. In addition, the 3D work is covered in the activities themselves, and hence is also omitted from the tutorial. If these are areas which you are interested in learning more about, or if you want to review any of the features covered in this tutorial, then I recommend accessing the Autograph Video Tutorials contained within the Help menu of Autograph, or visiting the Autograph Website (www.autograph-math.com).

STUDENT WORKSHEET   NAME:..

ACTIVITY 1

Autograph Buttons covered:

Enter equations

Edit axes

Zooming in and out

Dragging

Select Mode

Text boxes

Point Mode

Undo

Other Autograph Functions covered:

Solve f(x) = 0

Solve f'(x) = 0

Solve f(x) = g(x)

Change the line/curve appearance

Adjust the degree of accuracy

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are *not* in **Whiteboard Mode**.

Enter the equation: y = x² + 4x − 3
Note: To enter x², either use the little ⌐ button, press "**alt 2**" together, or type in "**xx**".

Predict   *What will this curve look like? Where will it cross the axes? Which direction will it slope?*

Click OK.

As you will see, not all of the curve fits on the page, so we need to edit our axes. Edit the axes as follows:
x: Minimum: −6   Maximum: 4   Numbers: 1   Pips: 0.2
y: Minimum: −8   Maximum: 6   Numbers: 1   Pips: 0.5

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Note: Before you attempt to re-scale the axes yourself, often pressing the **Default Scales** button will do a good job of sorting the scales out for you.

Now we have our curve on the page, let's see what Autograph can help us find out about it.

1. Where does the curve cross the x-axis?

Use the **Zoom In** function to take a closer look at the first crossing point which is close to −5.

Notice how the scale automatically adjusts the closer in you get.

Use the **Drag** function to move across the screen to find the other point where the curve crosses the x-axis.

Zoom back out so we are looking at the original graph again.

Note: Pressing **Undo** several times is often a quicker way of getting back to your original view.

Now, there is a way to find where the curve crosses the x-axis much more accurately:

Make sure you are in **Select Mode**.
Left-click on the curve (it should turn black).
Right-click to bring up a menu.
Select **Solve f(x) = 0**.

This will mark on the two points where the curve crosses the x axis, and give you their values to four significant figures in the **Status Bar** at the bottom of the page.

Left-click to select one of the crossing points (it should turn black).
Note: Because these two points are related, they will both be selected.

Click on **Text Box**.
In the text field, delete "Equation solver", and instead write "x-axis crossing points".
Click **OK**.
This displays the results in a moveable text box on the page.

Your page should look something like this:

x-axis crossing points: (-4.646, 0), (0.6458, 0)

Equation 1: y=x²+4x−3
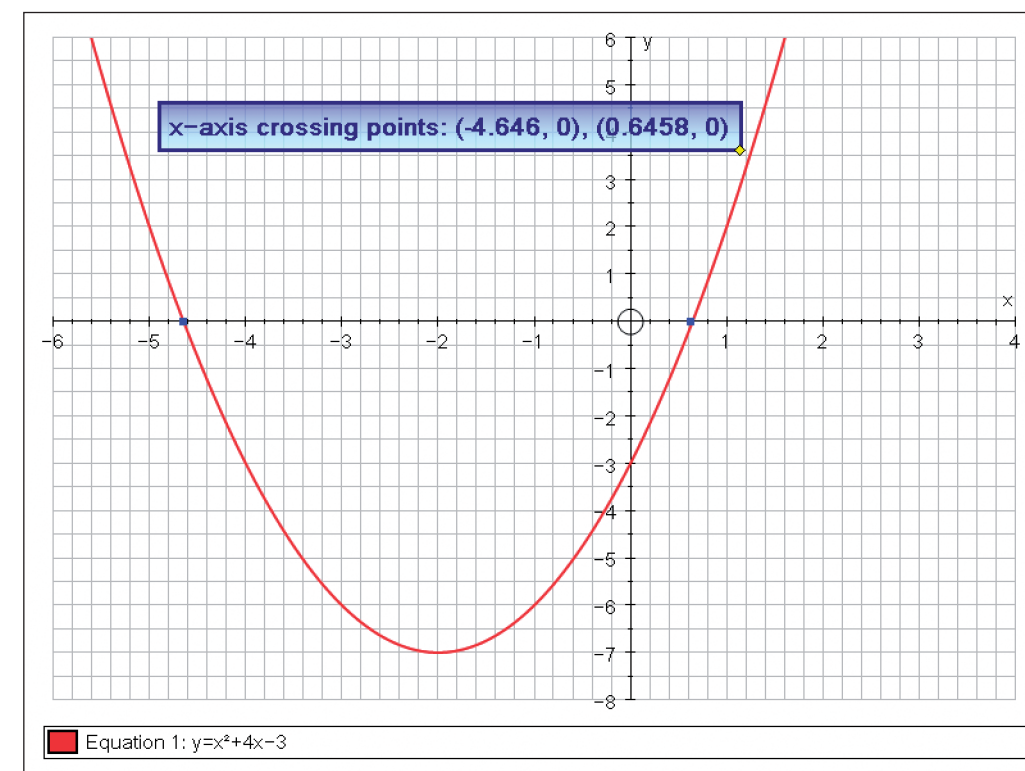
2. Where does the curve cross another line?

Enter another equation: y = −x − 4
Still on the enter Equation screen, click on **Draw Options**:
- Change the colour of the line to purple.
- Choose a *dashed* line style.
- Set the line thickness to 3 pts.

Predict | *What will this line look like? Where will it cross the axes? Which direction will it slope?*

Click **OK** twice.

You can see that it crosses our curve at two points. Again, we can zoom in to take a closer look at the points of intersection, but we can also use a similar technique to find out their co-ordinates more accurately:

After zooming in, press **Undo** until you return to the original view of the graph.

Ensure you are in **Select Mode**.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black).
Note: The use of the **Shift** button to select *more than one object* is very important!

Right-click, select **Solve f(x) = g(x)**, and once again the co-ordinates of the

crossing points are marked, and the results are displayed in the status bar.

If you want to improve the accuracy of the calculations:
Go to **Page** > **Edit Settings**.
Adjust the level of accuracy to 8 significant figures.

Another quite nice way to achieve all this is to have a moveable co-ordinate on the curve.

Select **Point Mode** and place a point somewhere on the left-hand side of the curve.
Note: When the cursor hovers over a section of the curve, it should turn from a cross into a black arrow.

Click on **Text Box**.
In the text field, delete "Point", and instead write "A".
Click **OK**.
This will keep a record of the current co-ordinates of point A.

Ensure you are in **Select Mode** and point A is selected (it should have a square around it).
Use the **left** and **right** arrow buttons on the keyboard to move the point along the curve.
Use the **up** and **down** arrow buttons on the keyboard to switch between the curve and the line.
Place the point somewhere on the curve.

Left-click twice on the point, and type in the x-value −5.

Your screen should look something like this:

x-axis crossing points: (-4.646, 0), (0.6458, 0)

A: (-5, 2)

Equation 1: y=x²+4x−3
Equation 2: y=−x−4
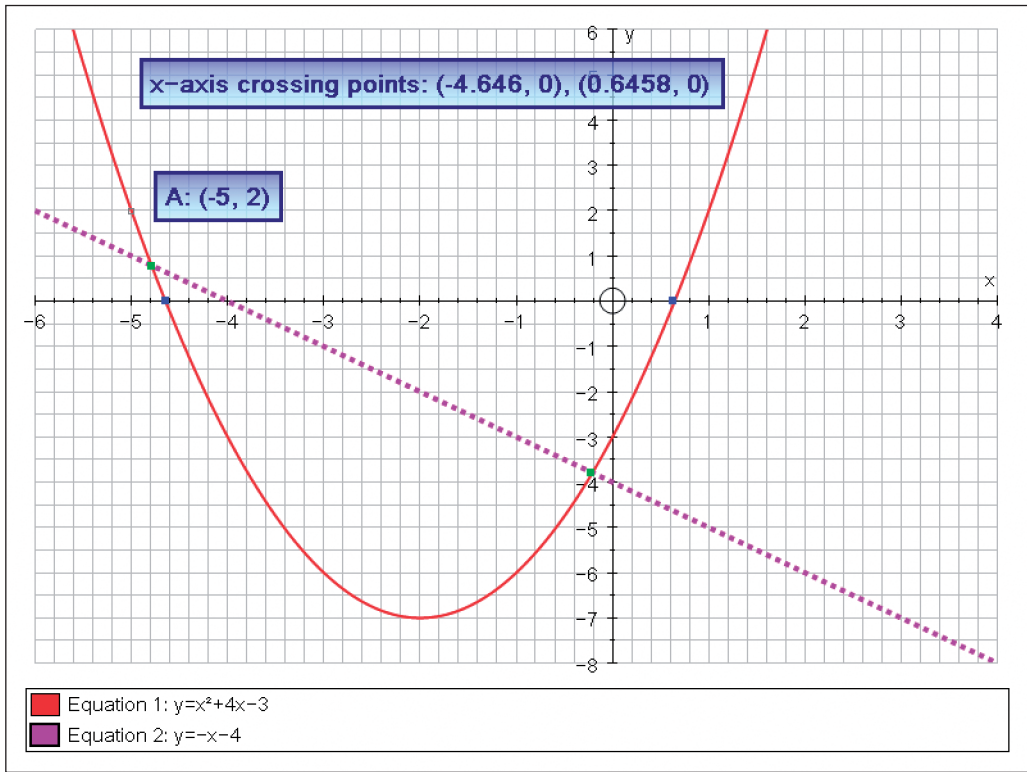
With the point still selected, if you **right-click** you also have the following options:
- **Move to next f(x) = 0** which will move you to the next crossing of the x axis
- **Move to next f'(x) = 0** which will move you to the next stationary point on the curve.

ACTIVITY 2

Autograph Buttons covered:

Slow Plot

Re-plot, Pause, Fast Forwarding

Animate Object

Gradient Function

Area Under a Curve

Enter Co-ordinates

Other Autograph Functions covered:

Table of Values

Results Box

Delete Objects

Move to Next Intersection

Open up a **New 2D Graph Page**

Edit the axes as follows:
     x: Minimum: **−4**   Maximum: **4**   Numbers: **1**   Pips: **0.5**
     y: Minimum: **−10** Maximum: **15**   Numbers: **2**   Pips: **1**
Remove all of the green ticks underneath **Auto**.

Click on **Slow Plot**.

Enter the equation: $y = x^3 + 2x^2 - 5x - 3$
Note: To enter x^3, either use the little 3 button, press "**alt 3**" together, or type in "**xxx**"

Predict

> *What will this curve look like? Where will it cross the axes? Which direction will it slope?*

Click **OK**.

Autograph will begin to plot the graph from left to right.

Pressing **Pause Plotting** at anytime stops and starts the plotting and enables you to focus on the key features of the graph.
Note: Pressing the **spacebar** on your keyboard does the same.

Pressing **Fast-Forward Plotting** immediately speeds to the end of the plotting.

When the curve has finished plotting, click on the **Replot** button.
You now have the option to adjust the plot settings, which means you can adjust both the x-step and the specific section of the graph you want to plot.

Left-click to select the curve (it should turn black).
Right-click and select **Table of Values** from the menu.
Change the **x-step** to 1 and leave the **x-min** and the **x-max** as they are.

This will now give you the corresponding y values for each integer x value on your graph in a **Results Box** at the side of the page.

Note: To view the Results Box at any stage, just click **Results Box**.
The Results Box can also be copied and pasted into another application should you wish.

Your screen should look something like this:

Equation 1: $y = x^3 + 2x^2 - 5x - 3$
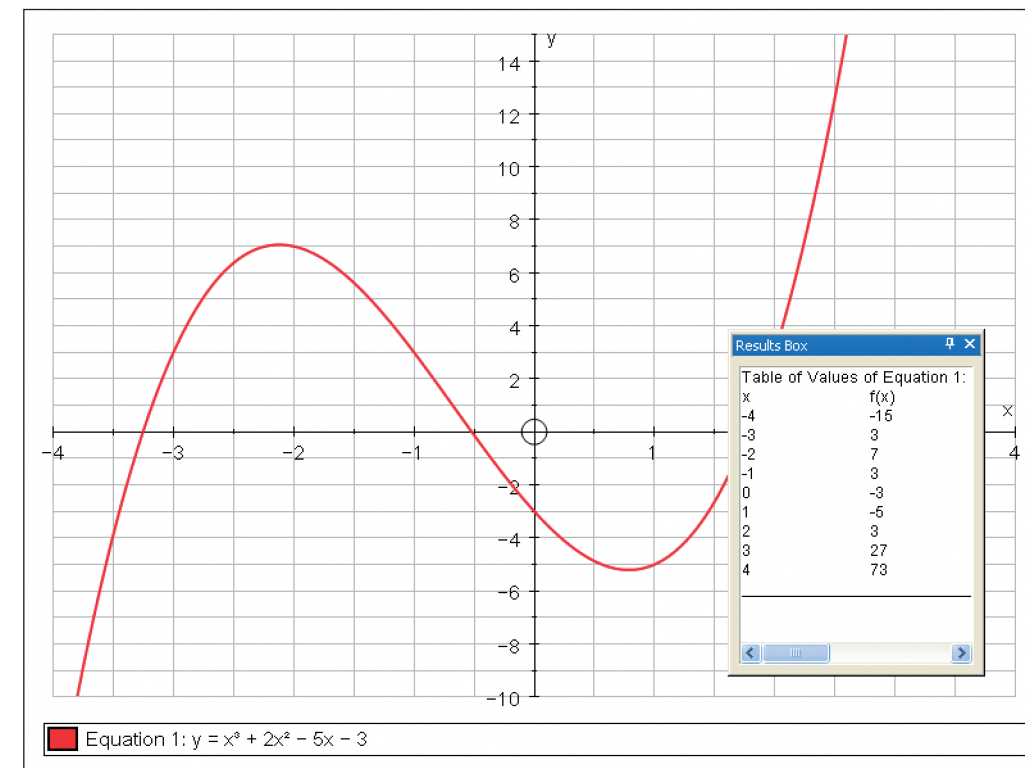
Predict

> *What would a **tangent** to this curve look like? What direction would it slope?*

Select **Point Mode** and place a point somewhere on the curve
Right-click and select **Tangent** from the menu.

Left-click to select the tangent (it should turn black).

Select **Text Box** and click **OK**.
The equation of the tangent should now be displayed.

Left-click to select the point on the curve (it should have a square around it).
Use the **left** and **right** arrow keys on the keyboard to move the point along the curve. Notice how the equation of the tangent adjusts.

There is a better way of moving points along a curve then using the left and right buttons.

Make sure the point is selected.

Select the **Animate Object** button, which brings up the **Adjust Position** box.
You now have the option of selecting the exact starting location for the point, and the size of the step which it increases in.
Select a position of **−2**, and a step of **0.1**.
Use the **left-right** buttons to move your point along the curve.

Still in the **Adjust Position** box, click on **Animate**.
Here you can set up Autograph to adjust the step automatically.

Try these settings on **Automatic - Repeat**:
- Animation speed as far to the left as it will go
- Start: **−3**   Finish: **2**   Step: **0.05**

Click the **Play** button.

Your screen should look something like this:
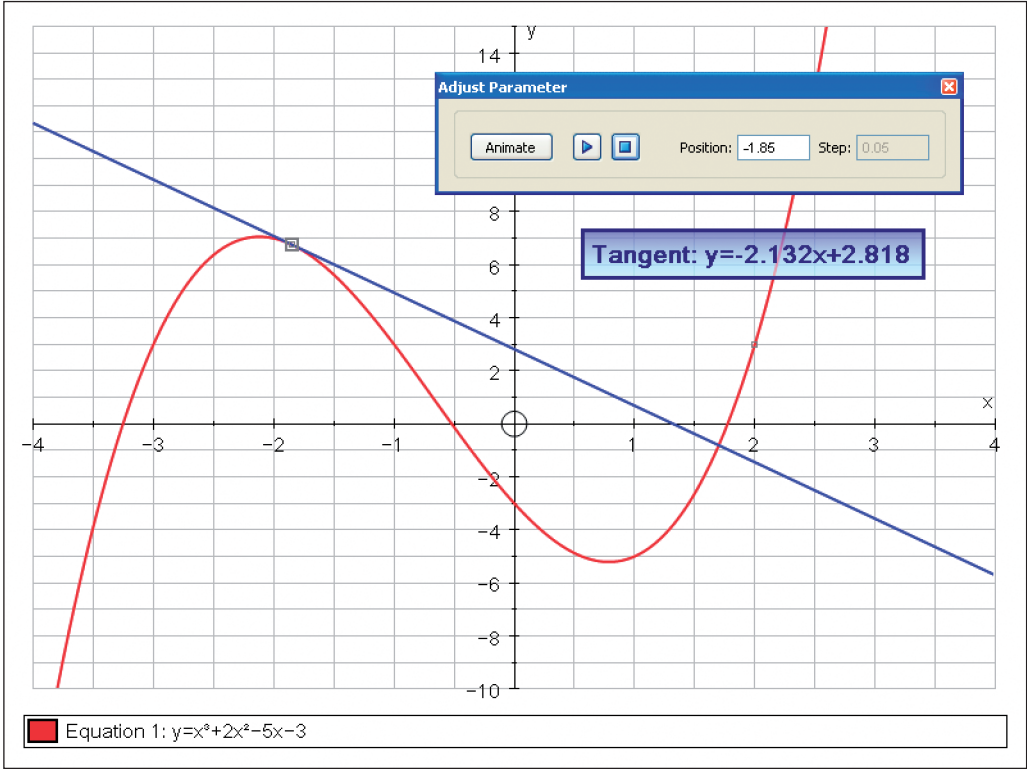
When you are ready:

Close the **Adjust Position** box by clicking the red cross in the corner.

Left-click to select the tangent (it should turn black).
Either **right-click** and select **Delete Objects** from the menu, or just press **Delete** on your keyboard.
Note: Don't worry about the warning. This is just to remind you that the Text Box is linked to the tangent, and so will be deleted as well.

You should now be only left with the curve.

Make sure you are still in **Slow Plot** mode (there should be a small blue square around the button).

Left-click to select the curve (it should turn black).

Click on **Gradient Function**.
This will automatically calculate the gradient of the tangent along the curve,

and plot these values as it goes.

The gradient function is set-up to pause at important points, such as each maximum and minimum point, and each point of inflexion.

Press **Pause Plotting** (or the **Spacebar**) to resume the plot.

Left-click to select this new curve – the gradient function (it should turn black).

Again, click on **Gradient Function**.
This will automatically plot the gradient function of the gradient function, otherwise known as the second gradient function of the original graph.

Note: These lines and curves can then be analysed like any other, whether it be finding points of intersection with the other curves, crossing points with the x-axis, or anything else.

Your screen should look something like this:
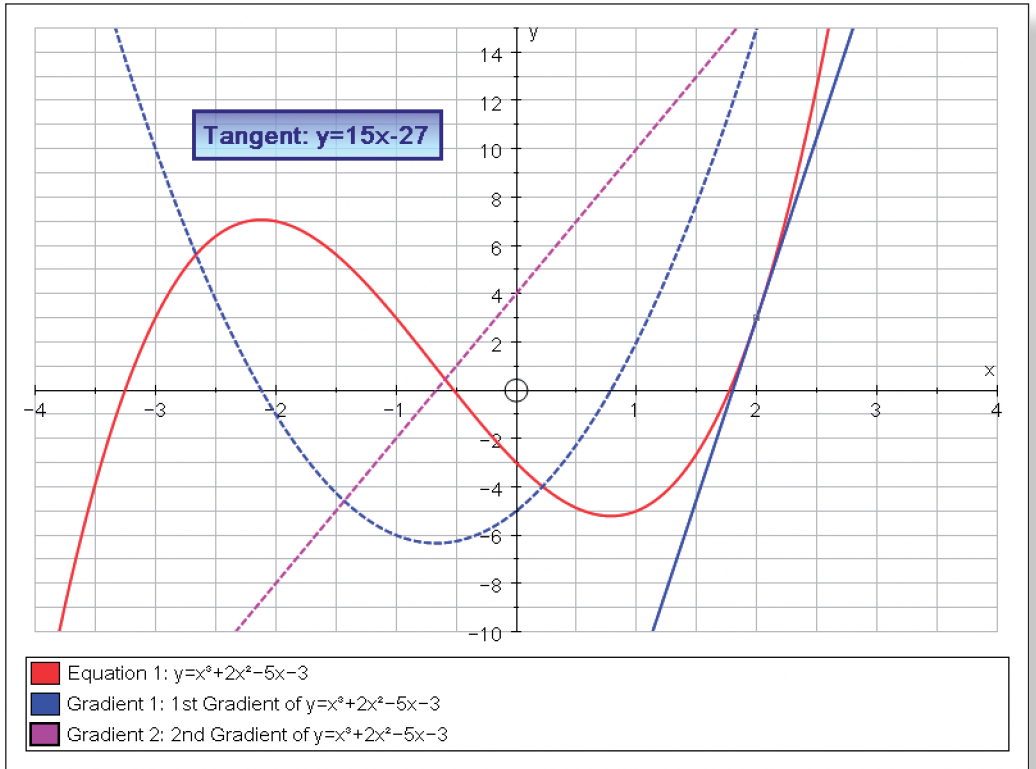
Open up a **New 2D Graph Page**.

Edit the axes as follows:
x: Minimum: **−4**   Maximum: **4**
y: Minimum: **−6**   Maximum: **6**
Leave all of the green ticks underneath **Auto**.

Make sure **Slow Plot** is turned off.

Enter the equation: y = 5 − x²

What will this curve look like? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Left-click on the curve (it should turn black).

Add a point on the curve with an x value of 0.

Add another point on the curve with an x value of 1.

Left-click on an unoccupied part of the graph area to de-select everything.

Hold down the **Shift** button to select more than one object:
Left-click on both these points (they should both have a square around them).

Right-click and select **Find Area** from the menu.
Use the following settings:
• Trapezium Rule
• Divisions: 5

What will this look like?

Click **OK**.

Left-click on the marked area under the curve.

Select **Text Box** from the menu and click **OK**.
This will display a numeric estimation for the area under the curve.

Left-click on one of the points.
Use the **left** and **right** arrow buttons on the keyboard to move the point along the curve and observe how the area changes.

What effect will increasing the number of divisions have on the estimate for area?

Left-click to select the area under the curve.

Click on **Animate Object**.
You can now adjust the number of divisions and observe the effect it has on the estimate for area.

Your screen should look something like this:
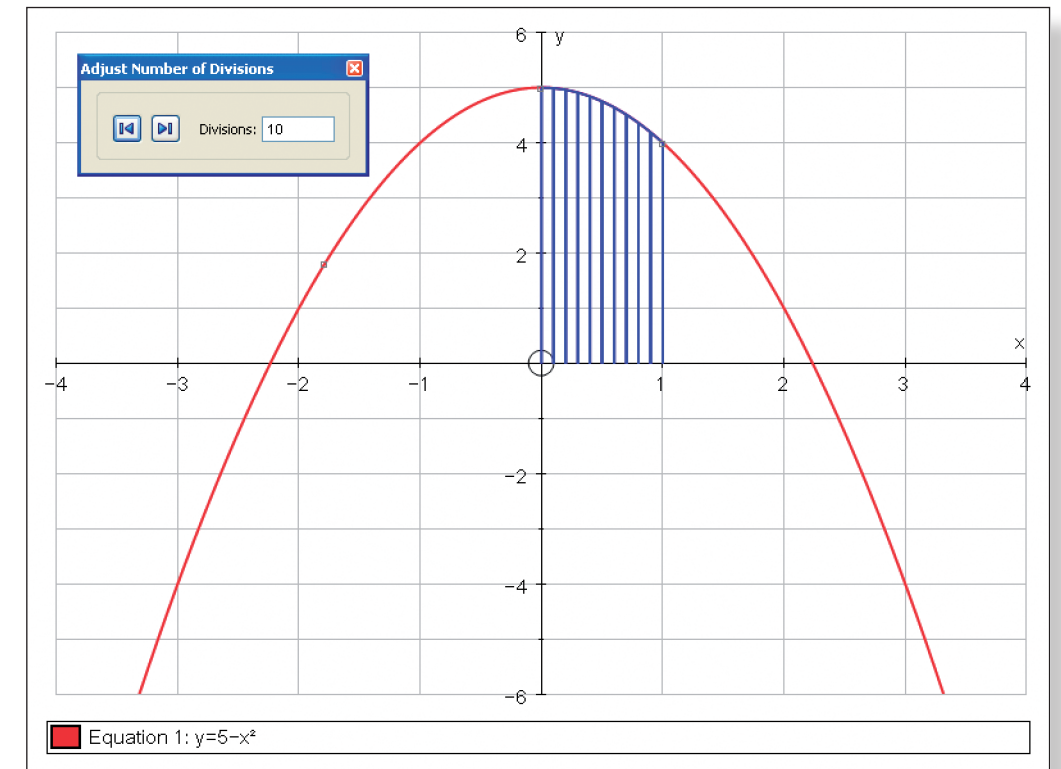
Left-click to select the area under the curve, and **Delete**.

Enter the equation: y = −x

What will this line look like?

Click **OK**.

Place a point on the left hand section of the curve which is below the line.

What do you think the feature Move to Next Intersection will do?

Make sure you are in **Select Mode**.

Use the **Shift** button to select the point *and* the line.
Right-click and select **Move to next Intersection** from the menu.

Place a point on the middle section of the line which is below the curve.

Make sure you are in **Select Mode**.

Use the **Shift** button to select the new point and the curve.

Right-click and select **Move to Next Intersection** from the menu.

Make sure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to de-select everything.

Use the **Shift** button to select both points.

Right-click and choose **Find Area**.
Use the following settings:
- Trapezium Rule
- Divisions: 50

The area between the two curves should now be marked.
You can now use the **Animate Object** function to experiment with different divisions as before.

Your page should look something like this:

Equation 1: y=5−x²
Equation 2: y=−x
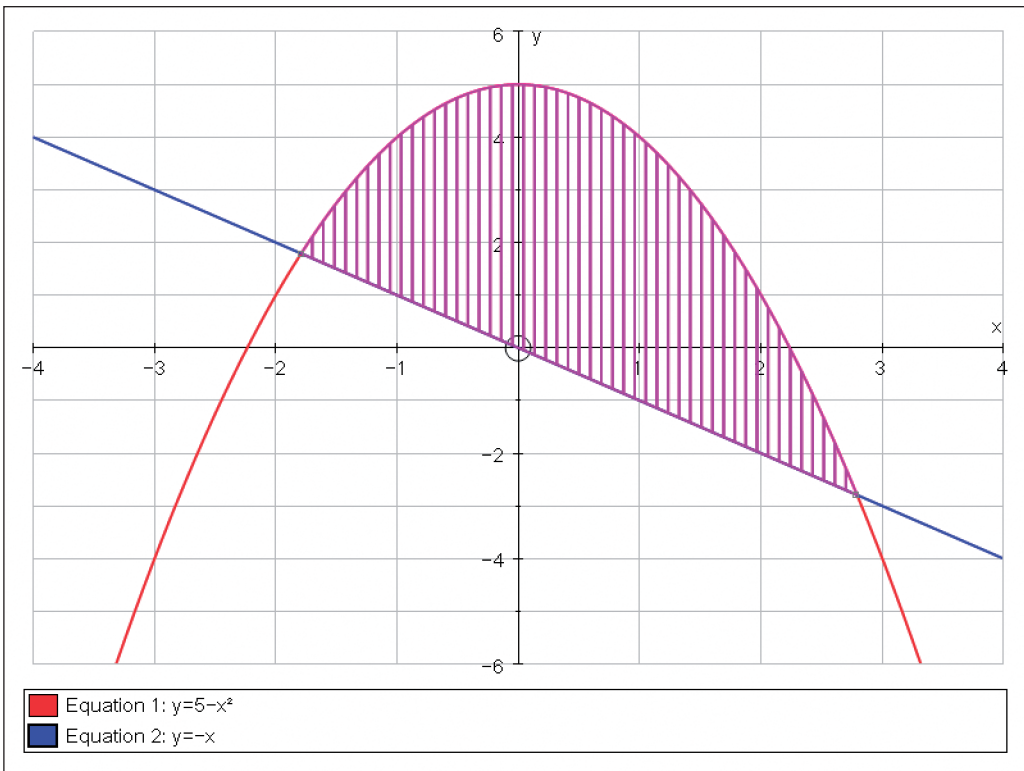
Activity 3

Autograph Buttons covered:

Enter equations (complicated ones!)

Default Scale

Degrees and Radians

Zoom In Box

Reset Axes

Often in mathematics we encounter some pretty nasty looking equations. Autograph can handle all of these, but it important you know how to enter them.

We are going to go through a number of equations and go through step-by-step how to enter each of them. Hopefully what you end up with on your screen is pretty similar to the graphs on the right.

Equation 1:

y = 4sin(3x)

Open up a **New 2D Graph Page**.

Ensure you are in **Degree** mode.

Type the following and then press **OK**:

y = 4sin(3x)

Click on **Default Scales** to improve the graph.

Edit the axes as follows:
   x: Minimum: **−360** Maximum: **360**
   y: Minimum: **−5**   Maximum: **5**

Equation 2:

x = cos(3t + π)

Open up a **New 2D Graph Page**.

Click on the small black arrow next to this button.

Change the axes to x and t.

Ensure you are in **Radian** mode.

Type the following and then press **OK**:

x = cos(3t + π)

Note: Either use the ▣ button or press "**alt p**" together.

Click on **Default Scales** to improve the graph.
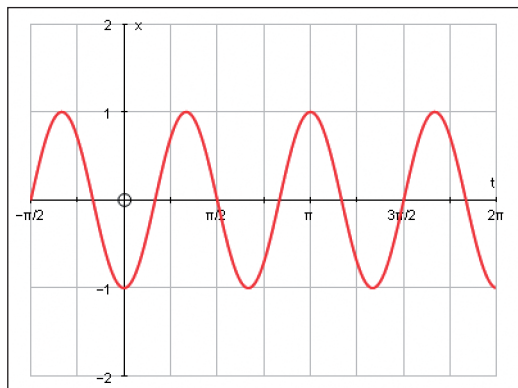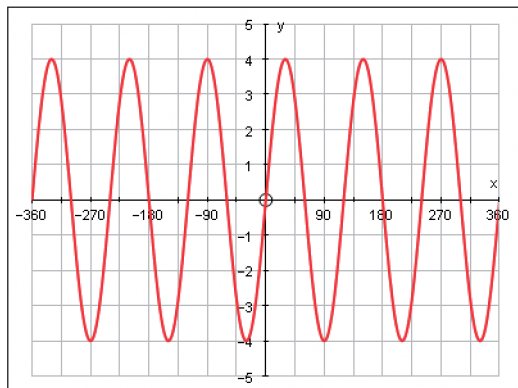
Equation 3:

$$y = \frac{(5x+3)^3}{2x}$$

Open up a **New 2D Graph Page**.

Type the following and then press **OK**:

y = (5x+3)³/2x

Note: For the cubed, use either the little 3 button or press "**alt 3**" together.

Edit the axes as follows:
    x: Minimum: **−5**   Maximum: **5**
    y: Minimum: **0**   Maximum: **20000**

Use the **Zoom In Box** function to take a closer look at the graph around the origin.
Notice how the scale automatically adjusts the closer in you get.

Use the **Drag** function to move across the screen.

Press **Undo** several times is to get back to your original view of the graph.

Equation 4:

$$y = x^{\frac{4}{5}}$$

Open up a **New 2D Graph Page**.

Click on **Slow Plot** mode.

Type the following and then press **OK**:

y = x^(4/5)

Note: To access the power/hat button press shift and the number which the symbol is above together (usually 6).

Edit the axes as follows:
    x: Minimum: **−6**   Maximum: **6**
    y: Minimum: **−2**   Maximum: **2**

Equation 5: Parametric

x = t² − 1, y = 2t

Open up a **New 2D Graph Page**.

Ensure **Slow Plot** mode is on.

Type the following and then press **OK**:
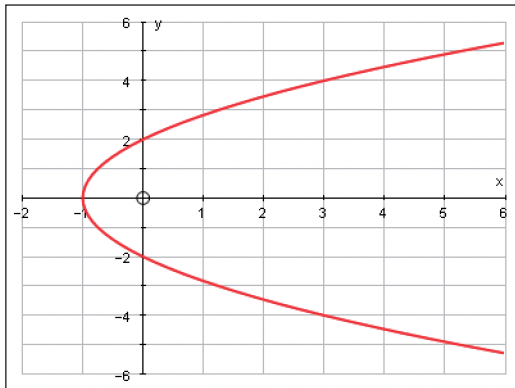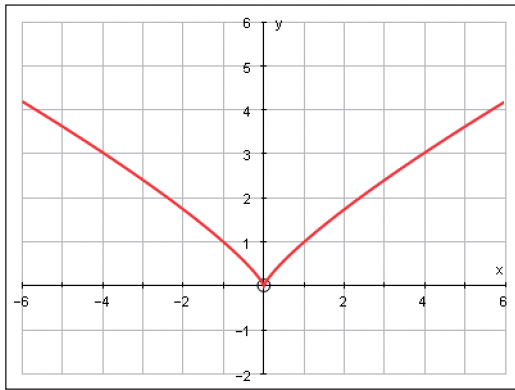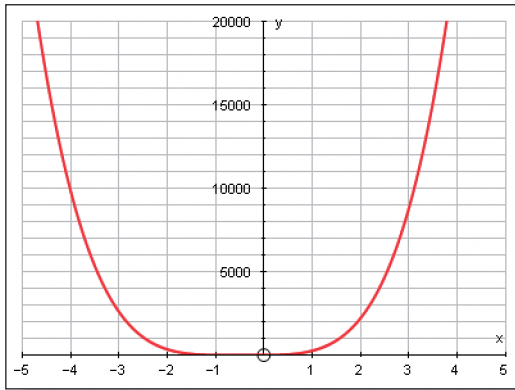
x = t² − 1, y = 2t

Edit the axes as follows:
    x: Minimum: **−2**   Maximum: **6**
    y: Minimum: **−6**   Maximum: **6**

Equation 6:

$$y = |x| \pm \sqrt{4 - x^2}$$

A nice romantic one to finish…

Predict   *Can you guess what this graph will look like?*

Open up a **New 2D Graph Page**.

Ensure **Slow Plot** mode is on.

Type the following and then press **OK**:

y = |x| ± √(4 − x²)

Note: To access the modulus signs, the plus-minus, and the square root, simply press the corresponding buttons in the equation editor.

Note: For more examples of the different types of equations that can be entered into Autograph, see the relevant section in the Help menu.
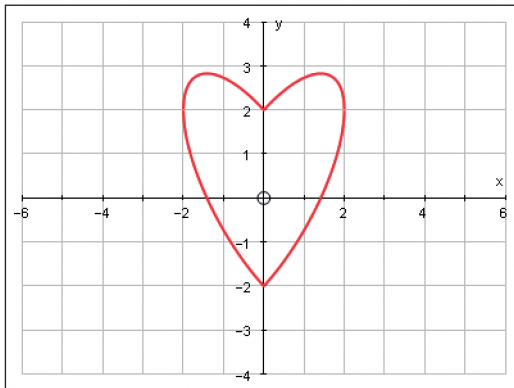
Activity 4

Autograph Buttons covered:	Other Autograph Functions covered:
Equal aspect	Graph paper theme
Line segment	Hiding the Key
Gradient	Circles and Tangents
Perpendicular Bisector	Marquee Select
	Use of Shift and Ctrl

Open up a **New 2D Graph Page**.

Click on **Equal Aspect Mode**.
This alters the x scale so that the axes are square.

Click on **Edit Axes**
Go to **Appearance**
Open the drop-down menu underneath **Themes**
Select **Graph Paper**
Click **OK**

On the top toolbar, click on **Axes**, and then **Show Key** to hide the key at the bottom of the screen.

Place a co-ordinate at (1, 3).

Place another co-ordinate at (4, 1).

Predict
What will a line segment between these two points look like? What direction will it slope? Will it cross the axes?

Make sure you are in **Select Mode**.

Use the **Shift** button to select both points.

Right-click and select **Line Segment** from the menu.

Predict
What do you think the Gradient function will do?

Again, use the **Shift** button to select both points.

Right-click and select **Gradient** from the menu.

Left-click to select the gradient triangle.

Select **Text Box** from the menu and click **OK**.
This displays the gradient of the line, as well as the equation.

Your screen should look something like this:

Gradient: -0.6667   △y=2, △x=3   y=-0.6667x+3.667
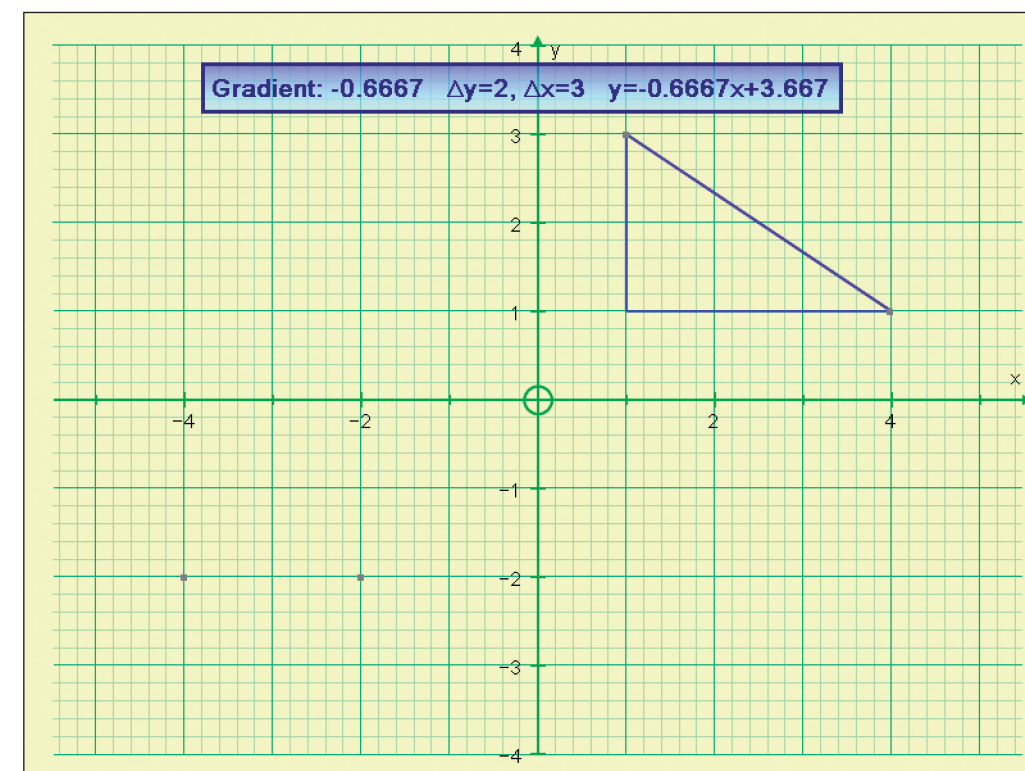
You can now select a point and move it using the **left-right** arrows on the keyboard to see the effect on the gradient and the equation of the line.

Note: The default in **Advanced Mode** is that the left-right arrows move points 1/10th of a grid spacing. If you hold **Shift** down whilst pressing the arrows, the point moves a whole "grid spacing". If you hold down **Ctrl** whilst pressing the arrows, the point moves by 1/100th of a grid spacing.

Predict
What do you think the Perpendicular Bisector function will do?

Again, use the **Shift** button to select both points.

Right-click and select **Perpendicular Bisector** from the menu.

Left-click to select the perpendicular bisector.

Select **Text Box** from the menu and click **OK**.
This displays the equation of the line.

Place a co-ordinate at (−4, −2).

Place another co-ordinate at (−2, −2).

Drag a rectangle that surrounds both points.
This is known as **Marquee Selection**, and is a useful way of selecting multiple objects in a given area.

Predict   *What do you think the Circle (2pts) function will do?*

Right-click and select **Circle (2pts)** from the menu.

Predict   *What will a tangent to this circle look like? Where will it cross the axes?*

Place a point somewhere on the circle.

Left-click so just this new point is selected.
Right-click and select **Tangent** from the menu.

Left-click so just the tangent is selected.

Select **Text Box** from the menu and click **OK**.
This displays the equation of the tangent.

You can now use the arrow buttons on the keyboard to move the tangent around the circle.

Your screen should look something like this:

Gradient: -0.6667   △y=2, △x=3   y=-0.6667x+3.667

Perpendicular Bisector: y=1.5x-1.75

Tangent: y=0.09849x+0.2067
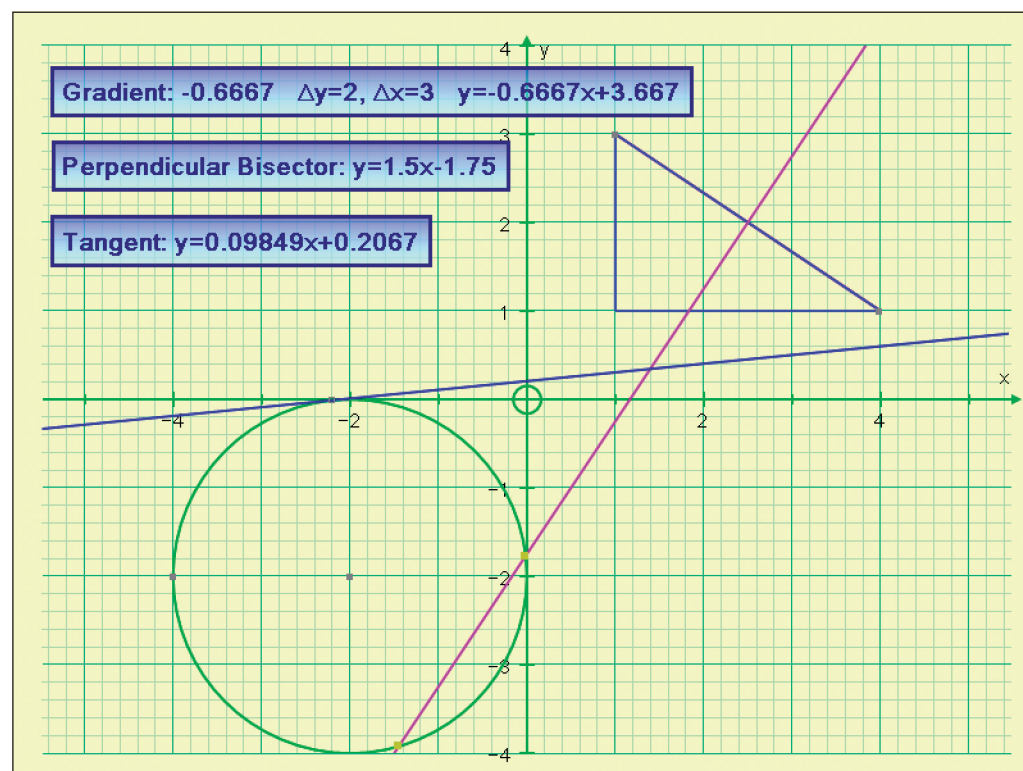
Note: With these various shapes and lines on the page, it is also possible to find the points of intersection in exactly the same way as described in Activity 1.

ACTIVITY 5

Autograph Buttons covered:

k   Constant Controller

f(g(Function Definitions

   Manage List

Open up a **New 2D Graph Page**.

Edit the axes as follows:
     x: Minimum: −6   Maximum: 6
     y: Minimum: −10   Maximum: 10
Leave the green ticks as they are to allow Autograph to select the best grid spacings.

Enter the equation: y = ax² + bx + c
Still on the **Enter Equation** screen, click on **Edit Constants**, and set the values as follows:
     a = 1   b = 4   c = 4

Predict   *What will this curve look like? Where will it cross the axes? Which direction will it slope?*

Click **OK** twice.

Left-click to select the curve (it should turn black).

Select **Text Box**.
Tick the box next to **Show Detailed Object Text** and click **OK**.
The equation of the curve and the current values of the constants should now be displayed in a moveable **Text Box**.

Click on **Constant Controller**.
You can now change the values of any of the three constants and observe the effect it has on the shape and position of the curve.
The drop-down menu is used to switch between constants.
The **up-down** buttons alter the value of the constant.
The **left-right** buttons alter the size of the step.

Experiment by adjusting the values of each constant, trying to predict what effect it will have on the shape and position of the graph.

Your page should look something like this:

Equation 1: y=ax²+bx+c
a = 2
b = 4
c = -1.8

Constant Controller - [Tutorial]

Options   c   = -1.8   Step: 0.1

Equation 1: y=ax²+bx+c

Equation 1: y=ax²+bx+c   c=-5
a = 1
b = 1
c = Min: -5, Max: 5, Step: 1

Equation 1: y=ax²+bx+c
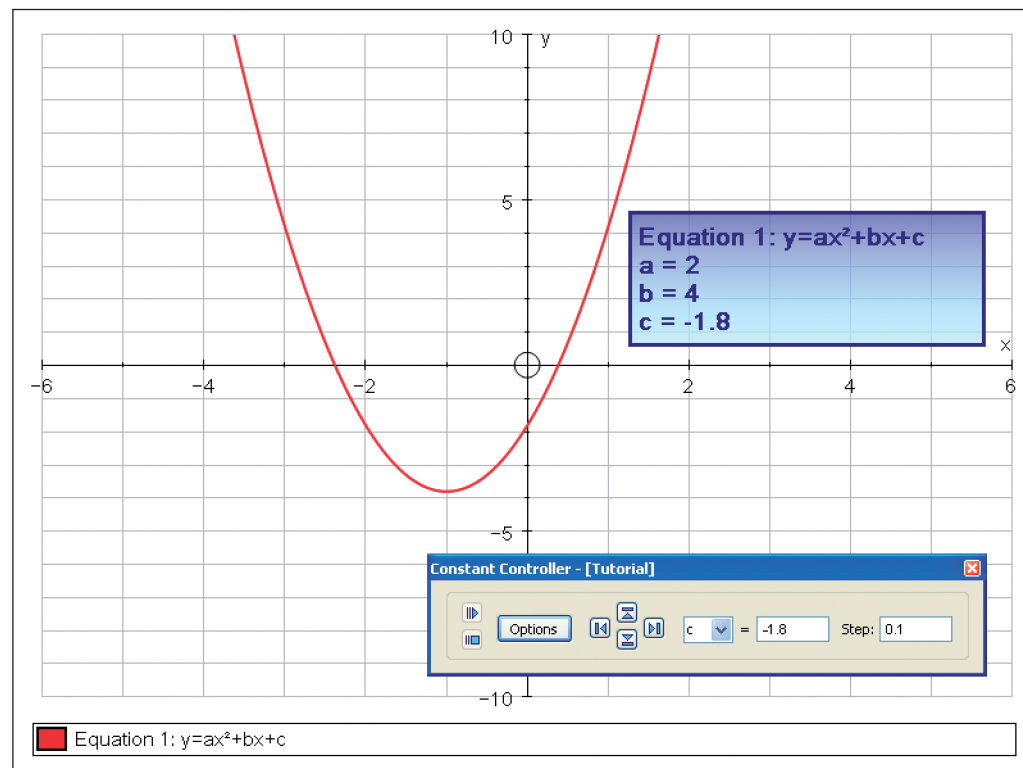
Set the value of each constant to 1.
Select constant c.
Click on Options on the **Constant Controller**.
Select **Family Plot**.
Adjust the parameters as follows:
    Start: −5   Finish: 5   Step: 1

Predict

What will this look like?

Click **OK**.

Your screen should look something like this:
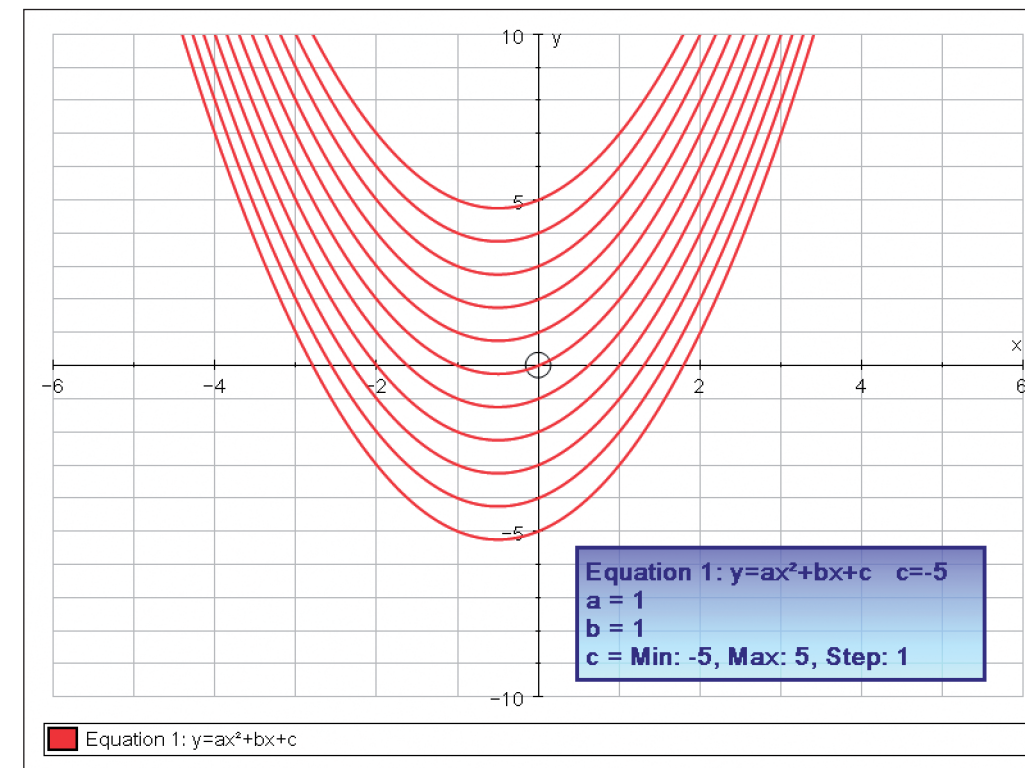
Click on **Options** again.
This time select **Animation**.
Keep the **Animation Speed** and the **Start** and **Finish** the same, but adjust the **Step** to 0.1.
Click **OK** and press the **Play** button.
This automatically adjusts the value of c for you so you can observe its effects on the shape and position of the curve.

Open up a **New 2D Graph Page**.

Edit the axes as follows: x from −10 to 10 and y from −40 to 40.

Click on **Function Definitions**:
  • Define f(x) to be 10√x
Note: to type the √ sign, press "**alt r**" together.
  • Define g(x) to be x² − 4

Ensure **Slow Plot** mode is on.

Enter the equation: y = f(x)

Predict

What will this look like?

Click **OK**.
Enter another equation: y = g(x)

Predict

What will this look like?

Click **OK**.
Enter the equation: y = f(x) + g(x)

What will this look like?

Click **OK**.
Enter another equation: y = f(x) − g(x)

What will this look like?

Click **OK**.

Your screen should look something like this:

Equation 1: y=f(x)
Equation 2: y=g(x)
Equation 3: y = f(x) + g(x)
Equation 4: y = f(x) − g(x)
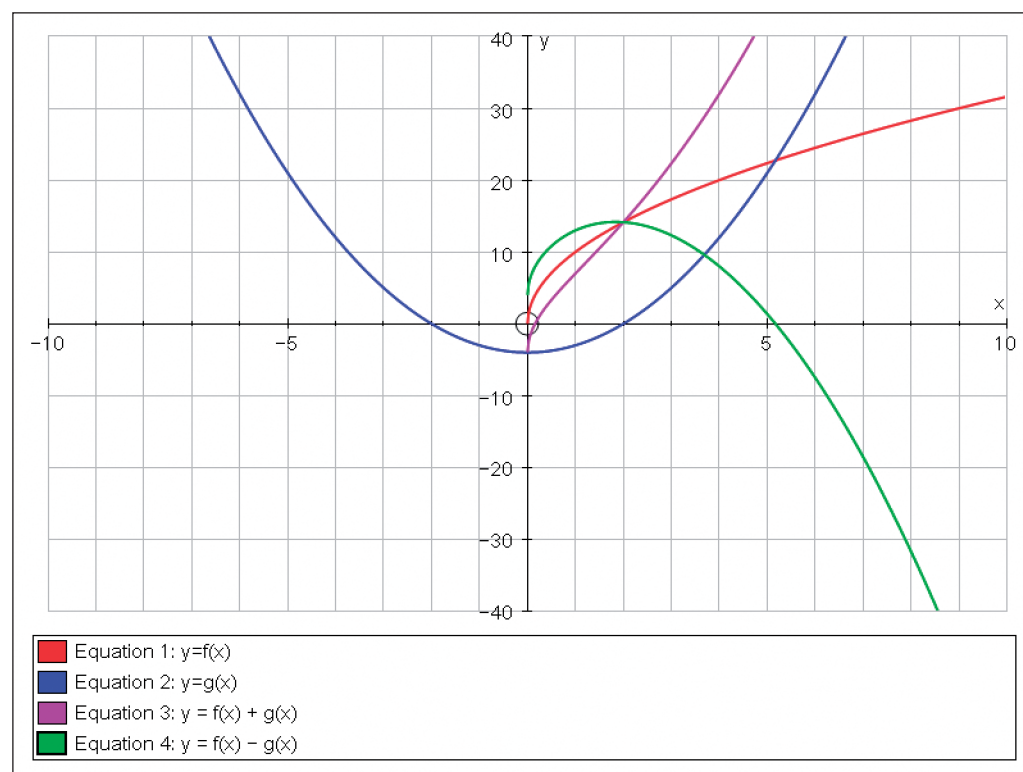
Click on **Manage List**.

Select Equation 3: y = f(x) + g(x)
Delete it.

Do the same with Equation 4: y = f(x) − g(x)

Click **OK**.

You should now be left with the original two curves.

Carefully enter the equation: y = f(g(x))

What will this look like?

Click **OK**.

Carefully enter another equation: y = g(f(x))

What will this look like?

Click **OK**.

Your screen should look something like this:

Equation 1: y=10f(x)
Equation 2: y=g(x)
Equation 3: y=f(g(x))
Equation 4: y=g(f(x))
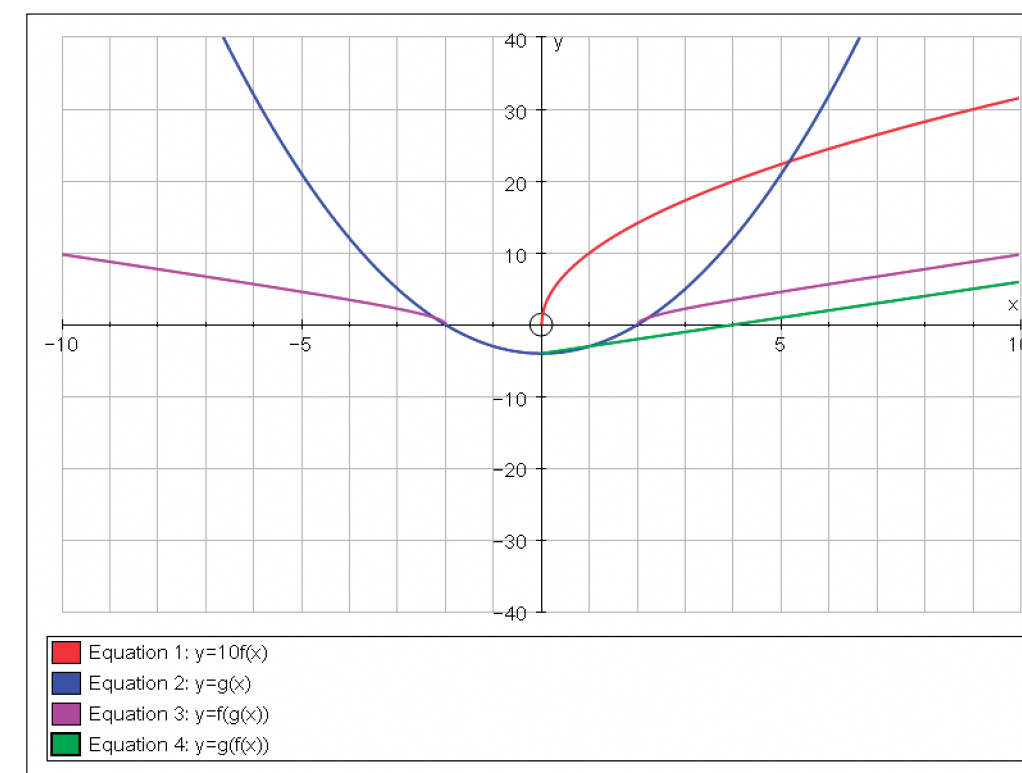
Use the zoom and **Drag** functions to have a closer look at the intersections of these curves.

ADDITIONAL TEACHER TUTORIAL

WHITEBOARD MODE AND ON-SCREEN KEYBOARD

Note: This additional tutorial is designed to help you get the most out of Autograph on the Interactive Whiteboard (IWB) for classroom demonstrations, and hence ideally it should take place on an IWB.

New Autograph Buttons covered:

Whiteboard Mode

Keyboard

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Select **Whiteboard Mode**.

Edit the axes as follows:
  x: Minimum: **−12**   Maximum: 12   Numbers: 1   Pips: 1
  y: Minimum: **−6**   Maximum: 6   Numbers: 1   Pips: 1
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under Auto are removed or Autograph will attempt to re-scale your axes for you.

Select **Equal Aspect Mode**.
This will automatically adjust the x-axis so the axes are square.

In the **View** menu on the top toolbar, select **Keyboard**.
The on-screen keyboard should now appear.
Click on the **Text** button to display a greater range of the keyboard buttons.
Drag the on-screen keyboard to a convenient place on the screen.

Place a point at (3, 3).

The point should still be selected (it should have a square around it), but if not simply **left-click** to select it again.

Click on **Text Box**.
Use the on-screen keyboard to change the word "Point" to "A" and click **OK**.
Drag the **Text Box** to a convenient space on the screen.

Place a point at (−2, −4).

Label the point "B".

Place a point at (−4, −3).

Label the point "C".

Note: At this stage, all three points are selected (they all have little squares around them). If you want to work on just one (or two) points, you must first de-select everything.

VERY IMPORTANT: DE-SELECTING EVERYTHING

Whilst in **Select Mode**, **left-click** on an unoccupied part of the graph area.

This de-selects everything and is extremely useful in all Autograph activities in **Whiteboard Mode**.

If you don't de-select everything before moving onto another task, you might find that Autograph thinks you still want to work with a certain point or line, and this could lead to problems!

Note: Pressing the **Esc** button on the keyboard performs the same function.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select Point B (it should have a square around it).

Click on the **Text** button again to show a more limited range of the keyboard buttons.

We are now going to move Point B using the **arrow buttons** on the on-screen keyboard.

The default in **Advanced Mode** is that the left-right arrows move points 1/10th of a gird spacing.
If you hold **Shift** down whilst pressing the arrows, the point moves a whole "grid spacing".
If you hold down **Ctrl** whilst pressing the arrows, the point moves by 1/100th of a grid spacing.
Note: It is important when using the on-screen keyboard to remember that the **Shift**, **Ctrl**, **Alt** and **Caps Lock** buttons remain pressed down until you click on them again.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select point A and then point C (they should both have squares

around them)

Note: In **Whiteboard Mode**, it is enough to simply click on objects in turn to select them.

Right-click and choose **Line Segment** from the menu.

Click on the **Data** and **Text** buttons to show the full range of the on-screen keyboard buttons.

Use the on-screen keyboard to enter the equation: y = x² and click **OK**.
Note: Hitting the **Enter/Return** button on the keyboard is often a quicker way of opening the **Add Equation** screen.
Note: To enter x², either use the little ⊡ button, press "**alt 2**" together, or type in "**xx**".

At the top of the screen go to **Axes** > **Show Key**.

This should make the key at the bottom of the screen disappear and is extremely useful when you don't wish your students to see the equations of lines and curves on the screen.

Note: This can also be done by **right-clicking** on the **Key** towards the bottom of the screen where it says "Equation 1: y = x²", and from the menu **left-click** on **Show Key**.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click to select the line segment and the curve (they should both turn black).
Right-click and choose **Solve f(x) = g(x)** from the menu.

The points of intersection of the curve and the line segment should now be marked on the graph.

Left-click on an unoccupied part of the graph area to de-select everything.

Left-click on one of the points of intersection (they should both turn black).

Click on **Text Box**.
Use the on-screen keyboard to change the words "Equation Solve" to "Intersection" and click **OK**.
Drag the **Text Box** to a convenient space on the screen.

Your screen should look something like this:
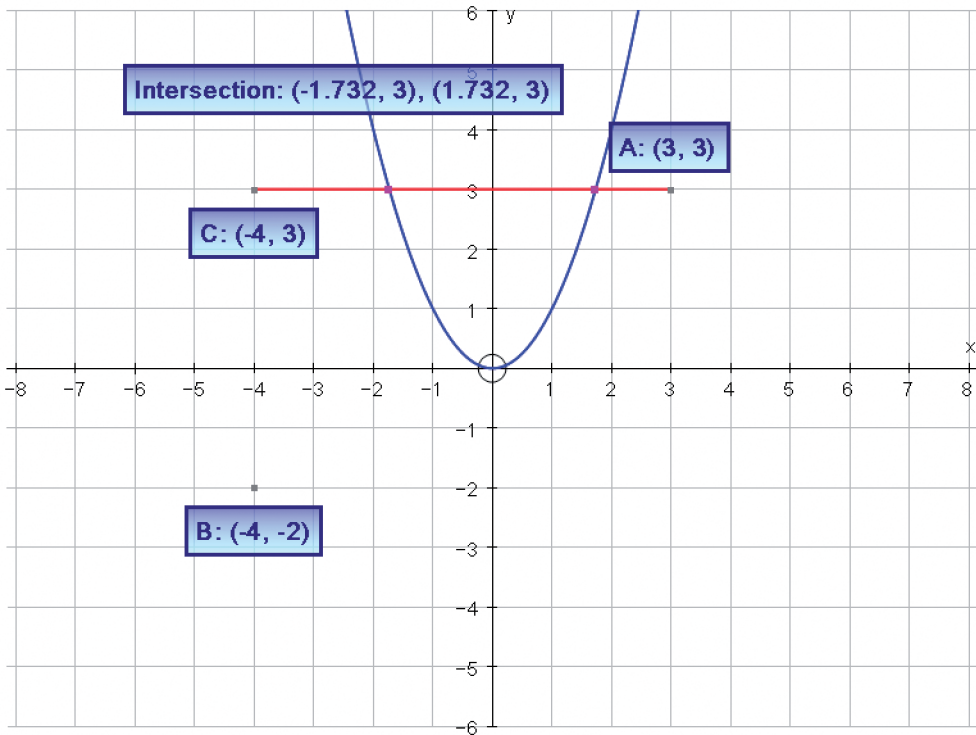
Note: The **onscreen keyboard** is extremely useful as it can be used with any other application, not just Autograph, and the maths symbols in the Extra panel will also work in other applications. Only a couple of characters are exclusive to Autograph font.

The more you use Autograph, the more little short-cuts you find to make life even easier for yourself. Here is a collection of some of my favourite handy hints and tips.

KEYBOARD SHORTCUTS

The following keyboard shortcuts are particularly useful when entering equations, writing in Text Boxes, or defining functions:

π	Alt-p	σ	Alt-s	x^2	Alt-2 or xx
√	Alt-r	µ	Alt-m	x^n	Alt-n
||	Alt-l (modulus)	α	Alt-a	x^x	Alt-x
θ	Alt-t	β	Alt-b	½	Alt-h

Spacebar pauses and restarts during **Slow Plot** mode.

Enter takes you straight from the graph area to the **Enter Equation** screen.

THE USE OF UNDO

Undo is not only useful for correcting mistakes, and returning to the original view of a graph following several zooms and drags, it can also be used as a means of revealing a pre-prepared set of objects. Simply prepare the page how you want, delete the objects you don't want the class to see, and then whenever you are ready for them to see them, just hit Undo!

MOVING POINTS

If you select a point and use the left-right arrows on the keyboard to move it, then by default Autograph will move the point 1/10th of a grid space. But if you want to move it more (or less) then try this:

If you hold **Shift** down whilst pressing the arrows, the point moves a whole grid spacing.

If you hold down **Ctrl** whilst pressing the arrows, the point moves by 1/100th of a grid spacing.

MANAGE EQUATION LIST

If you have two or more lines drawn on top of each other, and you want to be sure you have the selected the equation you want, just click on **Manage Equation List**, and you will be able to access all equations for editing or deleting

from here.

Note: Double **left-clicking** on the "top" equation in the graph area should also automatically select the one "below" it, but the above method is fail-safe!

The Marque Select Tool

When in **Select Mode**, you can drag a rectangular shape across the screen, and any objects contained within this rectangle will automatically become selected.

Selecting All Objects

If you wish to select the vast majority of objects on a screen, it can often be quicker to select all the objects at once, and then de-select a few of those.

In the **Edit** menu, go to **Select All**.

Now just left-click on any objects you don't want to select to de-select them!

Note: If you are NOT in **Whiteboard Mode**, you must hold down **Shift** whilst de-selecting objects.

Viewing in 3D

When working in 3D, the **Drag** function is very handy for having a look around your shapes. But this function is even better when combined with a few buttons:

+ **Ctrl** Zooms in and out

+ **Shift** Shifts the camera left and right

And remember, you can restore the original view of your 3D page with one click of a button (**x-y-z Orientation**).

Putting Hide Objects onto the Toolbar

Hide Object and **Reveal Object** are two extremely useful functions in Autograph, and if you follow these steps you can have these options just one click away on your toolbar:

Enter the equation y **=** x

Left-click to select the line (it should turn black).

Right-click in the black space to the right of the top toolbar and a small menu should appear.

From this menu, select **Customise…**

With the **Customise** window still open, click on the **Object** menu from the top toolbar, and **Hide Object** should be one of the options.

You should now be able to drag **Hide Object** to the top toolbar, where it will remain as an option.

You can add **Reveal Object** to the toolbar (or indeed any function) in exactly the same way.

The Autograph Keyboard

Not only is the keyboard incredibly useful to use when running Autograph on an interactive whiteboard or via a graphics tablet, it can also be used in other applications such as Word, Excel or even when writing an email. So, your days of searching for that elusive theta or fractional notation may be over!

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

To be able to recognise and sketch each of the six types of transformations:

$y = f(x) + a$, $y = f(x + a)$, $y = -f(x)$, $y = f(-x)$, $y = af(x)$, $y = f(ax)$

To be able to deduce the equation of a function given knowledge about the transformation that created it.

EXTENSION MATERIAL:

To be able to combine transformations to arrive at given graphs.

REQUIRED PRIOR KNOWLEDGE

- Be able to recognise the shapes of quadratic and cubic graphs, as well as those of sine and cosine (in **Degrees** only).

- Be familiar with the f(x) notation.

- Understand the difference between a constant and a variable.

DESCRIPTION OF THE ACTIVITY

- Students will be introduced in turn to each of the six types of transformations. They will be able to make predictions, and then use Autograph's **Constant Controller** to easily manipulate constants and observe the effect it has on the shape of the graph, which should deepen their level of understanding.

- Once they understand each type of transformation, they will be required both to sketch and recognise examples, as well as deducing the equation of a function given the transformation.

- As a by–product of this activity, students will also learn that the sine and cosine graphs are actually transformations of each other, and that some functions remain the same when transformed.

- The extension material begins to look at combining transformations and challenges students to use their knowledge of transformation to reach a target graph by manipulating constants.

PRE–ACTIVITY SETUP

None required.

SUGGESTED INTRODUCTION

"The ability to transform functions, and to spot transformations, is one of the most important concepts you will learn in this course. What you will learn today will be the basis of a lot of the work that we will cover later on, across many different topics. There are six different types of transformations to learn, and a tricky little challenge at the end. If you understand each of the transformations, then you will be able to sketch the graph of any function the examiners could possibly throw at you."

SUGGESTED PROMPTS

- Once the students become familiar with defining functions, entering constants, and then manipulating them with the **Constant Controller**, they shouldn't have too much trouble with the core material of this activity. Therefore, just be on hand to help steer any struggling students in the right direction during the early stages of the activity.

- It is also a good idea to reinforce that the students can easily check their answers to the questions at the bottom of each page by simply typing the equation into Autograph. This will hopefully boost their confidence and enable them to work more independently.

- In terms of the extension material, suggesting to students that they think what effect each constant on its own has on the shape of the graph should be enough to get them on the right track.

SUGGESTED PLENARY

- Can anybody summarise each of the six types of transformations?

- Which do you find hardest to spot?

- Which do you find hardest to remember?

- Has anybody got any tips for sketching the transformations?

- Why are some of the answers to the tasks in this activity non-unique?

IDEAS FOR FURTHER WORK

- Exam style questions on transformations.

- Using their knowledge of transformations to sketch complicated equations. For example, given that they know what $y = x^2$ looks like, can they sketch $y = (x + 3)^2 - 2$?

- Look at whether the order in which multiple transformations are carried out effects the resulting graph.

STUDENT NOTES

NAME:..

ACTIVITY 1: $y = f(x) + a$

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Select **Function Definitions** from the top toolbar.

Define $f(x)$ as x^2, and leave $g(x)$ as 1.

Note: To enter x^2 use the little 2 button, or press "**alt 2**" together, or type "**xx**".

Click on **Slow Plot** mode (there should be a blue square around the button).

Enter the equation: $y = f(x)$.

Predict — *What will the graph of this function look like? Which direction will it slope? Where will it cross the axes?*

Click **OK** and the graph should begin to plot on the screen.

Turn off **Slow Plot** mode by clicking the button again.

Enter another equation: $y = f(x) + a$
Still on the **Enter Equation** screen, click on **Edit Constants**.
Change the value of a to 0 and click **OK**.

Click on the **Constant Controller** on the top toolbar.
The **up-down** arrows alter the value of the constant.
The **left-right** arrows adjust the size of the step.

Predict — *How will altering the value of a affect the graph of the function?*

Use the **Constant Controller** to adjust the values of constant a and observe the effect on the graph of the function.

Open a new **2D Graph Page**.

Make sure you are working in **Degrees**.

Edit the axes as follows:
x: Minimum: −360 Maximum: 360 Numbers: 90 Pips: 30

y: Minimum: **−2**   Maximum: **2**   Numbers: **1**   Pips: **0.25**

Remove all of the green ticks underneath **Auto**.

Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

f(g(

This time define f(x) to be sin(x).

Click on **Slow Plot** mode.

Enter the equation: y = f(x)

Predict

What will the graph of this function look like? Will it pass through the origin? Where will it cross the axes? Where will it reach its maximum and minimums?

Click **OK** and the graph should begin to plot on the screen.

Turn off **Slow Plot** mode by clicking the button again.

Enter another equation: y = f(x) + a

Still on the **Enter Equation** screen, click on **Edit Constants**.

Change the value of a to 0 and click **OK**.

Predict

How will altering the value of a affect the graph of the function?

Use the **Constant Controller** to adjust the values of constant a and observe the effect on the graph of the function.

TASK 1: Briefly describe what effect the value of the constant has on the transformation from y = f(x) to y = f(x) + a. Can you express this as a vector?

TASK 2: The diagram below shows the graph of y = f(x) where $f(x) = x^3 - 2x^2$.

On the same diagram sketch the graph of: y = f(x) − 2
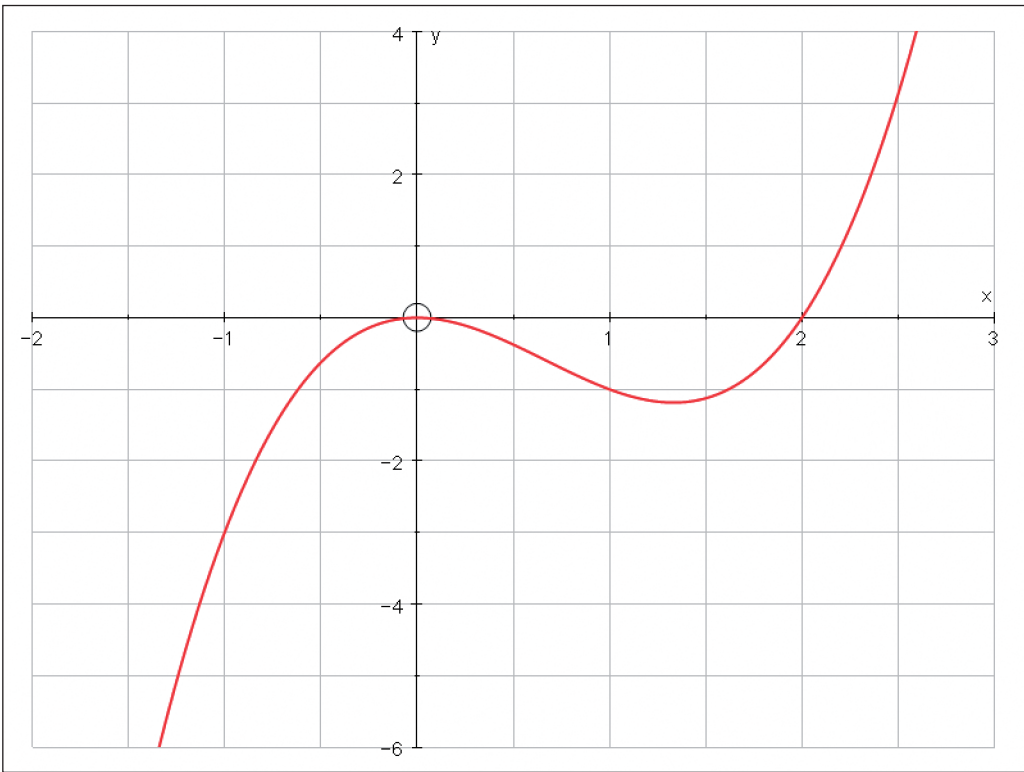
TASK 3: The red line below has the equation: $f(x) = x^3 - 4x^2 - x + 5$

What is the equation of the blue line?

Equation 1: $y = x^3 - 4x^2 - x + 5$
Equation 2: $y = x^3 - 4x^2 - x + 15$
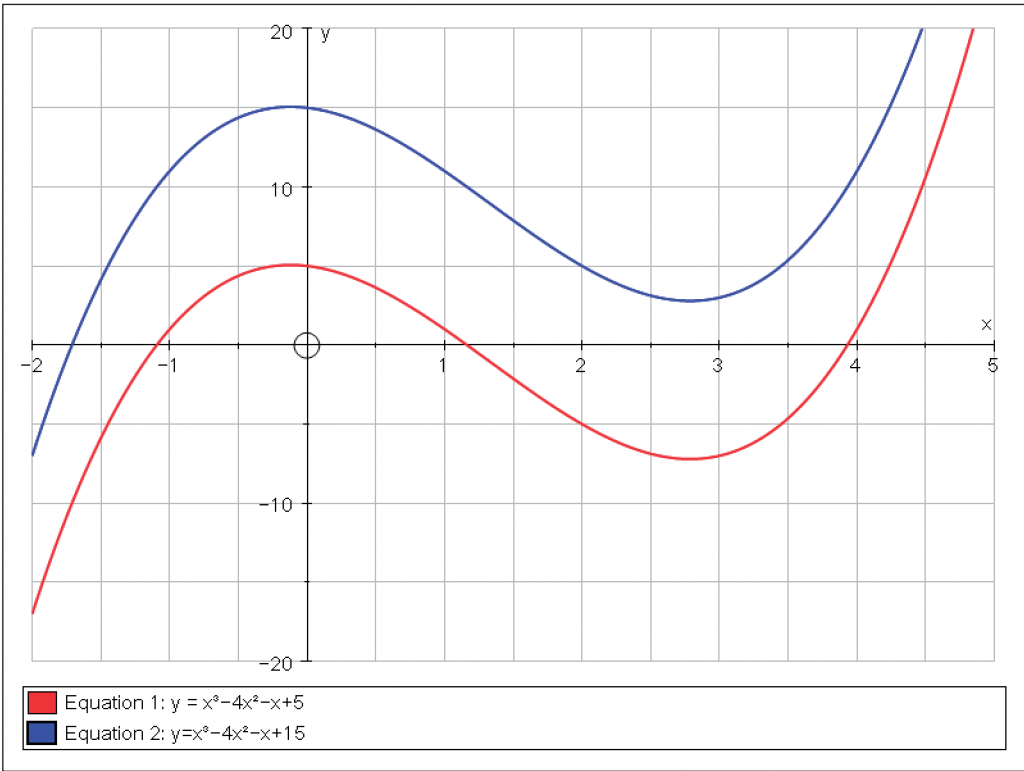
Check your answers by drawing the graphs in Autograph before moving on.

ACTIVITY 2: y = f(x + a)

Open a new **2D Graph Page**.

Select **Function Definitions** from the top toolbar.

Define f(x) as x³, and leave g(x) as 1.
Note: To enter x³ use the little 3 button, or press "**alt 3**" together, or type "**xxx**".

Click on **Slow Plot** mode.

Enter the equation: y = f(x)

Predict

What will the graph of this function look like? Which direction will it slope? Where will it cross the axes?

Click **OK** and the graph should begin to plot on the screen.

Turn off **Slow Plot** mode by clicking the button again.

Enter another equation: y = f(x + a)
Still on the **Enter Equation** screen, click on **Edit Constants**.
Change the value of a to 0 and click **OK**.

Predict

How will altering the value of a affect the graph of the function?

Use the **Constant Controller** to adjust the values of constant a and observe the effect on the graph of the function.

Open a new **2D Graph Page**.

Make sure you are working in **Degrees**.

Edit the axes as follows:
    x: Minimum: −540 Maximum: 540 Numbers: 90   Pips: 30
    y: Minimum: −2   Maximum: 2   Numbers: 1   Pips: 0.25
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

This time define f(x) to be cos(x).

Click on **Slow Plot** mode.

Enter the equation: y = f(x)

Predict

What will the graph of this function look like? Will it pass through the origin? Where will it cross the axes? Where will it reach its maximum and minimums?

Click **OK** and the graph should begin to plot on the screen.

Turn off **Slow Plot** mode by clicking the button again.

Enter another equation: y = f(x + a)

Still on the **Enter Equation** screen, click on **Edit Constants**.

Change the value of a to 0 and click **OK**.

Predict

How will altering the value of a affect the graph of the function?

Use the **Constant Controller** to adjust the values of constant a and observe the effect on the graph of the function.

TASK 4: Briefly describe what effect the value of the constant a has on the transformation from y = f(x) to y = f(x + a). Can you express this as a vector?

TASK 5: The diagram below shows the graph of y = f(x) where f(x) = x² − 4.

On the same diagram sketch the graph of: y = f(x − 3)

On your sketch you should be able to mark exactly where the curve crosses both the x and y axes.
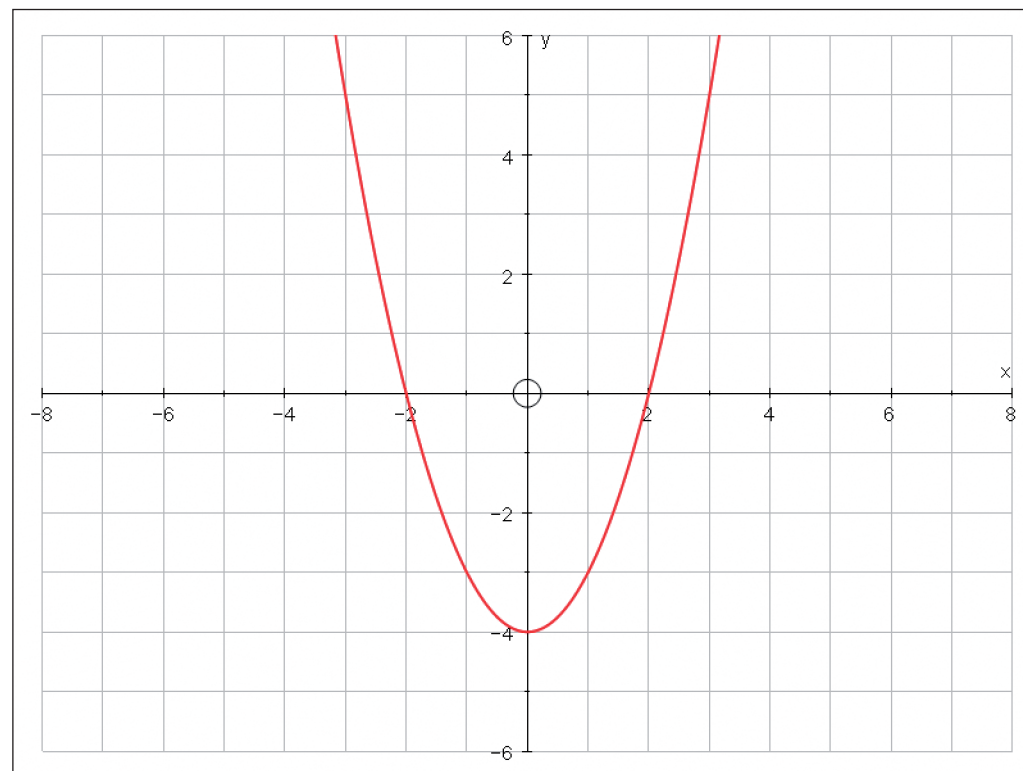
TASK 6: The red line below has the equation: $f(x) = \sin(x)$

Give two possibilities for the equation of the blue line and briefly explain why there are in fact an infinite number of correct answers!

Check your answers by drawing the graphs in Autograph before moving on.

ACTIVITY 3: $y = -f(x)$

Open a new **2D Graph Page**.

Select **Function Definitions** from the top toolbar.
Define $f(x)$ as x^2, and leave $g(x)$ as 1.

Click on **Slow Plot** mode.

Enter the equation: $y = f(x)$ and click **OK**.

Enter another equation: $y = -f(x)$

Predict   *What will the graph of this function look like? Where will it cross the axes?*

Click **OK** and the graph should begin to plot on the screen.

Observe the differences between the two graphs.

Open a new **2D Graph Page**.

This time define $f(x)$ to be $x^3 + 2$.

Enter the equation: $y = f(x)$.

Predict   *What will the graph of this function look like? Where will it cross the axes?*

Click **OK** and the graph should begin to plot on the screen.

Enter another equation: y = −f(x)

What will the graph of this function look like? Where will it cross the axes?

Click **OK** and the graph should begin to plot on the screen.

Observe the differences between the two graphs.

TASK 7: What is the equation of the last curve plotted? ...be careful!!!

Check your answer to Task 7 by drawing the graph in Autograph before moving on.

TASK 8: Describe what effect the negative sign has on the transformation from y = f(x) to y = −f(x).

TASK 9: The diagram below shows the graph of y = f(x) where f(x) = cos(x).

On the same diagram sketch the graph of: y = −f(x)
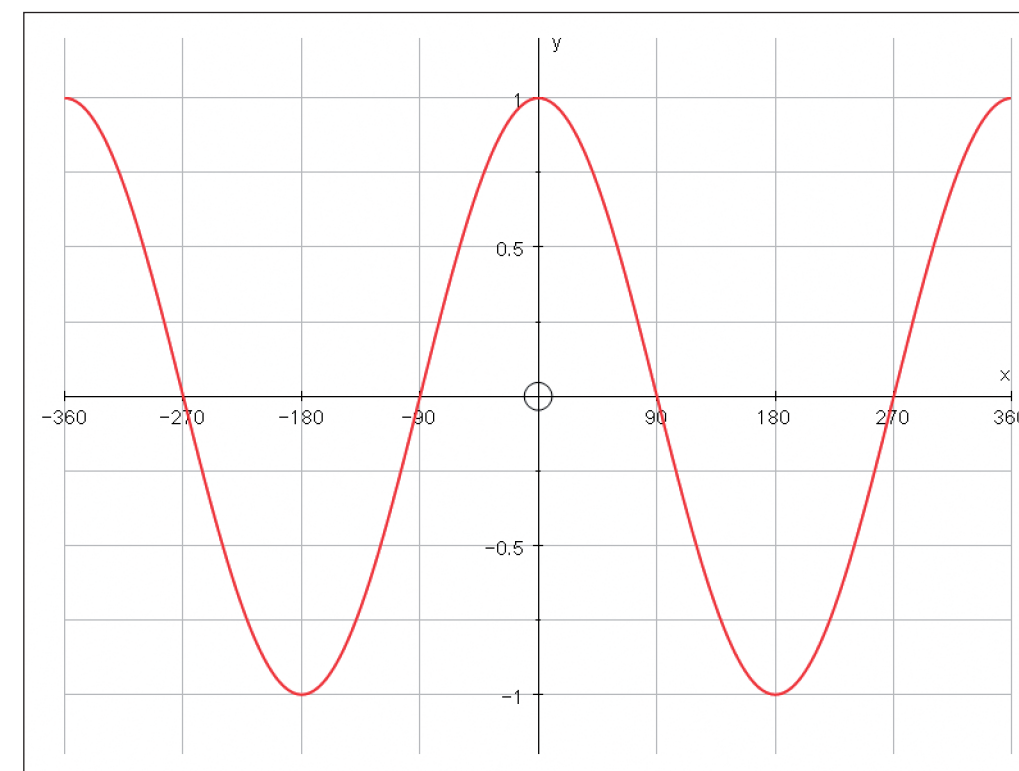
TASK 10: The red line below has the equation: $f(x) = 2x^3 + 3x^2 − 2x + 1$

What is the equation of the blue line?
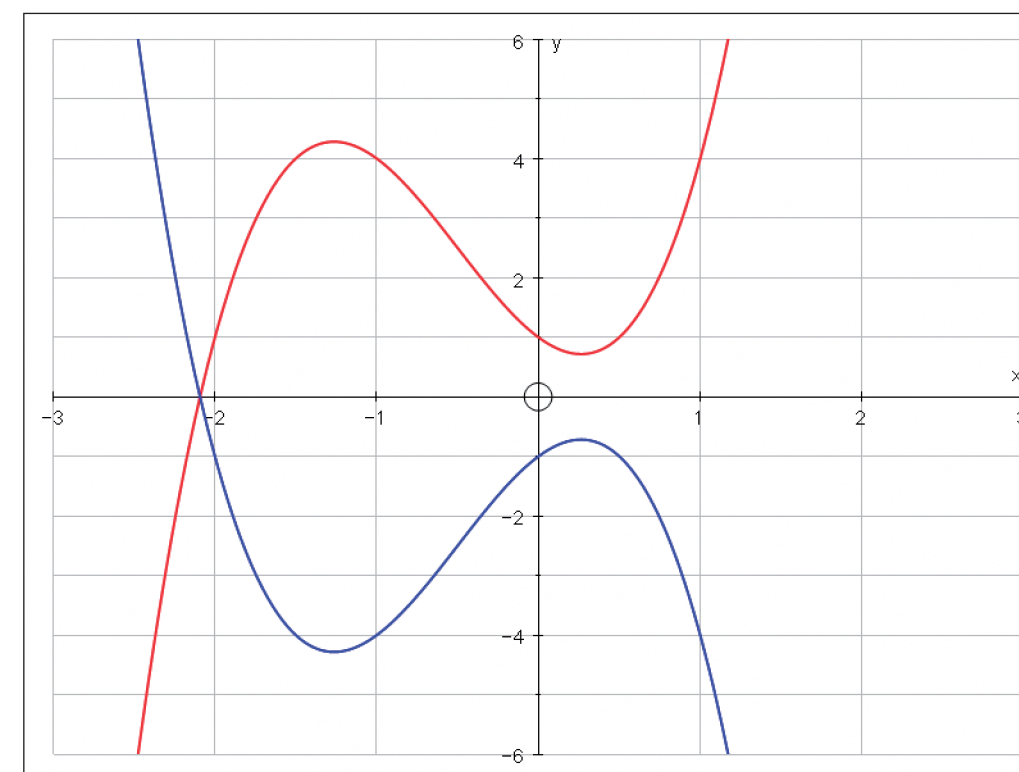
Check your answers by drawing the graphs in Autograph before moving on.

ACTIVITY 4: y = f(−x)

Open a new **2D Graph Page**.

This time define f(x) to be $x^3 - 1$.

Ensure **Slow Plot** mode is still turned on.

Enter the equation: y = f(x).

Predict — *What will the graph of this function look like? Where will it cross the axes?*

Click **OK** and the graph should begin to plot on the screen.

Enter another equation: y = f(−x)

Predict — *What will the graph of this function look like? Where will it cross the axes? Will it be the same as* y = −f(x)?

Observe the differences between the two graphs.

Click **OK** and the graph should begin to plot on the screen.

TASK 11: What is the equation of the last graph plotted? …again, be careful!!! Briefly explain how this would be different to the equation for y = −f(x).

Check your answer to Task 11 by drawing the graph in Autograph before moving on.
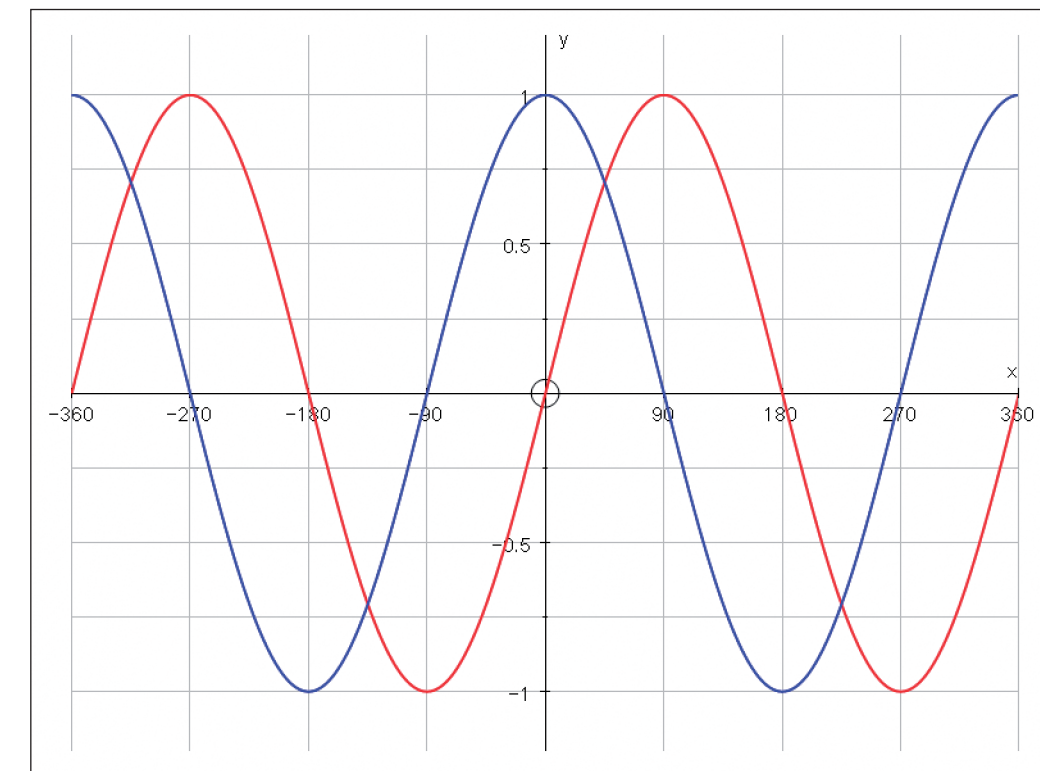
Open a new **2D Graph Page**.

This time define f(x) to be $x^2 - 1$.

Enter the equation: y = f(x).

Predict — *What will the graph of this function look like? Where will it cross the axes?*

Click **OK** and the graph should begin to plot on the screen.

Enter another equation: y = f(−x)

Predict — *What will the graph of this function look like? Where will it cross the axes?*

Click **OK** and the graph should begin to plot on the screen.

TASK 12: Explain why the two graphs look the same. Would this be true of any quadratic function?

Hint: Try defining f(x) as a different quadratic, and then plotting y = f(x) and y = f(−x).

TASK 13: Describe what effect the negative sign has on the transformation from y = f(x) to y = f(−x).

TASK 14: The diagram below shows the graph of y = f(x) where f(x) = $x^3 - 5x^2 - 2x + 2$.

On the same diagram sketch the graph of: y = f(−x) and next to it write its equation.
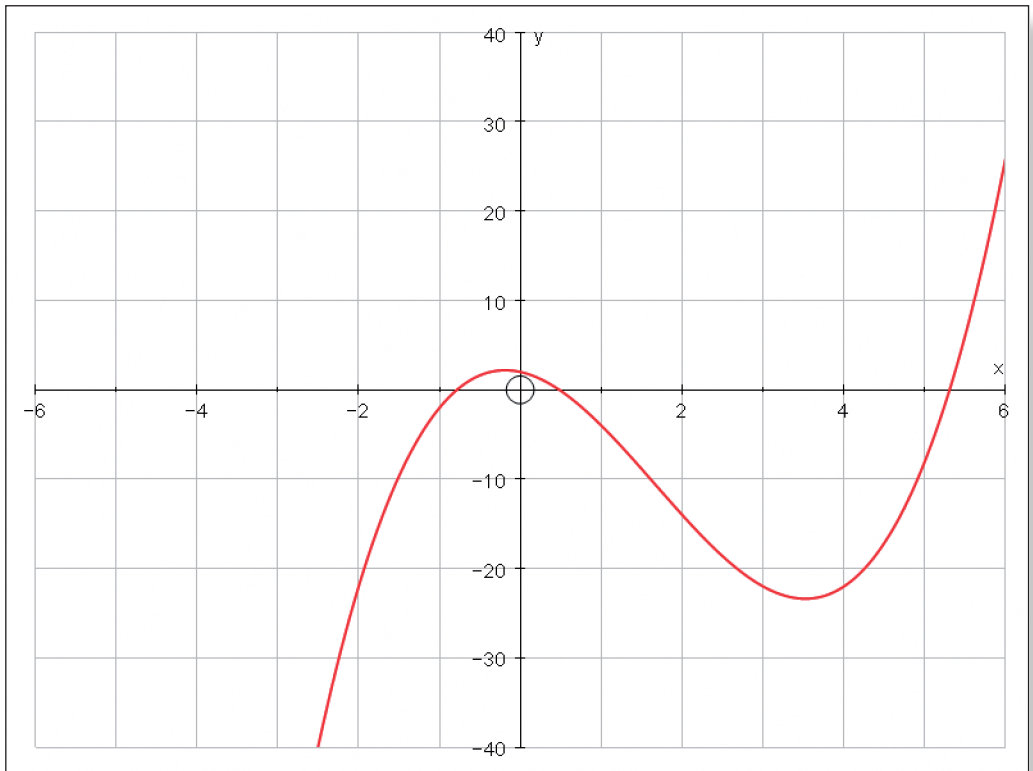
TASK 15: Which of these two pairs of graphs look the same and why?

1. y = sin(x) and y = sin(−x) or:

2. y = cos(x) and y = cos(−x).

Check your answers by drawing the graphs in Autograph before moving on.

ACTIVITY 5: y = af(x)

Open a new **2D Graph Page**.

Make sure you are working in **Degrees**.

Edit the axes as follows:
x: Minimum: −360 Maximum: 360 Numbers: 90   Pips: 30
y: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.5
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

This time define f(x) to be sin(x).

Enter the equation: y = f(x)

Click **OK** and the graph should begin to plot on the screen.

Turn off **Slow Plot** mode by clicking the button.

Enter another equation: y = af(x)
Still on the **Enter Equation** screen, click on **Edit Constants**.
Change the value of a to 1 and click **OK**.

Predict   *How will altering the value of a affect the graph of the function?*

Use the **Constant Controller** to adjust the values of constant a and observe the effect on the graph of the function.

First set the **Step** to 1 and try values of a between 1 and 4.
Then set the **Step** to 0.1 try values of a between 0 and 1.
Note: Don't worry about negatives… for now!

Observe what happens to the graph as you change the value of a.

TASK 16: Describe what effect the value of a has on the transformation from y = f(x) to y = af(x).

TASK 17: The diagram below shows the graph of y = f(x) where f(x) = cos(x).

On the same diagram, sketch and label the graphs of y = ½f(x) and y = 3f(x).

Equation 1: y=f(x)
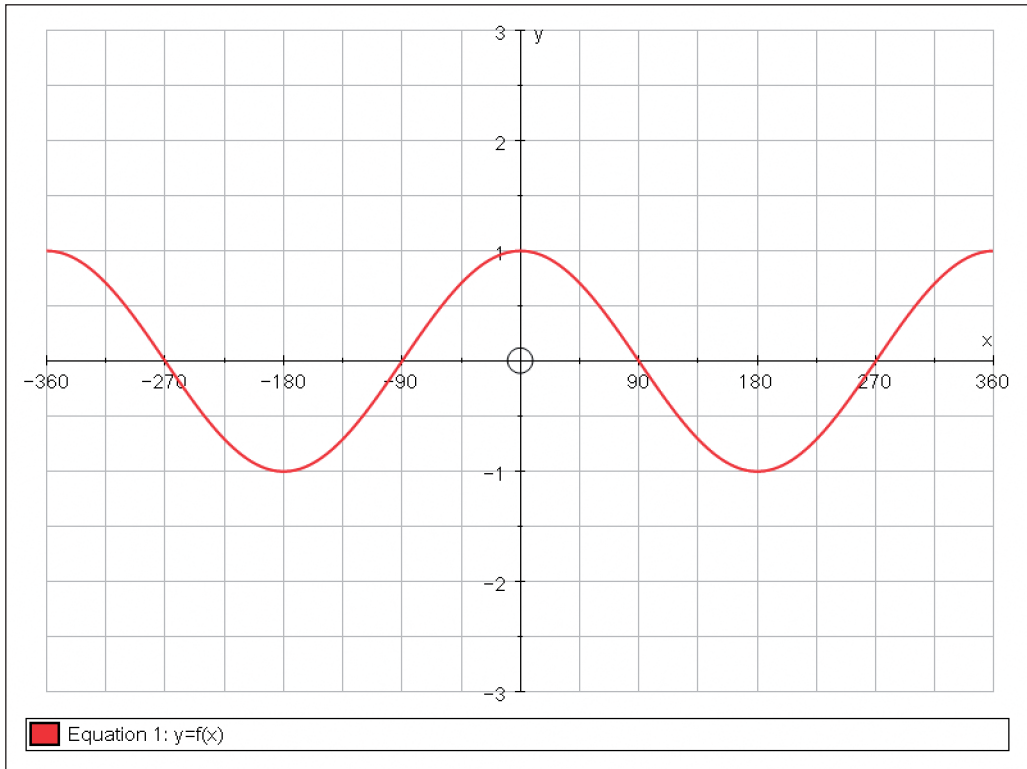
Check your answers by drawing the graphs in Autograph before moving on.

ACTIVITY 6: y = f(ax)

Open a new **2D Graph Page**.

Make sure you are working in degrees

Edit the axes as follows:
x: Minimum: −720 Maximum: 720 Numbers: 180 Pips: 30
y: Minimum: −2   Maximum: 2   Numbers: 1   Pips: 0.25
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

This time define f(x) to be cos(x).

Enter the equation: y = f(x)

Click **OK** and the graph should begin to plot on the screen.

Ensure **Slow Plot** mode is turned off.

Enter another equation: y = f(ax)
Still on the **Enter Equation** screen, click on **Edit Constants**.
Change the value of a to 1 and click **OK**.

Predict

How will altering the value of a affect the graph of the function? Will it be the same as af(x)?

Use the **Constant Controller** to adjust the values of constant a and observe the effect on the graph of the function.

First set the **Step** to 1 and try values of a above 1.
Then set the **Step** to 0.1 and try values of a between 0 and 1.
Note: Don't worry about negatives… for now!

Observe what happens to the graph as you change the value of a.

TASK 18: Describe what effect the value of a has on the transformation from y = f(x) to y = af(x).

TASK 19: The diagram below shows the graph of y = f(x) where f(x) = sin(x).

On the same diagram, sketch and label the graphs of: y = f(3x) and y = f(½x).

Equation 1: y=sin(x)
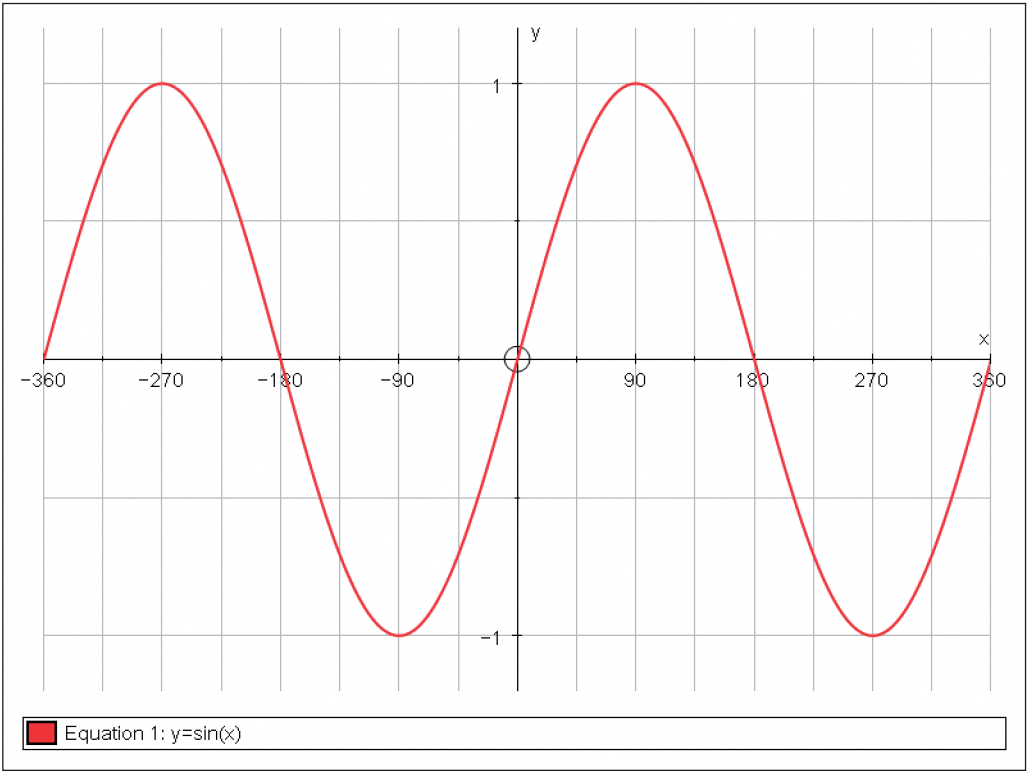
Check your answers by drawing the graphs in Autograph before moving on.

ACTIVITY 7: PUTTING IT ALL TOGETHER

For each of these questions you begin by following the instructions below to create the red graph on the left. You must then use your knowledge of the six different types of transformations covered in this investigation to change the values of a, b, c and d in order to make the blue graph on the right. Once you know the values of a, b, c and d, you can use them to write down the equation of the transformed graph.

Follow these instructions before you begin each task:

Open a new **2D Graph Page**.

Make sure you are working in **Degrees**.

Set the axes to be whatever it says underneath the left hand graph.

Define f(x) to be whatever it says underneath the left hand graph.

Enter the equation: y = f(x)

Enter another equation, being very careful of the brackets: y = af(b(x + c)) + d

Left-click this new graph (it should turn black), select **Text Box** from the tool-bar.
Tick the box that says **Show Detailed Object Text**, then click **OK.**

Click on the **Constant Controller**.

Begin changing your values of **a**, **b**, **c** and **d** to try and make the graph on the right.

Task 20:
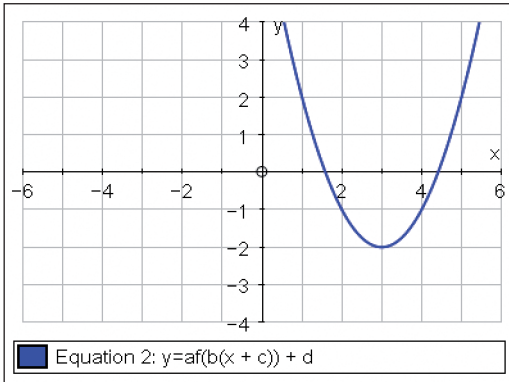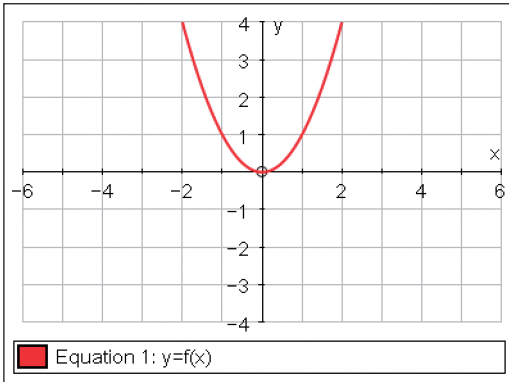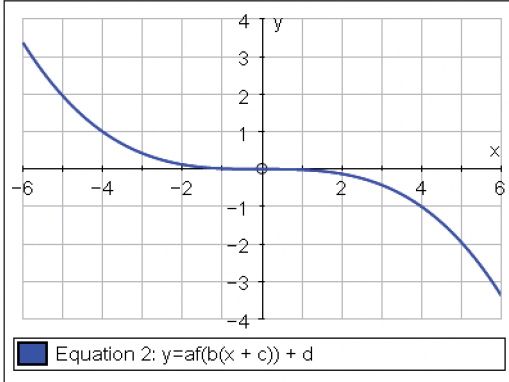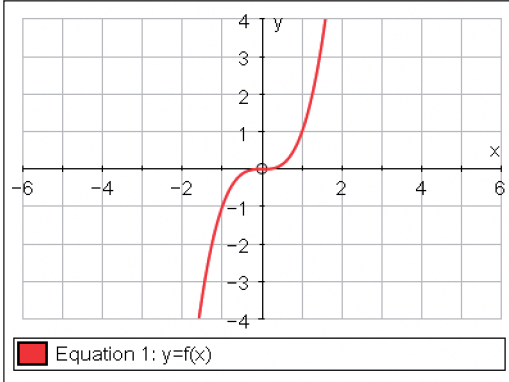
Equation 1: y=f(x)

Equation 2: y=af(b(x + c)) + d

$f(x) = x^2$

x range: −6 to 6   y range: −4 to 4

a = ☐   b = ☐   c = ☐   d = ☐

Equation: ☐

Task 21:

Equation 1: y=f(x)

Equation 2: y=af(b(x + c)) + d

$f(x) = x^3$

x range: −6 to 6   y range: −4 to 4

a = ☐   b = ☐   c = ☐   d = ☐

Equation: ☐

Task 22:

Equation 1: y=f(x)

Equation 2: y=af(b(x + c)) + d

$f(x) = \sin(x)$

x range: −360 to 360   y range: −3 to 3

a = ☐   b = ☐   c = ☐   d = ☐

Equation: ☐

Task 23:

Equation 1: y=f(x)

Equation 2: y=af(b(x + c)) + d

$f(x) = \cos(x)$
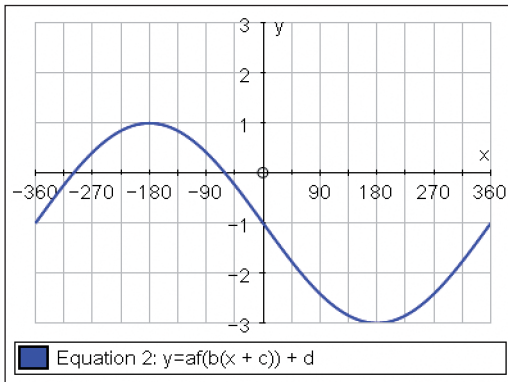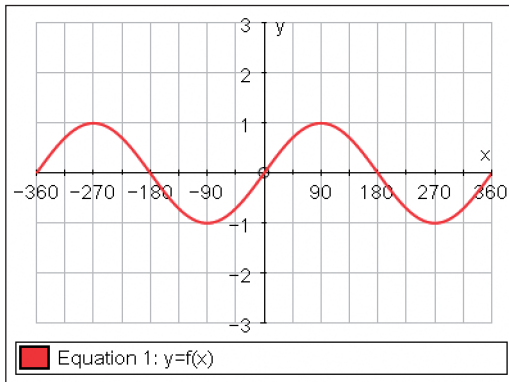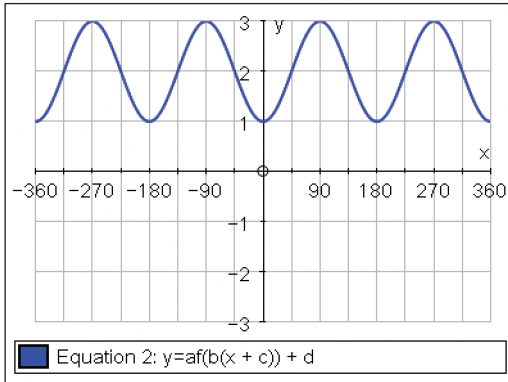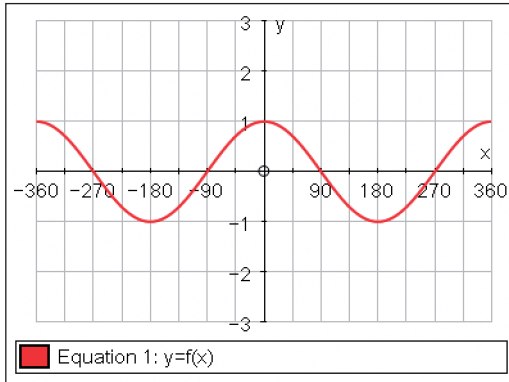
x range: −360 to 360   y range: −3 to 3

a = ☐   b = ☐   c = ☐   d = ☐

Equation: ☐

Summary

Use the space below to summarise what you have learnt during this activity:

ANSWERS

ACTIVITY 1: $y = f(x) + a$

TASK 1: $y = f(x) + a$ is a vertical translation of $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

TASK 2:

Equation 1: $y = x^3 - 2x^2$
Equation 2: $y = x^3 - 2x^2 - 2$
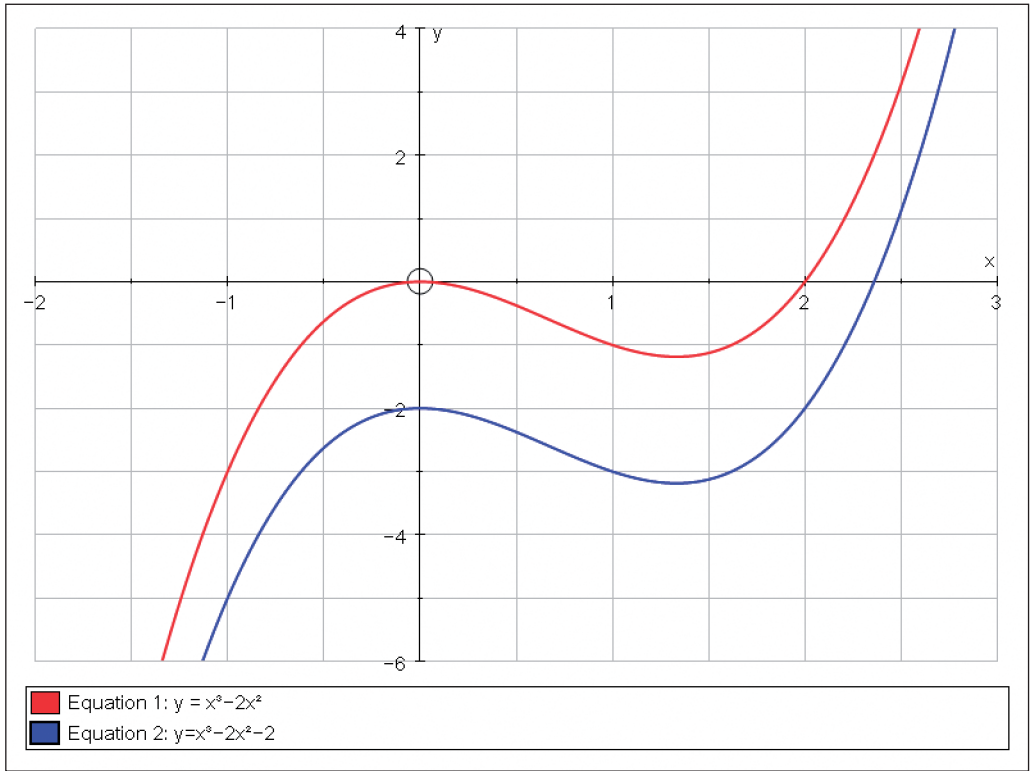
TASK 3: The equation of the blue line is: $y = x^3 - 4x^2 - x + 15$

ACTIVITY 2: $y = f(x + a)$

TASK 4: $y = f(x + a)$ is a horizontal translation of $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

TASK 5:
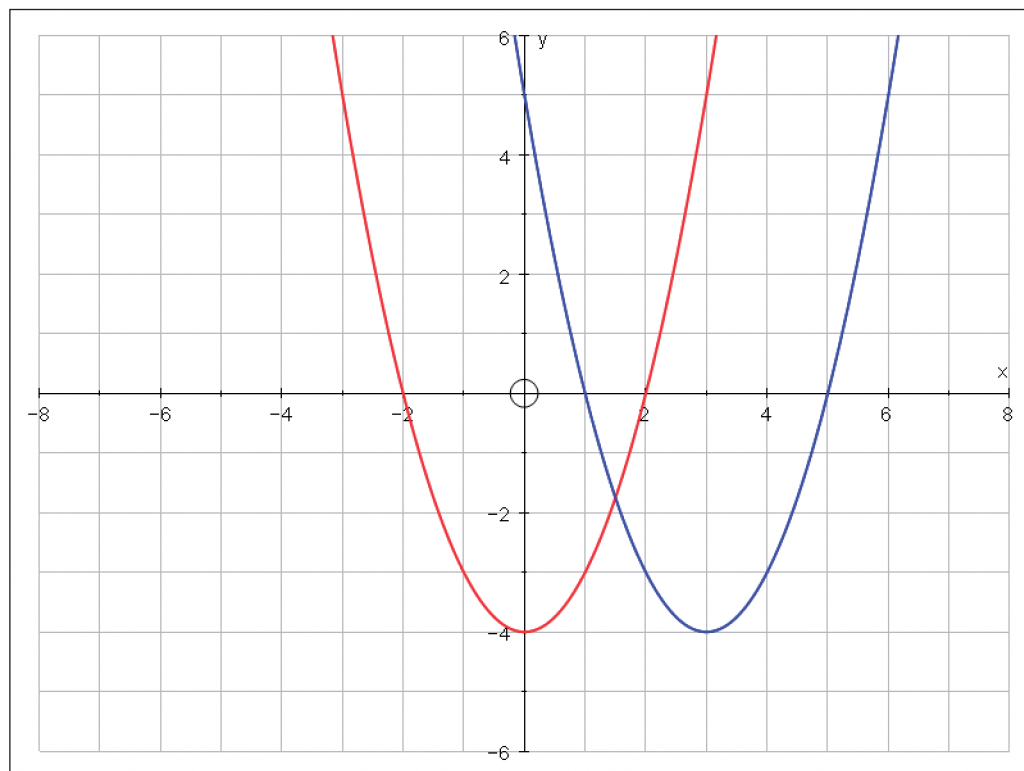
TASK 6: Two possibilities for the equation of the blue line are: y = sin(x + 90) or y = cos(x). y = sin (x − 270) and y = sin (x + 450) are probably the most common alternatives given. This reveals that the sine and cosine graphs are just horizontal translations of each other, and also illustrates the important identities: cos(x) = sin(x + 90) and sin(x) = cos(90 − x).

As with all trigonometrically equations there are infinite correct alternatives as the curves repeat themselves every 360 degrees.

ACTIVITY 3: y = −f(x)

TASK 7: The equation of the graph is y = −x³ − 2 and *not* y = −x³ + 2 as the whole of the expression is made negative.

TASK 8: y = −f(x) is a reflection in the x-axis of y = f(x).

TASK 9:

Equation 1: y=cos(x)
Equation 2: y = −cos(x)
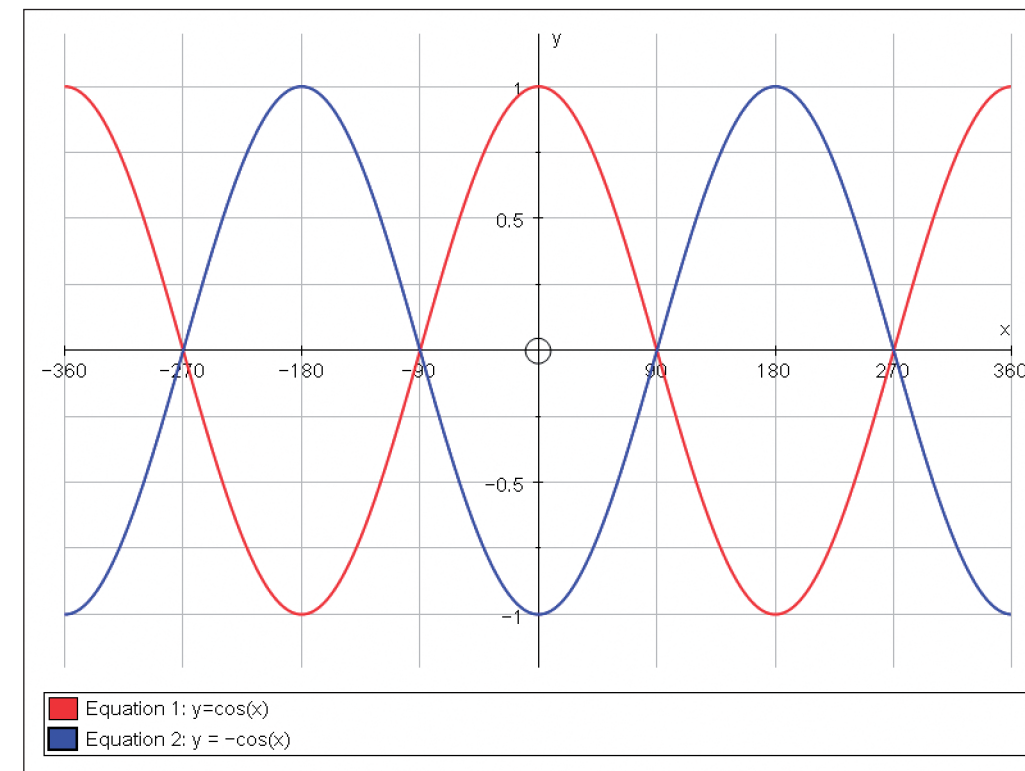
TASK 10: The equation of the blue line is: y = −2x³ − 3x² + 2x − 1 or simply y = −(2x³ + 3x² − 2x + 1).

ACTIVITY 4: y = f(−x)

TASK 11: The equation of the graph is y = −x³ − 1 and *not* y = −x³ + 1 as only the x term is made negative.

TASK 12: The x term is made negative, but then squared, so it becomes positive again, and hence the equation is exactly the same, and so the graphs are identical. This only works for quadratics which do not have an x term. For example if f(x) = x² + 3x − 2, then f(−x) = x² − 3x − 2.

TASK 13: y = −f(x) is a reflection in the y-axis of y = f(x).

TASK 14: Equation of f(−x): y = −x³ − 5x² + 2x + 2

Equation 1: y=x³−5x²−2x+2
Equation 2: y=−x³−5x²+2x+2
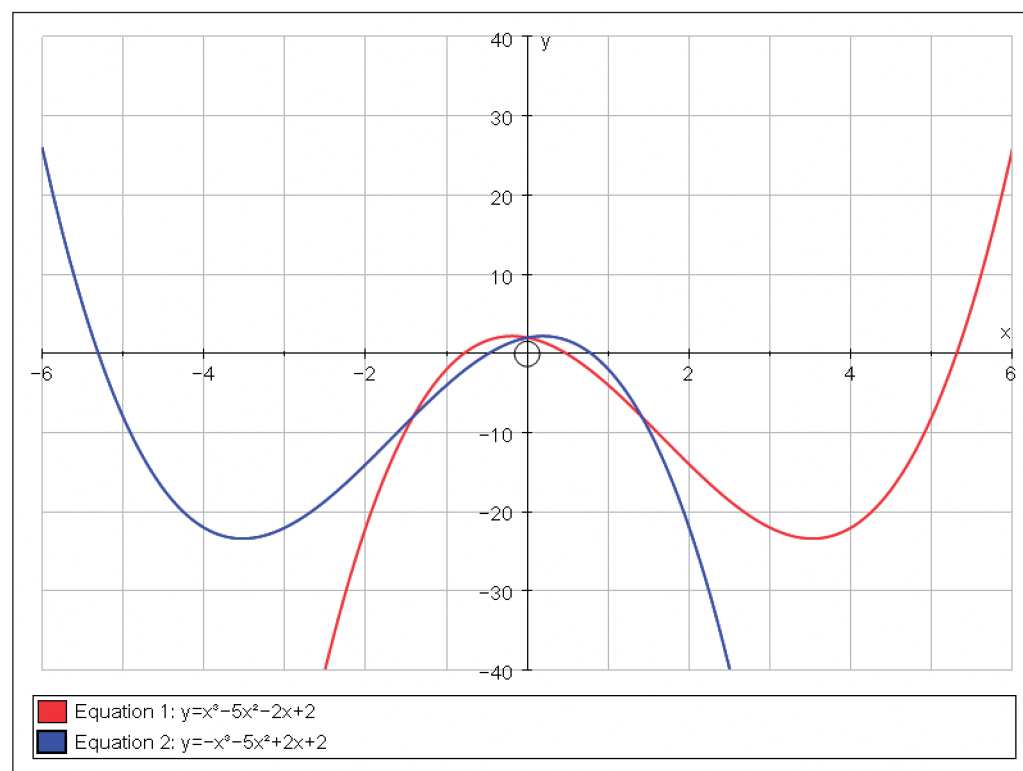
TASK 15: y = cos(x) and y = cos(−x) look the same because the graph of y = cos x is symmetrical about the y-axis. This is the reason why the cosine of positive angles is the same as their negative equivalents. For example, cos(60) = 0.5 and cos(−60) = 0.5.

ACTIVITY 5: y = af(x)

TASK 16: y = af(x) is a stretch in the y direction of scale factor a of y = f(x).

TASK 17:

Equation 1: y=f(x)
Equation 2: y=½f(x)
Equation 3: y=3f(x)
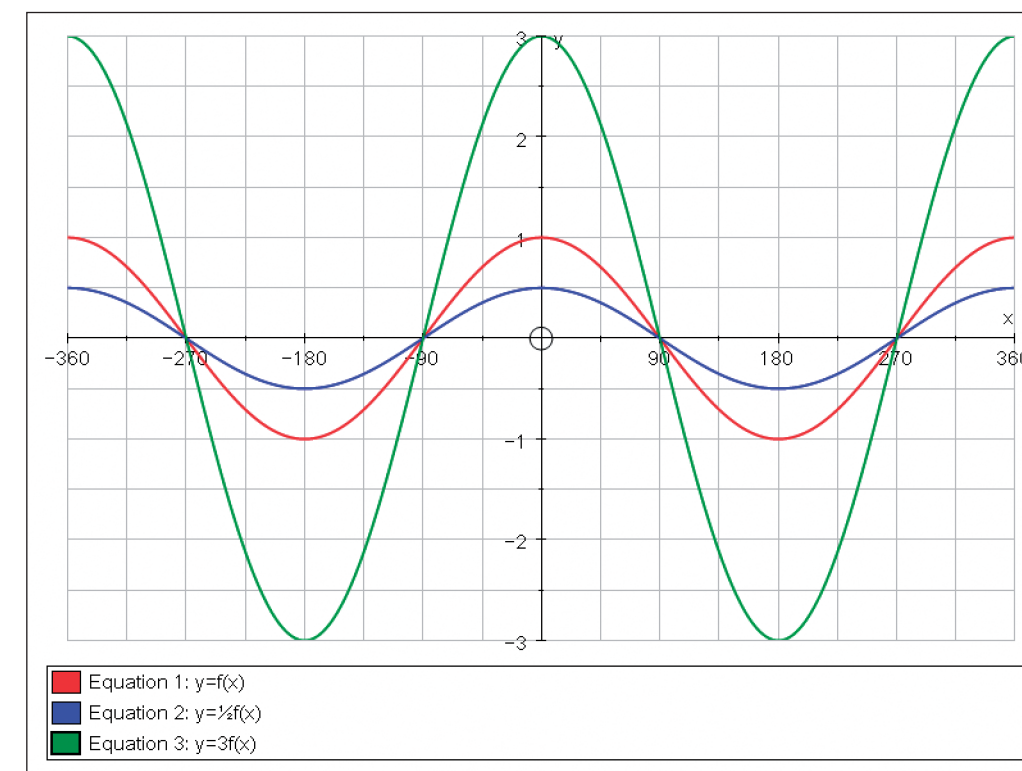
ACTIVITY 6: y = f(ax)

TASK 18: y = f(ax) is a stretch in the x direction of scale factor 1/a of y = f(x).

TASK 19:

Equation 1: y=sin(x)
Equation 2: y=sin(3x)
Equation 3: y=sin(½x)
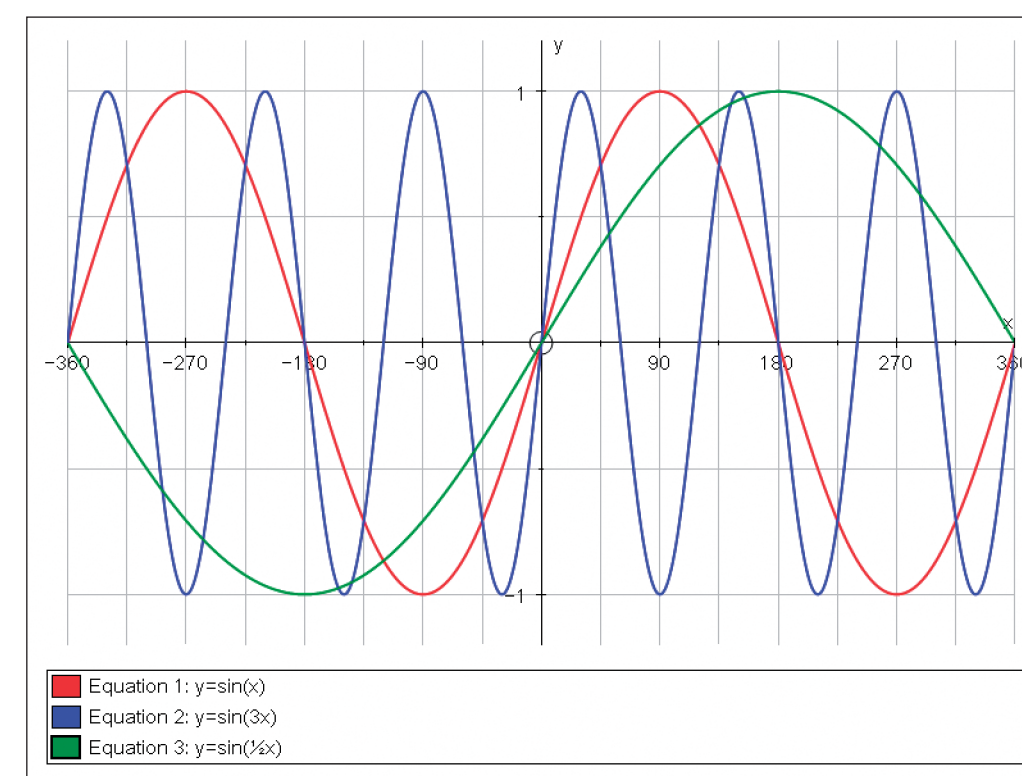
ACTIVITY 7: Putting it all Together

Note: As with all trigonometric graphs, there are an infinite number of correct alternatives. This provides a nice discussion point about non-uniqueness. The students (or the teacher) can quickly verify each one by typing the equation into Autograph.

TASK 20:

Equation 1: y=f(x)
Equation 2: y=af(b(x + c)) + d
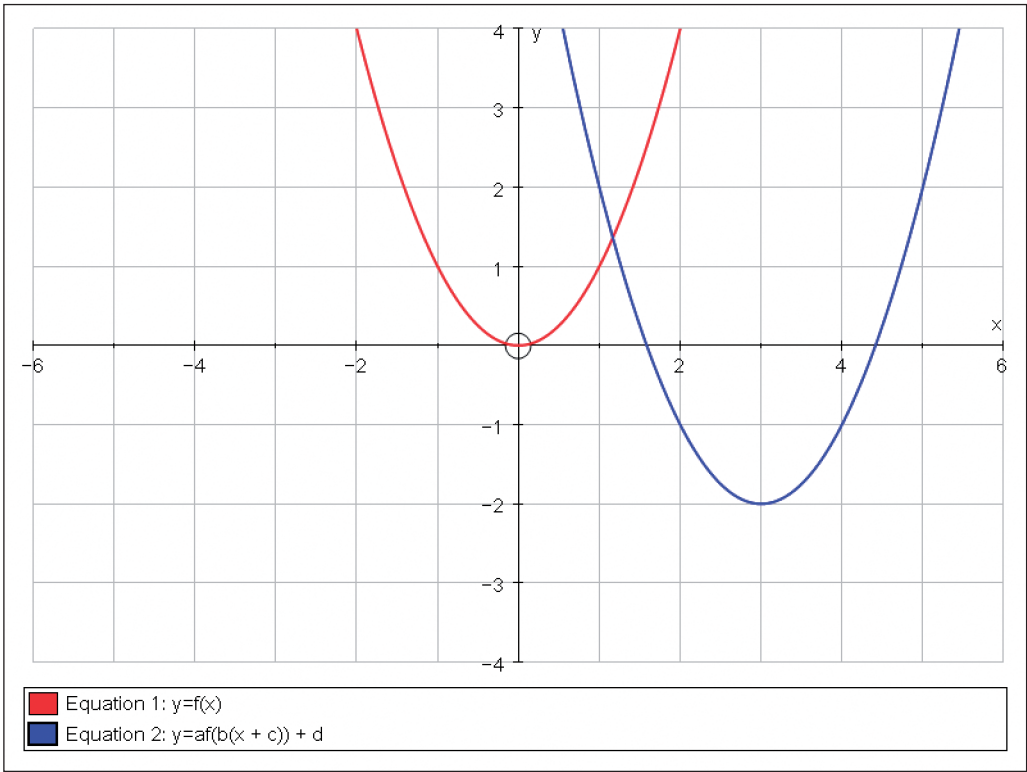
a = 1   b = 1   c = −3   d = −2
Equation: y = (x − 3)² − 2

TASK 21:

Equation 1: y=f(x)
Equation 2: y=af(b(x + c)) + d
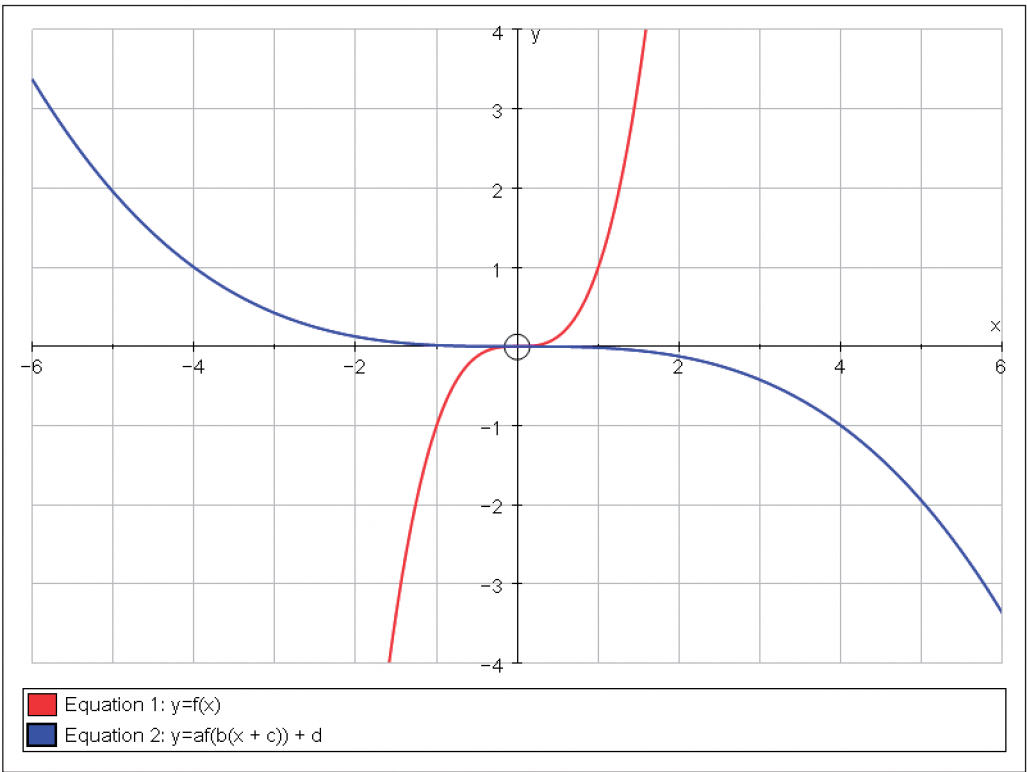
a = −1   b = 0.25   c = 0   d = 0
Equation: y = −0.25x³

TASK 22:

Equation 1: y=f(x)
Equation 2: y=af(b(x + c)) + d

a = −2   b = 0.5   c = 0   d = −1
Equation: y = −2sin(0.5x) − 1

TASK 23:

Equation 1: y=f(x)
Equation 2: y=af(b(x + c)) + d
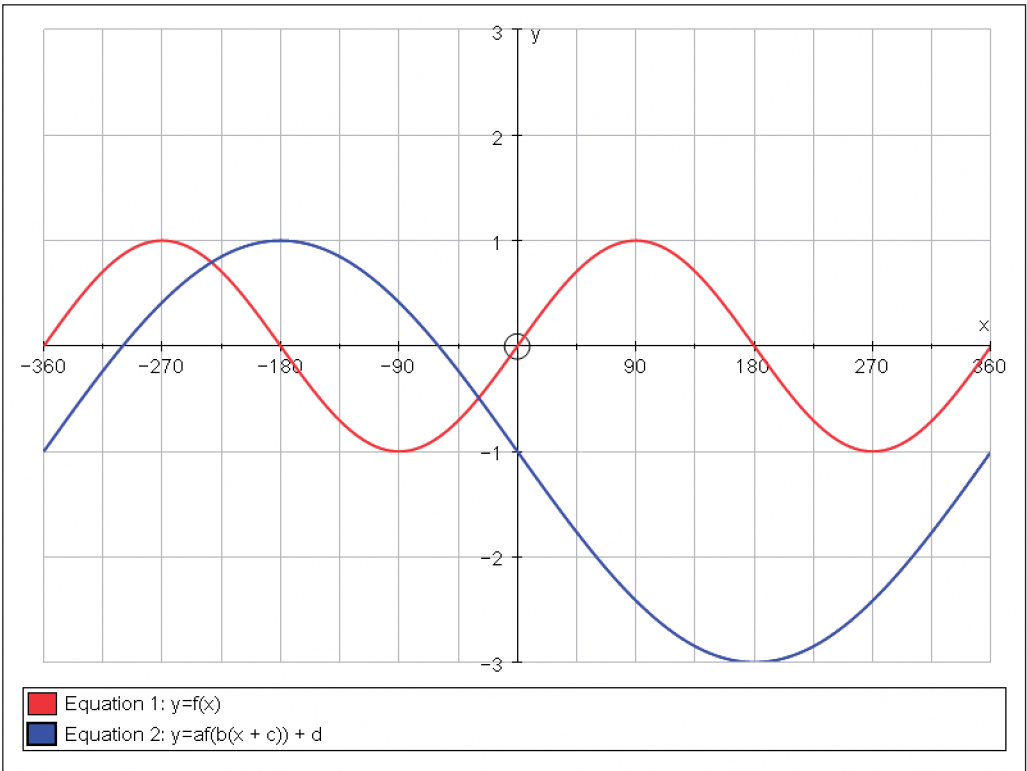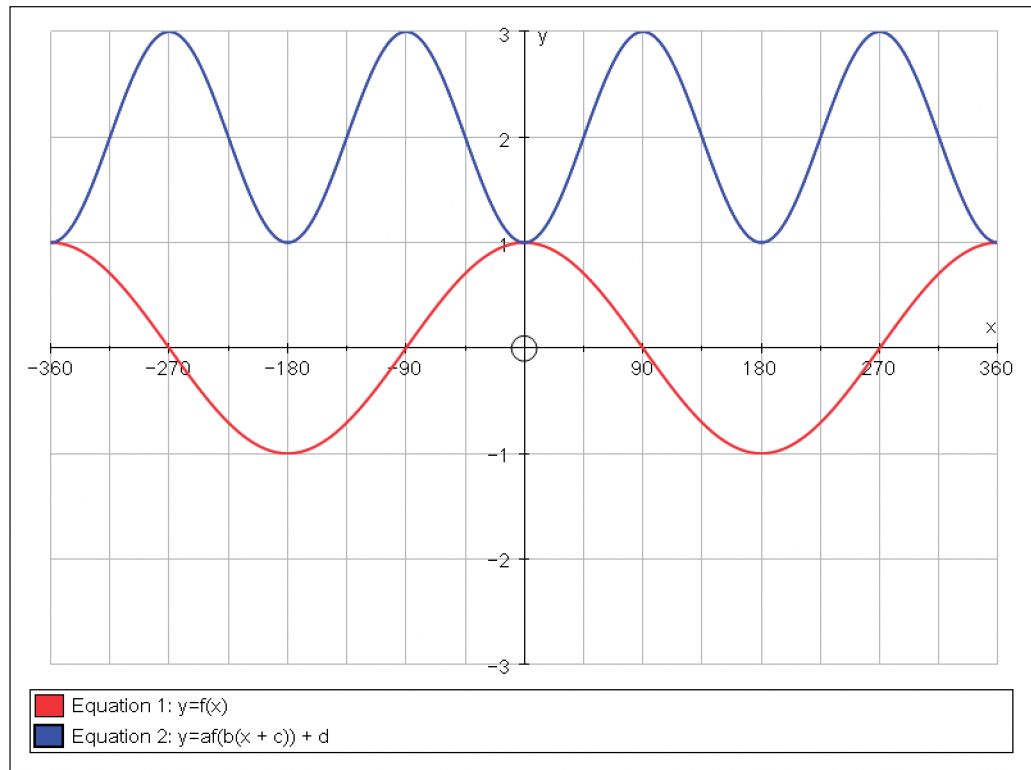
a = 1   b = 2   c = −90   d = 2
Equation: y = cos(2(x − 90)) + 2 or y = cos(2x − 180) + 2

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

- To consolidate and deepen understanding of the following skills and concepts:

  » To find the area bounded by a portion of a curve and the x-axis, and a portion of a curve and the y-axis.

  » To find the volumes of the solids produced by rotating these areas 360 **Degrees** about the two axes.

- To be able to predict the shape of the three-dimensional solid formed and then dynamically test these predictions.

- To be able to generalise the results from volume of revolution questions to derive the formulae for the volume of a cone and a sphere.

- To deepen understanding of the shapes of functions.

EXTENSION MATERIAL:

- To find the area formed between the intersections of two curves.

- To find the volume of the solid produced by rotating this area 360 **Degrees** about the x-axis.

- To use this result to find the volume of a solid rotated around the line y = − ½.

REQUIRED PRIOR KNOWLEDGE

- To be able to use integration to find the area bounded by a portion of a curve and the two axes, as well as between two curves.

- To be able to find the volume of the solids formed by rotating these areas 360 **Degrees** about the axes.

- To be comfortable with the following notation: $V = \pi \int_a^b y^2 \, dx$

- To understand the concept of a constant.

- To be able to re-arrange equations to make x the subject.

- Some knowledge of transformations would be useful, but is not essential.

Note: Teacher Demonstration 13: Volumes of Revolution Introduction pro-

vides a solid foundation from which to tackle this investigation.

Description of the Activity

- This activity should provide an opportunity for students to consolidate and deepen their understanding of both integration and volumes of revolution. The investigation is designed to be both dynamic and interactive, so students can really get a feel or the three-dimensional solids formed and how they relate to the original functions. Throughout all the activities the students are challenged to predict the shape of the solids formed, and then are able to use Autograph's unique 3D interface to investigate the exact nature of these solids. This should be both challenging and engaging.

- In Activity 1, students are presented with a straight line and asked to find the area bounded by a portion of this line and the x-axis, and then the volume of this area when it is rotated 360 **Degrees** about the x-axis. The activity then encourages the students to manipulate the constants in the straight line in order to find the volume of a cylinder, and eventually find the general formula for the volume of a cone.

- In Activity 2, students are challenged to find the volume of the solid produced by rotating a curve about the x-axis. This activity builds up to finding the general formula for the volume of a sphere.

- Activity 3 challenges the student to find the area bounded by a portion of a curve and the y-axis, and then the volume of this area when it is rotated 360 **Degrees** about the y-axis.

- Activity 4 contains the extension material. Here, students are asked to find the area bounded by two curves, and then the volume of the solid produced when this area is rotated about the x-axis. The final question challenges them to use this knowledge to find the volume of a solid rotated about the line y = −½.

Pre–Activity Setup

None required.

Suggested Introduction

"Today we are going to enter the world of the third dimension, and take a proper look at some of the solids formed when portions of curves are rotated about both sets of axes. Hopefully, by the end of this investigation you will have a better understanding of these solids and how they relate to the original functions."

Suggested Prompts

- The entire investigation is built around Autograph's 3D interface, which may be unfamiliar to students. If they carefully follow the instructions,

they should be fine, but it might be a good idea to run through the activity yourself to make sure you are on hand to sort out any difficulties.

- One specific point worth emphasising is that students must ensure they return to select mode after having zoomed in and used the **Drag** tool to have a good look around a solid. This is stressed in the instructions, but is often missed by eager students!

- The students can check the answers to most of the tasks by using Autograph when prompted, and it is often worthwhile reminding them of this fact before they ask you whether they are right or wrong.

- If students are struggling with the techniques of finding areas underneath curves, or the volume of revolution, then it might be a good idea to have a couple of worked solutions to different examples visible at the front of the class.

- Activity 4 is quite challenging, especially the last tasks, and it is up to you how much guidance you give your students. Mentioning translations and encouraging them to alter the equations of the line and curve should be enough to set them on their way.

Suggested Plenary

- Can anybody summarise the key points they have learnt from this investigation?

- What are some of the things we need to look out for when finding both the area underneath a curve and the volume of revolution?

- Which type of functions give solid 3D shapes when rotated about the axes, and which give hollow ones?

- Has anybody got any top tips, or way they have found successful, to help us avoid tripping up with work on integration and volume of revolution?

Ideas for Further Work

- This investigation is designed to cover all the main areas on this topic, but if students find they were getting stuck on a particular part of the activity, then it might be a good idea to go through some exam questions that cover the same material.

Note: Going through exam questions using Autograph can often help bring the questions to life and further deepen understanding.

STUDENT WORKSHEET

NAME:...

SETTING UP YOUR PAGE

In order to take full advantage of Autograph's unique 3D world, we first need to set up our page correctly. We will do this at the start, and then use the same page for the whole of the investigation.

Open up Autograph in **Advanced Mode**.

Open up a **New 3D Graph Page**.

Make sure you are not **Whiteboard Mode** (there should not be a blue square around the button).

Change the colour of the background to **Medium**.

Change the position of the key to the right-hand side of the screen.

Click on the little arrow next to **x-y-z Orientation**.

Select **x-y Orientation**.

Edit the axes as follows:
- **x:** Minimum: 0   Maximum: 4   Numbers: 1   Pips: 0.2
- **y:** Minimum: −12   Maximum: 12   Numbers: 2   Pips: 1
- **z:** Minimum: −12   Maximum: 12   Numbers: 2   Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Still on the **Edit Axes** screen, click on the **Options** tab.

Under **Axes**, remove the tick next to **Always Outside**.

Click **OK**.

Your screen should hopefully look something like this:
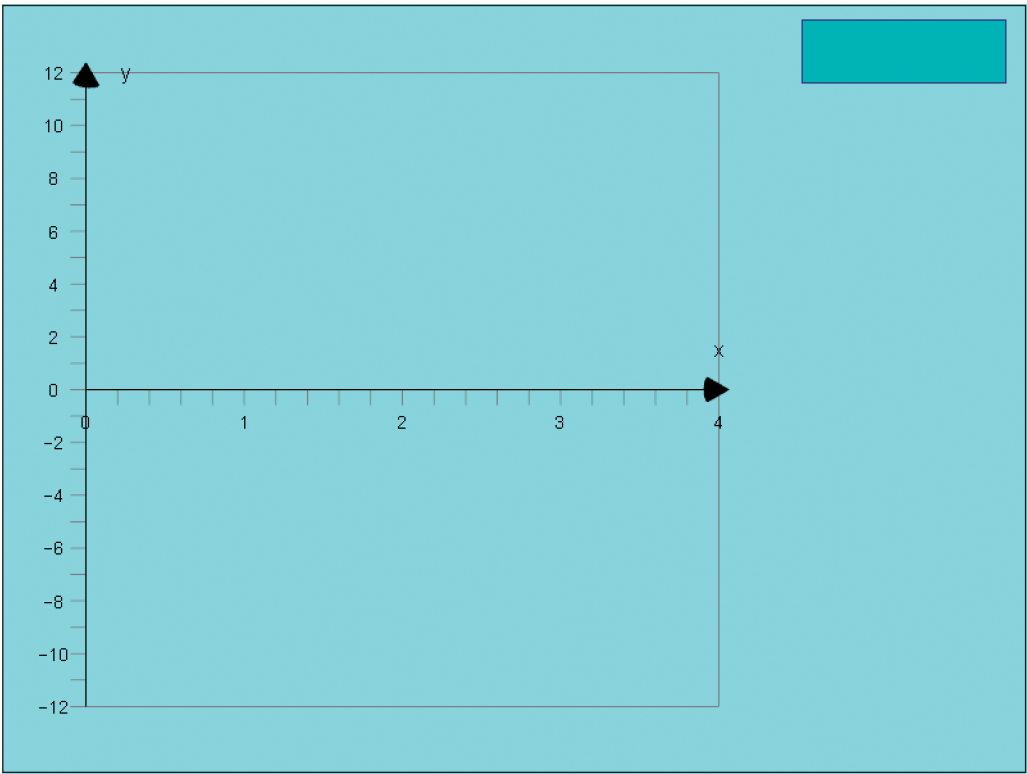
ACTIVITY 1

Enter the equation: y = ax + b

Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D equation**.

Click on **Edit Constants** and change the value of **a** to 3 and the value of **b** to 0.

Predict   *What will this line look like? Where will it cross the axes?*

Click **OK** twice.

TASK 1: Work out the area of the triangle which would be formed by the x-axis, the line y = ax + b, and the line x = 3.

Hint: A = $\frac{bh}{2}$

TASK 2: Use integration to work out the same area.

Hint: A = $\int_0^3 3x \, dx$

Checking your answer:

Left-click on the curve (it should turn black).

Enter a co-ordinate with an x value of 0 (leave z unchanged).

Enter another co-ordinate, this time with an x value of 3 (leave z unchanged).

Two points should now be marked on the curve.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both points (they should both have little squares around them).

Right-click and select **Find Area** from the menu.

Select **Simpsons Rule**, leave the number of divisions at 5, and click **OK**.

In the **View** menu on the top toolbar, select **View Status Box**, and move it to a convenient position on the page.

The area underneath the line should be displayed here.

Check this matches your answers to Task 1 and Task 2 before moving on.

Note: The **Status Box** only ever displays information about objects or areas selected. If the **Status Box** is not displaying the information you require, simply **left-click** to select the objects or area again.

TASK 3: The area enclosed by the line y = ax + b, the x-axis, and the line x = 3 is to be rotated through 360 about the x-axis. Describe the shape of the solid that will be formed.

TASK 4: Find the volume of that solid, leaving your answer in terms of pi (π).

Hint: MATHS

Checking your answer:

Click on **Slow Plot** mode (there should be a blue button square around the button).

Left-click on the area underneath the curve (it should turn grey).

Right-click and select **Find Volume** from the menu, and click **OK**.

The three-dimensional solid should now begin to form.

Use the **Drag** tool to have a good look around the solid.

Note: You can also use the **Drag** tool as the solid is being rotated.

Press **Undo** a couple of times, and repeat the steps above to **View** the solid being formed from a different angle.

Note: The following functions may also prove useful when working with Autograph in 3D:

+**Ctrl**   Zooms in and out

+**Shift**   Shifts the camera left and right

Restores the original view of the graph (**x-y Orientation**).

The volume of the solid should be displayed in the **Status Box**.

If not, simply **left-click** to select the volume (it should turn grey).

Check this matches your answer to Task 4 before moving on.

Select **x-y Orientation** to return to the original **View** of the graph.

Make sure you are in **Select Mode**.

TASK 5: What other values of **a** and **b** would produce a cone with the same volume as this one? Write the values, and the equation of the line these values give us.

Checking your answer:

Click on the **Constant Controller** on the top toolbar.
The **drop-down** menu allows you to select different constants.
The **up-down** arrows alter the value of the selected constant.
The **left-right** arrows adjust the size of the step.

Use the **Constant Controller** to adjust the values of **a** and **b** to check your answer to Task 5.

The volume should be displayed in the **Status Box**.

Use the **Drag** tool to have a good look around the solid

TASK 6: Imagine the value of **a** was 0, and the value of **b** was 3. Describe the solid that would be formed.

TASK 7: Work out the volume of this solid using integration, and leaving your answer in terms of pi (π).

Hint: What is the equation of the line?

Checking your answer:

Use the **Constant Controller** to adjust the values of **a** and **b** to check your answer to Task 7.

The volume should be displayed in the **Status Box**.

Use the **Drag** tool to have a good look around the solid.

Select **x-y Orientation** to return to the original **View** of the graph

Make sure you are in **Select Mode**.

Change the value of **a** back to 3, and the value of **b** to 0.

Left-click on the solid (it should turn white).

Right-click and select **Delete** from the menu, or simply press **delete** on the keyboard.

You should now be left once again with just the area under the line marked on the graph.

TASK 8: Look at the triangle on your screen. Imagine instead that the base of the triangle was not 3, but h, and the height of the triangle was not 9, but r. If this was the case, what would be the equation of the line?

Hint: Think about how to work out the gradient of a straight line.

[box]

TASK 9: Use integration to work out the volume of the solid formed when the area underneath this line is rotated 360° about the x-axis.

Hint: $V = \pi \int_0^h y^2 \, dx$

[box]

TASK 10: What is your answer to Task 9 the equation for?

[box]

ACTIVITY 2

Close both the **Status Box** and the **Constant Controller** by clicking on the red cross in the corner.

In the **Edit** menu on the top toolbar, click on **Select all Points** and press **Delete**.

You should now be left with a clear set of axes.

Edit the axes as follows:

    x: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.2
    y: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.2
    z: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.2

Remove all of the green ticks underneath **Auto**.

Make sure **Slow Plot** mode is still on.

Enter the equation: $y = \sqrt{(3^2 - x^2)}$

Note: To enter x², either use the little 2 button, or type "**xx**", or press "**alt 2**" together.

Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D equation**.

Predict   *What will this line look like? Where will it cross the axes? Will it ever be negative?*

Click **OK** twice.

TASK 11: The area enclosed by the line $y = \sqrt{(3^2 - x^2)}$ and the x-axis is to be rotated through 360° about the x-axis. Describe the shape of the solid that will be formed.

[box]

TASK 12: Find the volume of that solid, leaving your answer in terms of pi (π).

Hint: $V = \pi \int_{-3}^{3} y^2 \, dx$, and be careful with negatives!

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Checking your answer:

Left-click on the curve (it should turn black).

Enter a co-ordinate with an x value of −3 (leave z unchanged).

Enter another co-ordinate, this time with an x value of 3 (leave z unchanged).

Two points should now be marked on the curve.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both points (they should both have little squares around them).

Right-click and select **Find Area** from the menu.

Select **Simpsons Rule**, leave the number of divisions at 5, and click **OK**

Left-click on the area underneath the curve (it should turn grey).

Right-click and select **Find Volume** from the menu, and click **OK**.

Use the **Drag** tool to have a good look around the solid.

Press **Undo** a couple of times, and repeat the steps above to **View** the solid being formed from a different angle.

In the **View** menu on the top toolbar, select **View Status Box**, and move it to a convenient position on the page.

The volume of the solid should be displayed here.

Check this matches your answers to Task 12 before moving on.

Select **x-y Orientation** to return to the original **View** of the graph.

Make sure you are in **Select Mode**.

Left-click on the solid (it should turn grey).

Right-click and select **Delete** from the menu, or simply press **delete** on the keyboard.

You should now be left once again with just the area under the curve marked on the graph.

TASK 13: Look at the graph on your screen. Imagine that the equation of the curve was actually y = √(r² − x²). Use integration to work out the volume of the solid formed when the area enclosed by this curve and the x-axis is rotated 360° about the x-axis.

Hint: $V = \pi\int_{-r}^{r} y^2 \, dx$, and be very careful with negatives!

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TASK 14: What is your answer to Task 13 the equation for?

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ACTIVITY 3

Close the **Status Box** by clicking on the red cross in the corner.

In the **Edit** menu on the top toolbar, click on **Select All** and press **Delete**.

You should now be left with a clear set of axes.

Edit the axes as follows:

x:	Minimum: −20	Maximum: 20	Numbers: 4	Pips: 1
y:	Minimum: −5	Maximum: 5	Numbers: 1	Pips: 0.2
z:	Minimum: −5	Maximum: 5	Numbers: 1	Pips: 0.2

Remove all of the green ticks underneath **Auto**.

Make sure **Slow Plot** mode is still on.

Enter the equation: $y = \pm \sqrt{(x - 2)}$

Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D equation**.

Predict

What will this line look like? Where will it cross the axes? What effect does the ± have?

Click **OK** twice.

TASK 15: Re-arrange $y = \pm \sqrt{(x - 2)}$ to make x the subject (i.e. x =).

Enter your answer to Task 15 as an equation.

Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D equation**.

Click **OK**.

The new curve should be plotted on top of the old one.

In the Key on the **right-hand** side of the screen, **left-click** to select Equation 1.

Right-click and select **Delete Current** from the menu.

You should now be left with just the new graph.

TASK 16: Use your answer to Task 15 to help find the area bounded by the curve $y = \pm \sqrt{(x - 2)}$, the y-axis, and the lines y = 1 and y = 3.

Checking your answer:

Left-click on the curve (it should turn black).

Enter a co-ordinate with an y value of 1 (leave z unchanged).

Enter another co-ordinate, this time with an y value of 3 (leave z unchanged).

Two points should now be marked on the curve.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both points (they should both have little squares around them).

Right-click and select **Find Area** from the menu.

Select **Simpsons Rule**, leave the number of divisions at 5, and click **OK**.

In the **View** menu on the top toolbar, select **View Status Box**, and move it to a convenient position on the page.

The area to the left of the curve should be displayed here.

Check this matches your answers to Task 16 before moving on.

TASK 17: This area is to be rotated 360° around the y-axis. Briefly describe the shape of the solid that will be formed.

TASK 18: Work out the volume of this solid, leaving your answer in terms of pi (π).

Hint: $V = \pi \int_1^3 x^2 \, dy$

Checking your answer:

🐢 Make sure **Slow Plot** mode is still on.

➤ **Left-click** on the area underneath the curve (it should turn grey).

◁ **Right-click** and select **Find Volume** from the menu, and click **OK**.

The three-dimensional solid should now begin to form.

✋ Use the **Drag** tool to have a good look around the solid.

↩ Press **Undo** a couple of times, and repeat the steps above to **View** the solid being formed from a different angle.

The volume of the solid should be displayed in the **Status Box**.

Check this matches your answer to Task 18 before moving on.

ACTIVITY 4

∟ Select **x-y Orientation** to return to the original **View** of the graph.

↖ Make sure you are in **Select Mode**.

Close the **Status Box** by clicking on the red cross in the corner.

In the **Edit** menu on the top toolbar, click on **Select All** and press **Delete**.

You should now be left with a clear set of axes.

🔧 Edit the axes as follows:

    x: Minimum: −2   Maximum: 2   Numbers: 1   Pips: 0.2
    y: Minimum: −2   Maximum: 2   Numbers: 1   Pips: 0.2
    z: Minimum: −2   Maximum: 2   Numbers: 1   Pips: 0.2

Remove all of the green ticks underneath **Auto**.

TASK 19: Find the point(s) of intersection of $y = x$ and $y = \sqrt{x}$.

Predict *Use your answer to Task 19 to help you predict what these two functions would look like when plotted together.*

🐢 Make sure **Slow Plot** mode is still on.

➕ Enter the equation: $y = \sqrt{x}$

Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D equation**.

Click **OK**.

➕ Enter the equation: $y = x$

Still on the **Enter Equation** screen, place a tick in the space next to **Plot as 2D equation**.

Click **OK**.

The graphs of the two functions should now appear on the screen.

TASK 20: Calculate the area that lies between the two functions.

Checking your answer:

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on the graph of both functions (they should both turn grey).

Right-click and select **Find Area** from the menu.

Select **Simpsons Rule**, change the number of divisions to 40, and click **OK**.

In the **View** menu on the top toolbar, select **View Status Box**, and move it to a convenient position on the page.

The area to the left of the curve should be displayed here.

Check this matches your answers to Task 20 before moving on.

TASK 21: This area is to be rotated 360° about the x-axis. Carefully describe the shape of the solid that will be formed.

Hint: Think about the centre of the object.

TASK 22: Find the volume of that solid, leaving your answer in terms of pi (π).
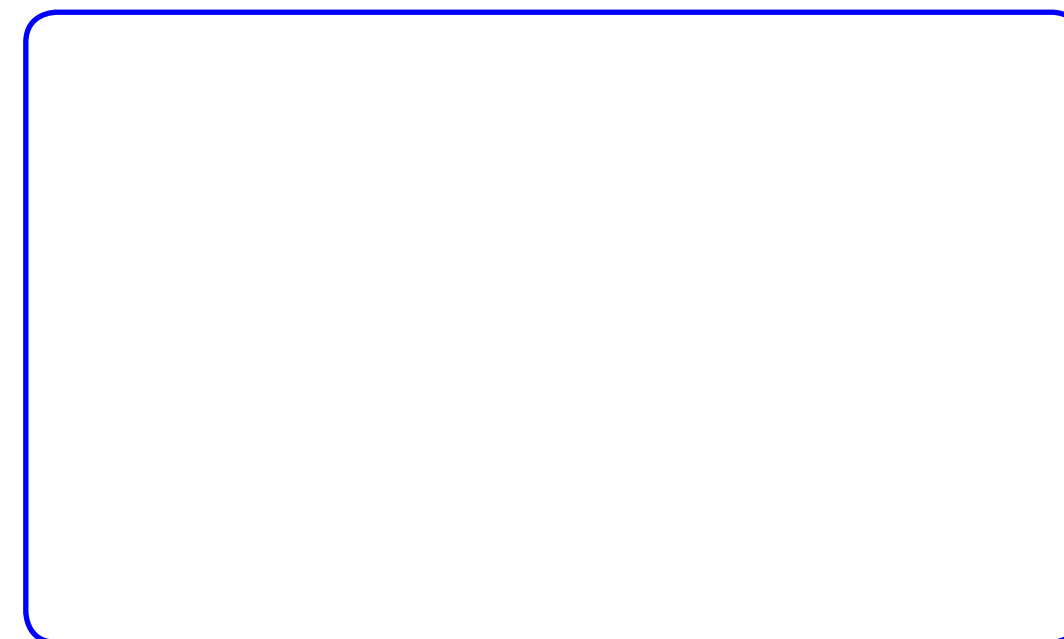
Checking your answer:

Make sure **Slow Plot** mode is still on.

Left-click on the area between the curves (it should turn grey).

Right-click and select **Find Volume** from the menu, and click **OK**.

The three-dimensional solid should now begin to form.

Use the **Drag** tool to have a good look around the solid.

Press **Undo** a couple of times, and repeat the steps above to **View** the solid being formed from a different angle.

The volume of the solid should be displayed in the **Status Box**.

Check this matches your answer to Task 22 before moving on.

Activity 5 – Only for the brave!

Press **Undo** enough times so just the area between the curves is marked on the graph.

Make sure you are in **Select Mode**.

Close the **Status Box** by clicking on the red cross in the corner.

TASK 23: This area is to be rotated 360° about the line y = −0.5. Carefully describe the shape of the solid that will be formed.

Hint: Think about the centre of the object.

Make sure **Slow Plot** mode is still on.

Left-click on the area between the curves (it should turn grey).

Right-click and select **Find Volume** from the menu, change the axis of rotation to y = −0.5, and then click **OK**.

The three-dimensional solid should now begin to form.

Use the **Drag** tool to have a good look around the solid.

Press **Undo** a couple of times, and repeat the steps above to **View** the solid being formed from a different angle.

TASK 24: Show that volume of the solid is exactly equal to $\frac{\pi}{3}$.

Hint: Think about how you could work out the volume of an identical shape, but which is in a different location on the axes.

Summary

Use the space below to summarise what you have learnt during this investigation:

ANSWERS

ACTIVITY 1
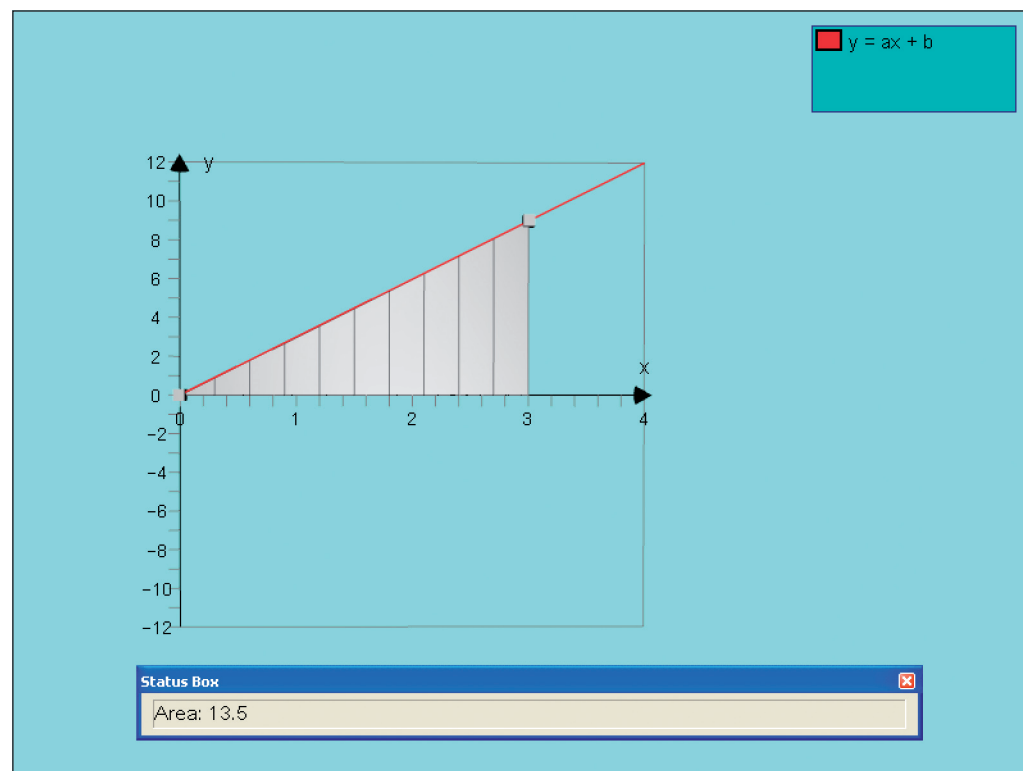
TASK 1: Area of triangle:

$A = \dfrac{bh}{2} = \dfrac{3 \times 9}{2} = 13.5$

TASK 2: Using integration:

$A = \int_0^3 3x\, dx = \left[\dfrac{3x^2}{2} \right]_0^3 = \dfrac{27}{2} - 0 = 13.5$

TASK 3: The solid formed will be a cone. The radius of the base is 9, and the height is 3.

TASK 4: Volume of the cone:

$V = \pi \int_0^3 y^2\, dx = \pi \int_0^3 (3x)^2\, dx = \pi \int_0^3 9x^2\, dx = \pi [3x^3]_0^3 = 81\pi$
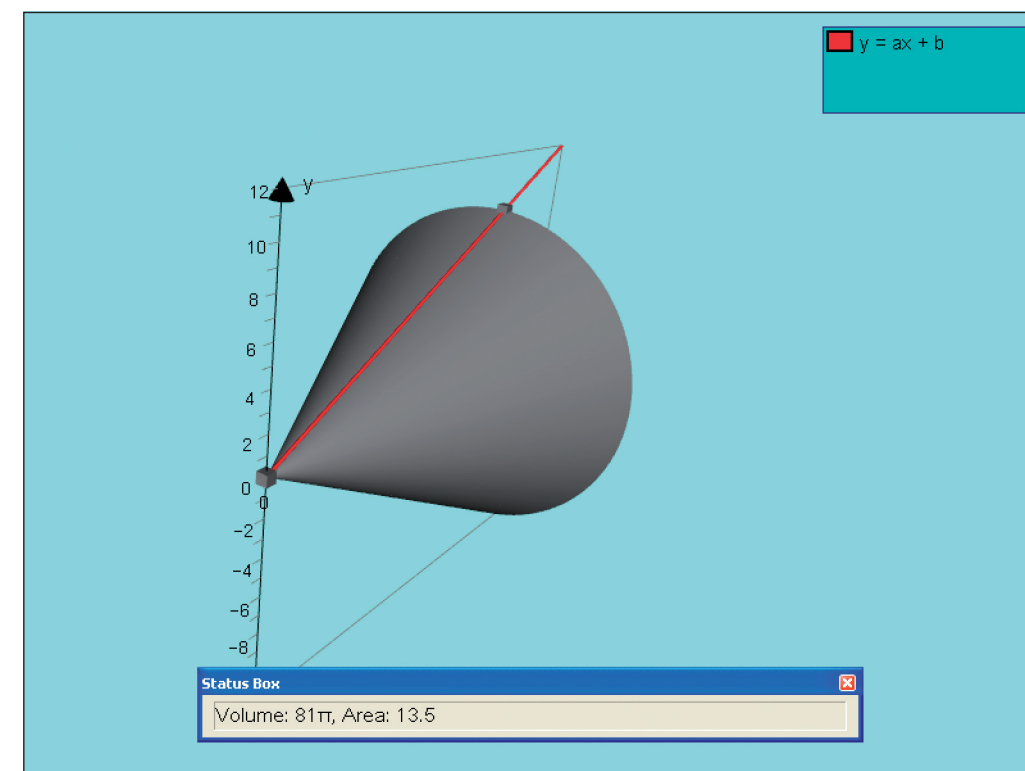
TASK 5: The ends of the cone would need to be switched around, so the pointed end is at (3, 0), and the flat circular face is along the y-axis. The line y = ax + b would need to have negative gradient, and cross the y-axis at (0, 9). The values of a and b which give this are a = −3, b = 9, giving y = −3x + 9.
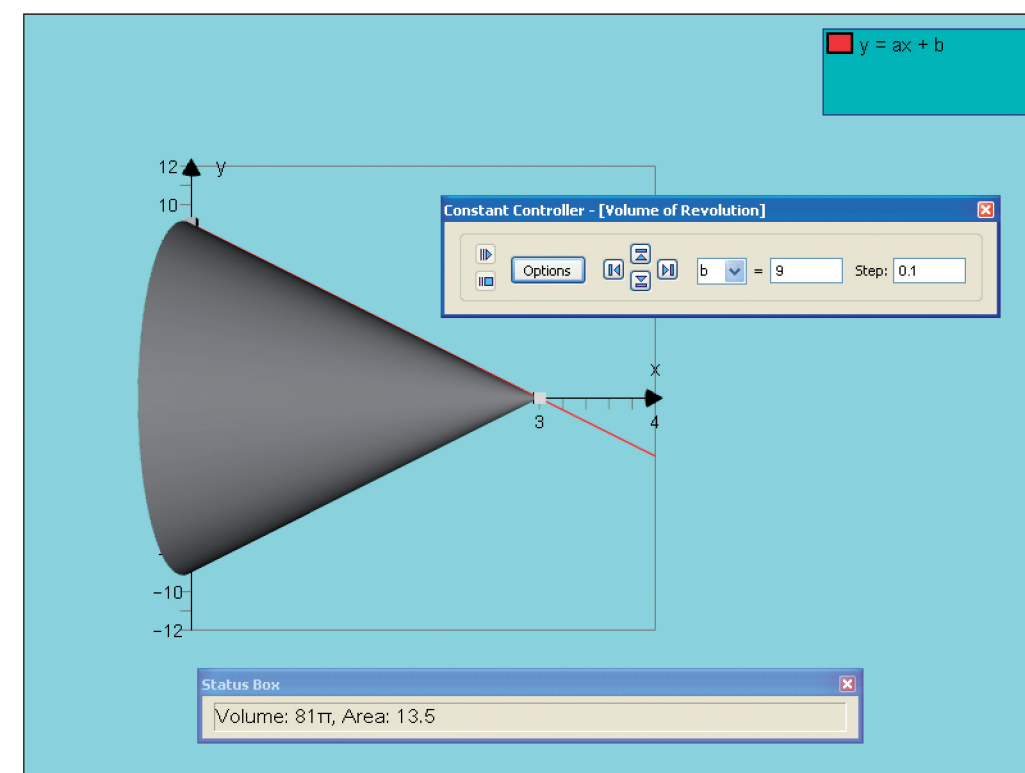
TASK 6: The solid formed will be a cylinder, with radius 4 and height/length 3.

TASK 7: Volume of the cylinder:

$$V = \pi\int_0^3 y^2\,dx = \pi\int_0^3 4^2\,dx = \pi\int_0^3 16\,dx = \pi[16x]_0^3 = 48\pi$$
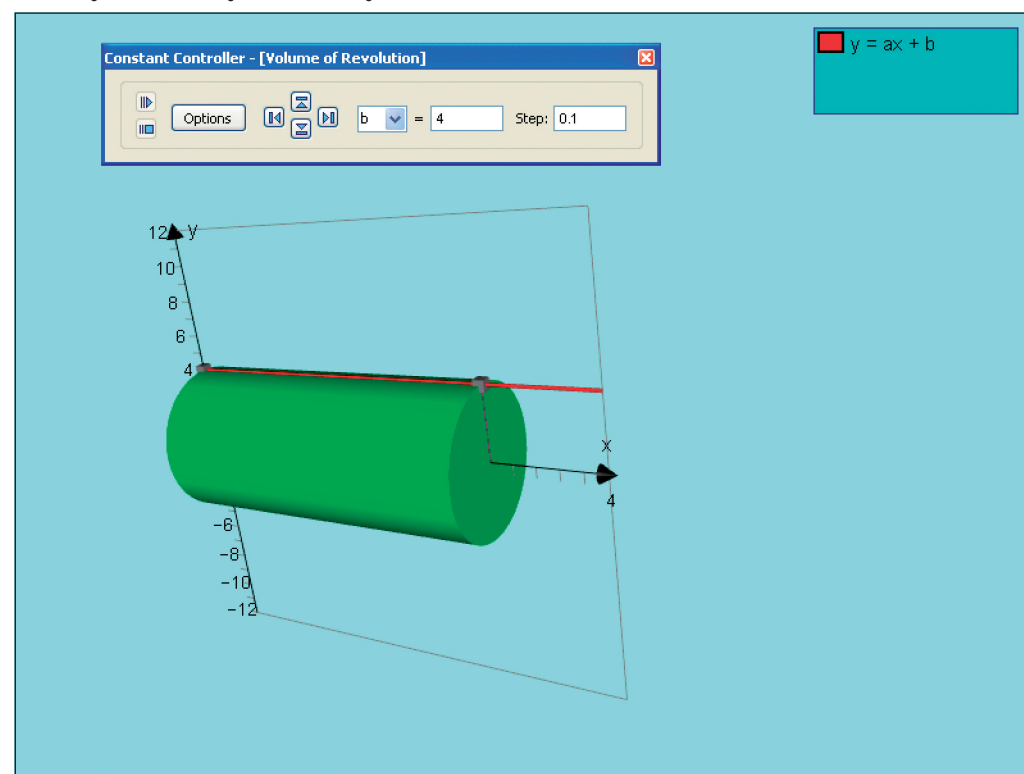
TASK 8: Equation of the line: $y = \dfrac{r}{h}x$

TASK 9: Volume of the cone:

$$V = \pi\int_0^h y^2\,dx = \pi\int_0^h \left(\tfrac{r}{h}x\right)^2 dx = \pi\int_0^h \tfrac{r^2}{h^2}x^2\,dx = \pi\left[\tfrac{r^2}{h^2}\tfrac{x^3}{3}\right]_0^h = \pi\left[\tfrac{r^2}{h^2}\tfrac{h^3}{3}\right] = \tfrac{\pi r^2 h}{3}$$

TASK 10: The answer to Task 9 is the formula for working out the volume of a cone with radius r and height h.

ACTIVITY 2
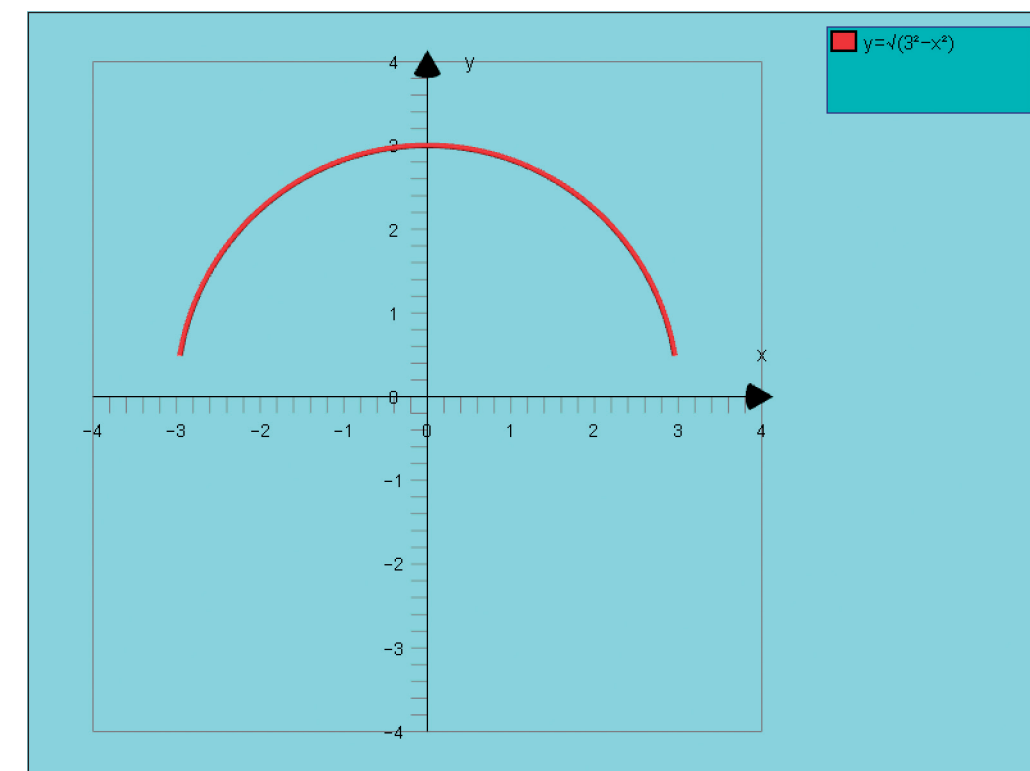
TASK 11: The solid formed will be a sphere, with radius 3.

TASK 12: Volume of the sphere:

$$V = \pi\int_{-3}^3 y^2\,dx = \pi\int_{-3}^3 (\sqrt{3^2 - x^2})^2\,dx = \pi\int_{-3}^3 9 - x^2\,dx = \pi\left[9x - \tfrac{x^3}{3}\right]_{-3}^3$$

$$= \pi[27 - 9 + 27 - 9]_{-3}^3 = 36\pi$$
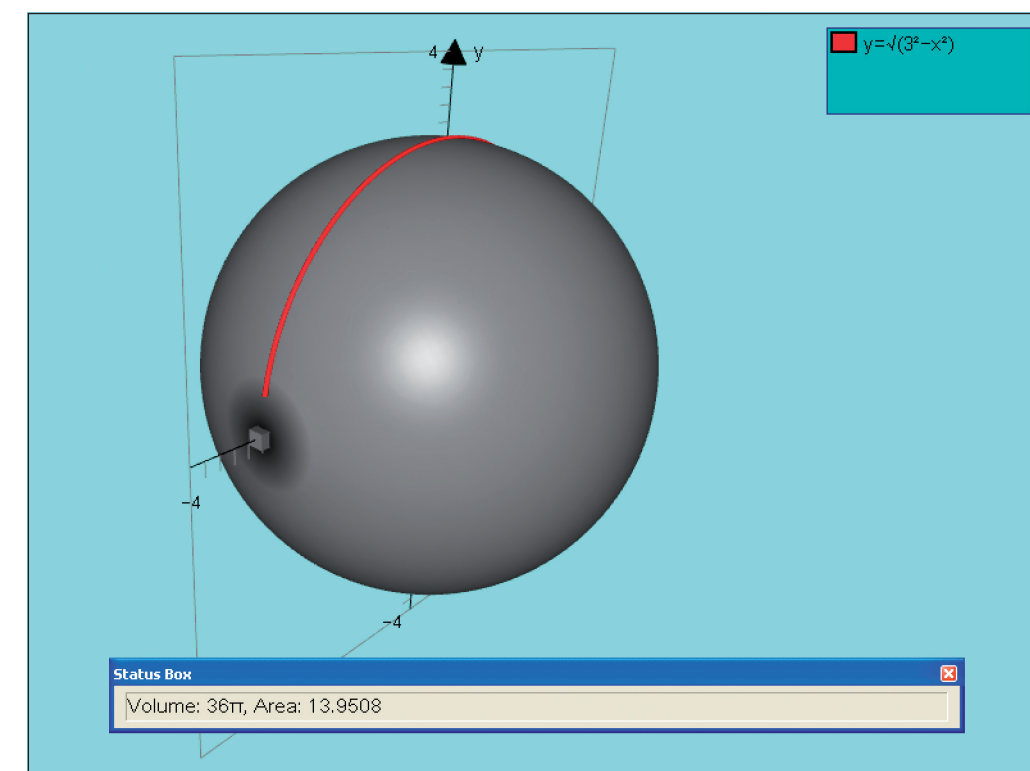
TASK 13: Volume of the sphere:

$$V = \pi\int_{-r}^{r} y^2 \, dx = \pi\int_{-r}^{r} (\sqrt{r^2 - x^2})^2 \, dx = \pi\int_{-r}^{r} r^2 - x^2 \, dx = \pi\left[xr^2 - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right] = \frac{4\pi r^3}{3}$$

TASK 14: The answer to Task 12 is the formula for working out the volume of a sphere with radius r.

ACTIVITY 3

y=±√(x−2)
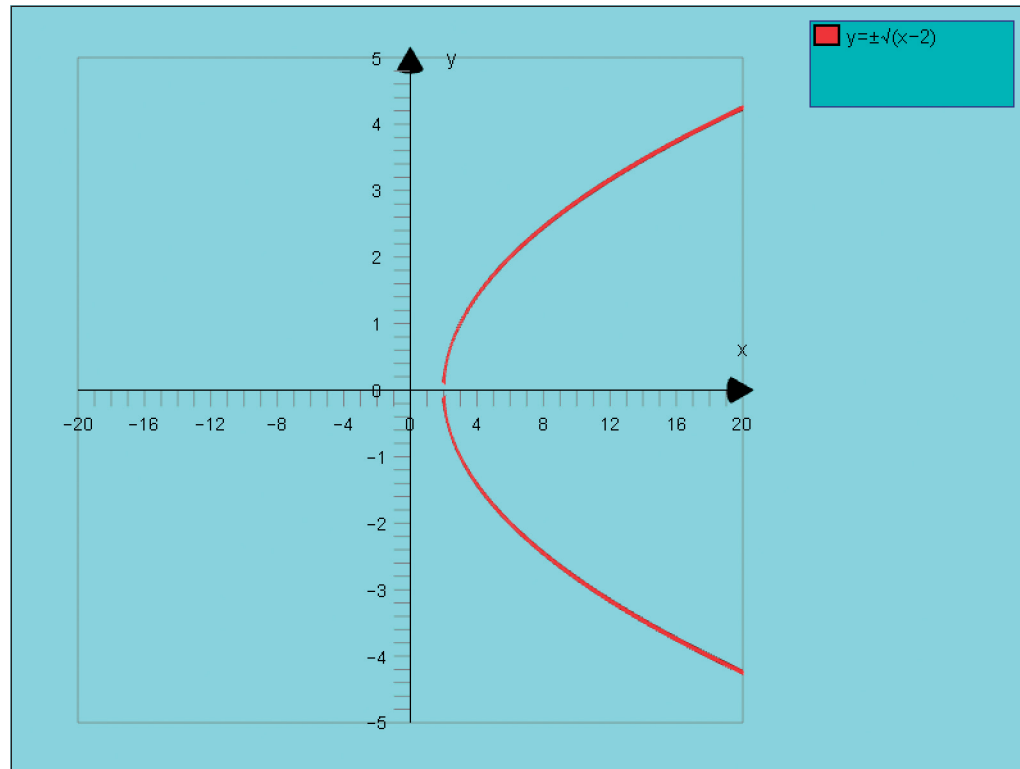
TASK 15:

y = ±√x − 2
y² = x − 2
x = y² + 2

TASK 16: Area of region:

$$A = \int_1^3 y^2 + 2 \, dx = \left[\frac{y^3}{3} + 2y\right]_1^3 = \left[9 + 6 - \frac{1}{3} - 2\right] = 12\frac{2}{3}$$

x=y²+2

Status Box

x = 11, y = 3, z = 0   Area: 12.67
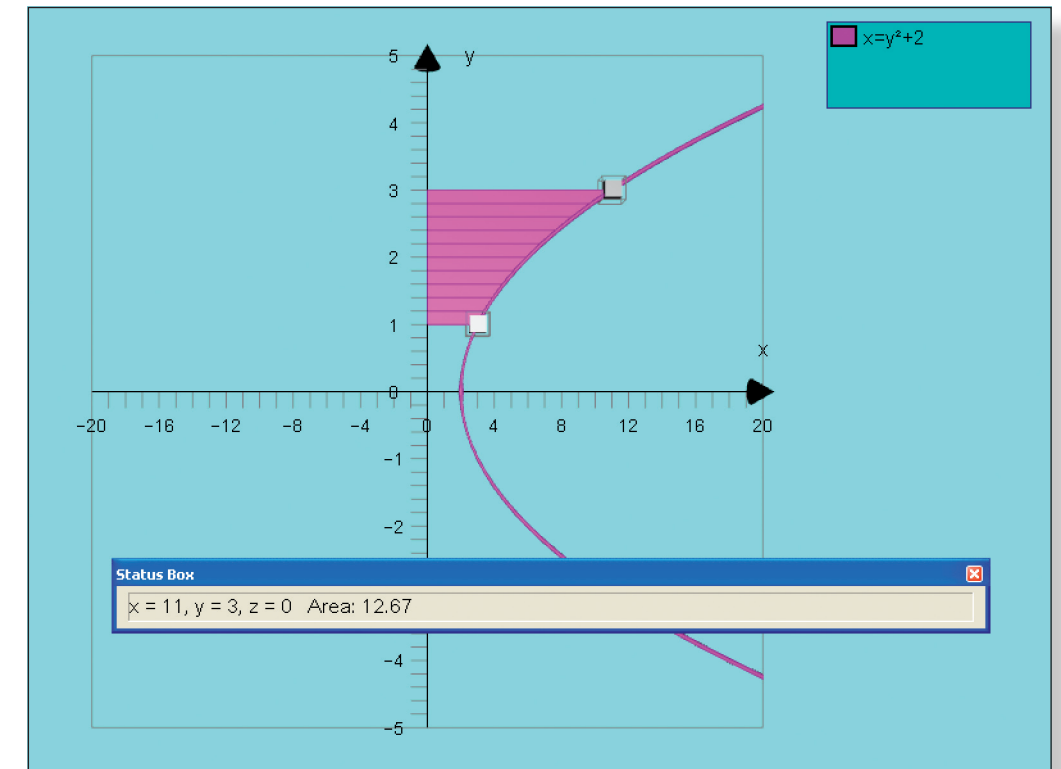
TASK 17: The solid formed is a vase shape

TASK 18: Volume of the solid:

$$V = \pi\int_1^3 x^2 \, dy = \pi\int_1^3 (y^2 + 2)^2 \, dy = \pi\int_1^3 y^4 + 4y^2 + 4 \, dy = \pi\left[\frac{y^5}{5} + \frac{4y^3}{3} + 4x\right]_1^3$$

$$= \pi\left[\frac{243}{5} + \frac{108}{3} + 12 - \frac{1}{5} - \frac{4}{3} - 4\right] = 91\frac{1}{15}\pi$$

ACTIVITY 4

$x = y^2 + 2$

Status Box

Volume: 91.07π, Area: 12.6667
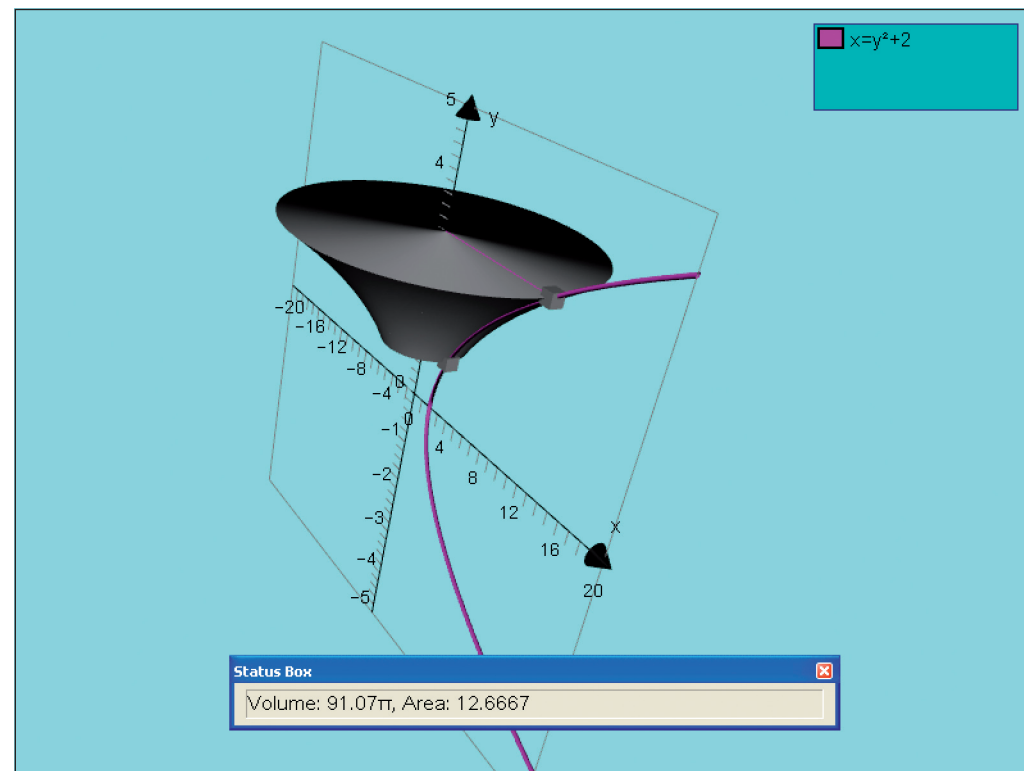
TASK 19: Points of intersection:

$x = \sqrt{x}$
$x = 1$
$(1, 1)$

$y = \sqrt{x}$
$y = x$
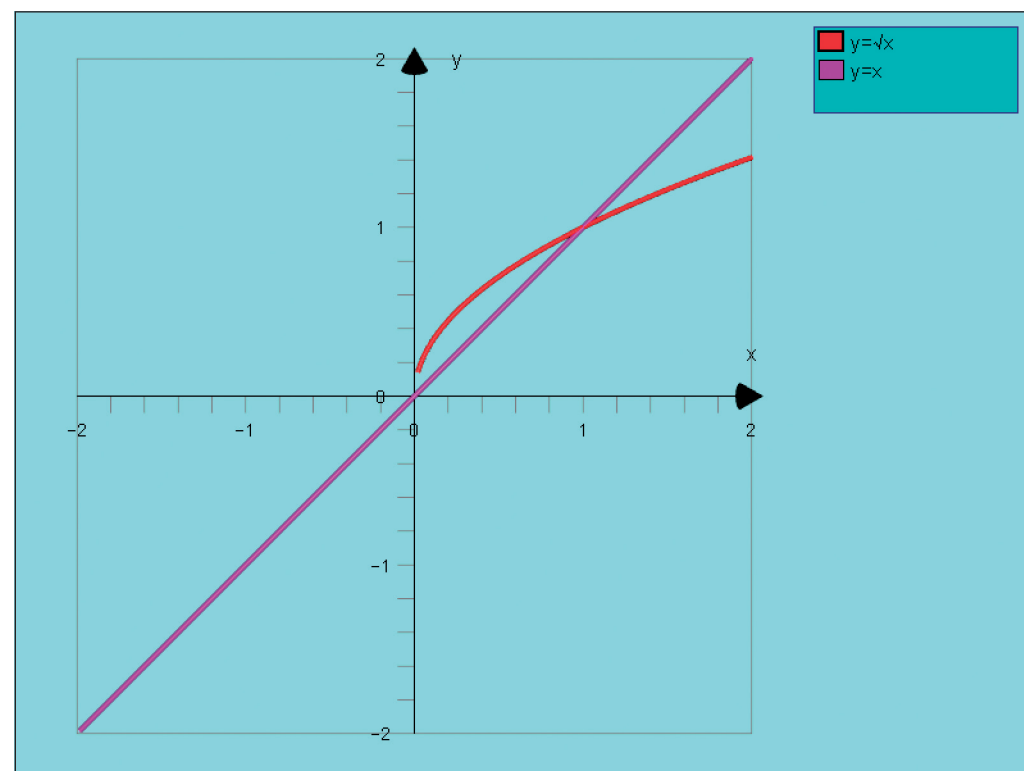
TASK 20: Area of region:

$$A = \int_0^1 \sqrt{x} - x \, dx = \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2} \right]_0^1 = \left[\frac{2}{3} - \frac{1}{2} \right] = \frac{1}{6}$$

$y = \sqrt{x}$
$y = x$

Status Box

Area: 0.1666
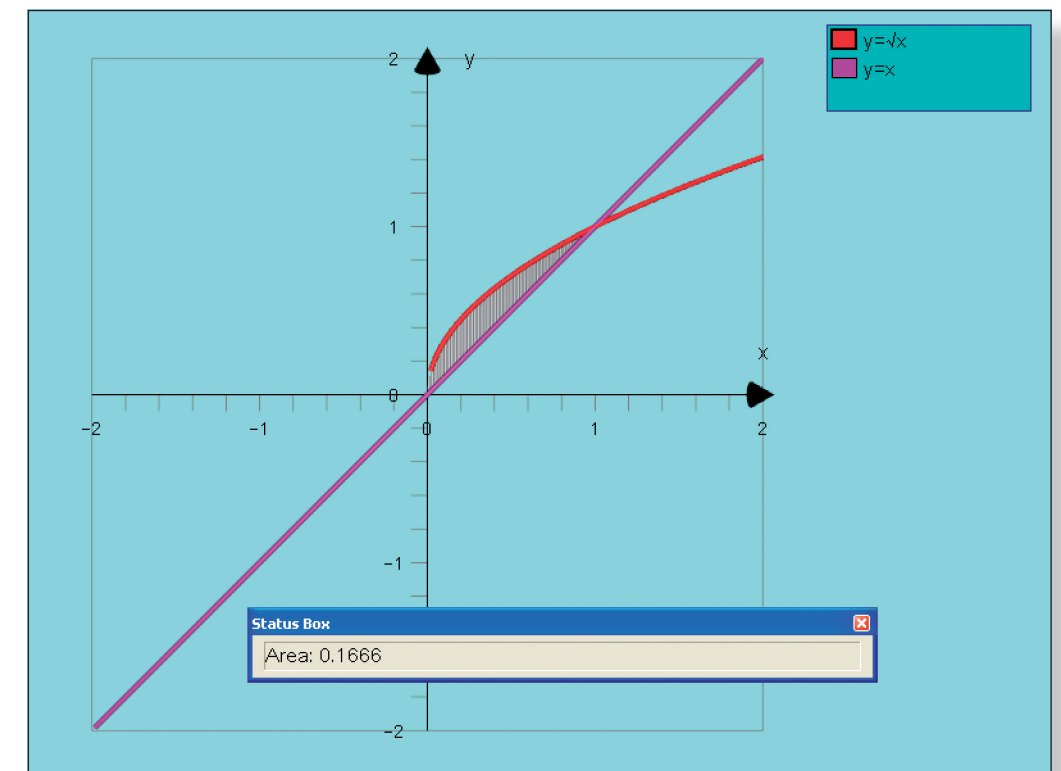
TASK 21: The solid formed will be a hollow bowl with a conical shaped hole down the middle.

TASK 22: Volume of the solid:

$$V = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 (\sqrt{x})^2 - x^2 \, dx = \pi \int_0^1 x - x^2 \, dx = \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$
$$= \pi \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{1}{6}\pi$$

Status Box

Volume: π/6, Area: 0.166553
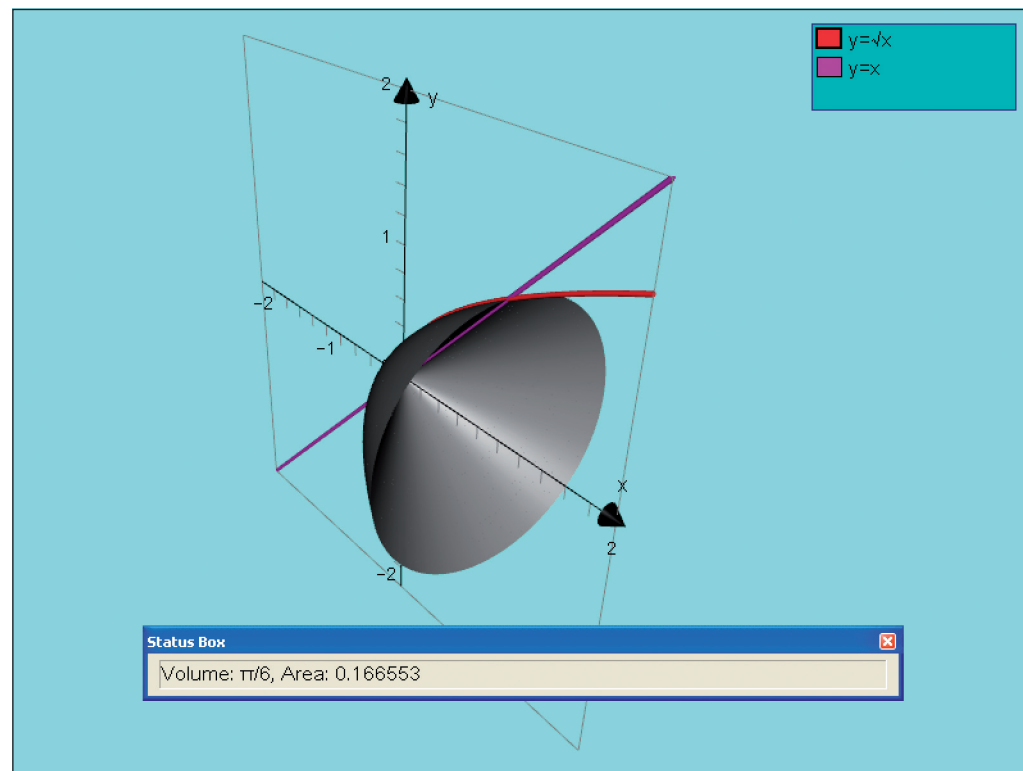
Activity 5: Only for the Brave!

TASK 23: The solid formed will be a similar shape to the one above, but with a cylindrical hole in the middle, of radius 0.5.
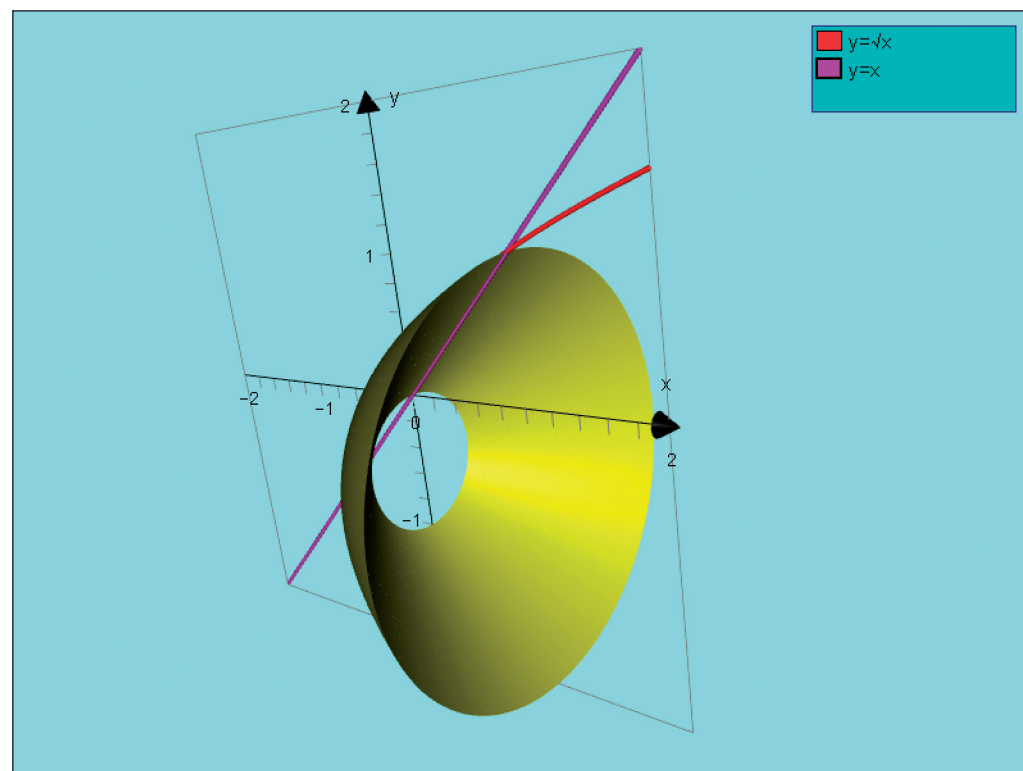
TASK 24: One way of approaching this question is to realise that the shape above is identical to the shape formed when rotating the area between the

curves $y = \sqrt{x} + \frac{1}{2}$ and $y = x + \frac{1}{2}$ about the x-axis. Translating both curves up by ½ unit will generate the solid with the required cylindrical gap as illustrated below:
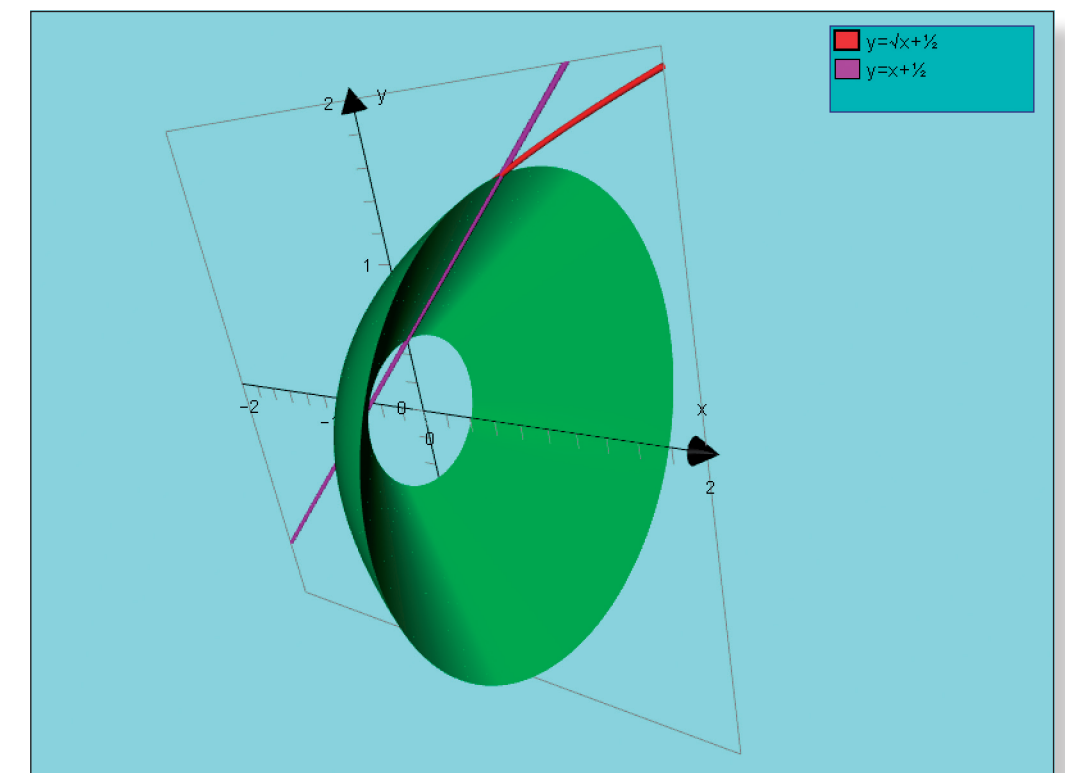
Hence, the volume of the solid can be found as follows:

$$V = \pi\int_0^1 y^2 \, dx = \pi\int_0^1 \left(\sqrt{x} + \frac{1}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2 dx = \pi\int_0^1 x + \sqrt{x} + \frac{1}{4} - x^2 - x - \frac{1}{4} \, dx$$

$$= \pi\int_0^1 \sqrt{x} - x^2 \, dx = \pi\left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1 = \pi\left[\frac{2}{3} - \frac{1}{3}\right] = \frac{1}{3}\pi$$

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

- To reinforce and practise the core skills of co-ordinate geometry and circle work.

- To get a more visual, interactive feel for the concepts involved.

EXTENSION MATERIAL:

- To be able to answer more investigative style questions on co-ordinate geometry relying on knowledge of tangents and geometrical properties of circles.

REQUIRED PRIOR KNOWLEDGE

- Be able to calculate:

  » The gradient of a line segment joining two points.

  » The length of a line segment joining two points.

  » The equation of a line given the gradient and one point on the line.

  » The mid-point of a line segment joining two points.

  » The gradient and equation of a perpendicular line.

  » The co-ordinates of a line's intersection with the x-axis.

- For circles, students need to be able to calculate:

  » The equation of a circle given the centre and a point on the circumference.

  » The centre and radius of a circle given its equation (the harder task requires the use of completing the square).

  » The equation of a tangent to a circle having been given the point of intersection and the gradient of the radius/normal.

- Show that a point lies on a circle.

- Be comfortable handling surds.

FOR THE EXTENSION MATERIAL:

  » Proving the a line is a tangent to a circle (requires knowledge of re-

peated roots).

» Finding the area of a triangle (requires knowledge of geometrical properties of circles).

Description of the Activity

- In Activity 1, students will be asked to complete a series of tasks, each of which covers a core skill of co-ordinate geometry. As they complete each task they will use Autograph to check their answers and reinforce visually exactly what each of these concepts mean. They also have the opportunity to make predictions, and then use Autograph's constant controller to confirm their answers.

- Similarly, in Activity 2, students will be asked to complete a series of interactive tasks covering the core elements of circles.

- Activity 3 is the extension material. This is more investigative and really stretches the students' knowledge of all aspects of co-ordinate geometry.

- The vast majority of the tasks have a self-checking mechanism at the end of them, so the students will immediately know whether their methods were correct or not. This will hopefully boost their confidence and enable them to work more independently. Further more, if they have made a mistake, they should be able to use Autograph to help them figure out where they might have gone wrong.

Pre–Activity Setup

None required.

Suggested Introduction

"Co-ordinate geometry is all about the location of points, lines and curves in relation to each other. During this investigation we will be reViewing and exploring all the important concepts of co-ordinate geometry. By conducting this activity on Autograph you will hopefully get a more visual feel for the concepts involved which will hopefully help to deepen your levels of understanding."

Suggested Prompts

- This activity uses fairly basic tools on Autograph. Once the students get to grips with defining the axes, selecting points, and entering equations, they shouldn't have too much trouble with the software itself. The use of the **Shift** button to select multiple objects is important, but students should grasp this quite quickly.

- The majority of the first two activities rely on the knowledge and application of the core formulae of co-ordinate geometry and circle work. If students are struggling, it might be an idea to have a set of formulae or some

old worked examples on hand to help jog their memory and steer them in the right direction.

- In terms of the extended material in Activity 3, it is up to you how much help and guidance you give the students. Mentioning repeated roots for Task 16 and angles made by tangents and radii in Task 17 should be enough to help them on their way.

Suggested Plenary

- Can anybody summarise the key skills we need to possess for success with co-ordinate geometry?

- Has anybody got any tips for tackling these types of questions?

Ideas for Further Work

- More exam style questions on co-ordinate geometry

- Moving on to look at how differentiation can help us with some co-ordinate geometry style questions, such as finding the co-ordinates of stationary points.

STUDENT WORKSHEET   NAME:..

ACTIVITY 1

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Select **Equal Aspect Mode** from the toolbar.

This alters the x scale so that the axes are square.

Edit the axes as follows:
     x: Minimum: −15   Maximum: 15   Numbers: 1   Pips: 1
     y: Minimum: −10   Maximum: 10   Numbers: 1   Pips: 1
Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re–scale your axes for you.

Enter a point at (5, 6).

Select **Text Box** from the toolbar.
Change the word "Point" to "A" and click **OK**.

The **Text Box** should now display the co-ordinates of this point which we have called A.

Enter another point at (10, 4).

Select **Text Box** from the menu.

Change the word "Point" to "B" and click **OK**.

TASK 1: In the space below work out the *distance* between points A and B. Give your answer both as a surd and to 4 significant figures.

TASK 2: In the space below work out the *gradient of a line segment* connecting points A and B.

TASK 3: Using $y - y_1 = m(x - x_1)$ work out the equation of a straight line going through points A and B, and re-arrange your answer so that it is in the form $y = mx + c$.

Predict   *What would a line segment between these two points look like?*

Left-click on an unoccupied part of the graph area to deselect everything.

Hold down the **Shift** button to select more than one object:

Left-click on both points to select them (they should both have squares around them).
Note: The use of the **Shift** button to select more than one object is very important!

Right-click and select **Line Segment** from the menu.

Left-click on the line segment (it should turn black).

Select **Text Box** from the toolbar.
Change the word "Distance" to "Line Segment AB" and click **OK**.

The **Text Box** should now display the both the distance between the points (the first value), and the equation of the line connecting them.

Check: Use this information to check your answers to Tasks 1 to 3 before moving on.

Click on hide axes on the top toolbar.

TASK 4: Without revealing the axes again, work out the co-ordinates of the mid-point of the line segment AB.

TASK 5: What would be the gradient of a line that is *perpendicular* to the line segment AB?

TASK 6: What would be the equation of the line which is perpendicular to the line segment AB and passed through the mid-point of AB? In other words, what is the equation of the *Perpendicular Bisector* of the line segment AB? Give your answer in the form y = mx + c.

Left-click on an unoccupied part of the graph area to deselect everything.

Hold down the **Shift** button to select more than one object:

Left-click on Points A and B (they should both have squares around them).

Right-click and select **Mid–Point** from the menu.

A dot should appear at the mid-point of the line segment AB.

Left-click on the dot at the mid-point (it should turn black).

Select **Text Box** from the toolbar.
Change the word "**Mid–Point**" to "M" and click **OK**.

The co-ordinates of the mid-point of the line segment AB should now be displayed.

Left-click on an unoccupied part of the graph area to deselect everything.

Hold down the **Shift** button to select more than one object:

Left-click on points A and B.

Right-click and select **Perpendicular Bisector** from the menu.

A line perpendicular to AB and passing through point M should now appear.

Left-click on the perpendicular bisector (it should turn black).

Select **Text Box** and click **OK**.

The equation of the perpendicular bisector to the line segment AB should now be displayed.

Check: Use this information to check your answers to Tasks 3 to 6 before moving on.

TASK 7: What are the exact co-ordinates of the point where the perpendicular bisector crosses the x-axis?

Left-click on the perpendicular bisector (it should turn black).

Right-click and select **Solve f(x) = 0** from the menu.

A new dot should now be displayed.

Left-click on this new dot (it should turn black).

Select **Text Box** from the toolbar.
Change the word "Equation Solver" to "x-axis crossing" and click **OK**.

The co-ordinates of the where the line crosses the x-axis should now be displayed.

Check: Make sure this matches your answer for Task 7 before moving on.

Click on **Cartesian Grid** on the toolbar to put this activity into perspective.

Left-click on Point B to select it.

Hold down **Shift** and use the **arrow keys** on the keyboard to move the point around.

Observe the effect it has on the information displayed in the **Text Boxes**.

Predict

Where would you have to move Point B so that the perpendicular bisector sits on top of the x-axis?

Where would you have to move Point B so that the perpendicular bisector sits on top of the y-axis?

Predict

Where would you have to move Point B so that the perpendicular bisector passes through the (1, 0)? Is this the only solution?

Check: Use Autograph to check your answers.

ACTIVITY 2

Open up a New **2D Graph Page**.

Select **Equal Aspect Mode** from the top toolbar.

This alters the x scale so that the axes are square.

Edit the axes as follows:
    x: Minimum: −30   Maximum: 30   Numbers: 2   Pips: 1
    y: Minimum: −16   Maximum: 16   Numbers: 2   Pips: 1
Remove all of the green ticks underneath **Auto**.

Enter a point at (0, 0).

Select **Text Box** from the toolbar.
Change the word "Point" to "C" and click **OK**.

The **Text Box** should now display the co-ordinates of this point which we have called C.

Enter another point at (4, 3).

Select **Text Box**.
Change the word "Point" to "D" and click **OK**

TASK 8: Point C is the centre of a circle, and Point D lies on the circumference of the circle. In the space below, work out the equation of that circle, giving your answer in the form $x^2 + y^2 = r^2$.

Enter a point at (−20, 4).

Select **Text Box** from the menu.
Change the word "Point" to "E" and click **OK**.

The **Text Box** should now display the co-ordinates of this point which we have called E

Enter another point at (−24, −2).

Select **Text Box** from the menu.
Change the word "Point" to "F" and click **OK**

TASK 9: Point E is the centre of a circle, and Point F lies on the circumference of the circle. In the space below, work out the equation of that circle, giving your answer in the form $(x − a)^2 + (y − b)^2 = r^2$. Give your answer both as a surd and round to 4 significant figures.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on Point D first and then Point C (they should both have squares

around them).

Right-click and select **Circle (2 pts)** from the menu.

A circle with the centre as Point C and passing through Point D should now appear.

Note: If your circle has Point D as its centre, you have probably selected the points the wrong way around. Click **Undo** and try again.

Left-click on the circle (it should turn black).

Select **Text Box** from the toolbar.
Change the word "Circle" to "Circle 1" and click **OK**.

The equation of the first circle should now be displayed.

Note: You cannot change the red text as this is dynamic, meaning it will change as you adjust the various points.

Repeat these steps to display the equation of the second circle, naming it "Circle 2".

Check: Make sure these match your answer for Tasks 8 and 9 before moving on.

Left-click on Point C.

Hold down **Shift** and use the **arrow keys** on the keyboard to move the point around.

Observe the effect it has on the equation of the circle.

Predict

Where would you have to move Point C so that the radius of the circle is twice as big?

Check: Use Autograph to check your answer.

Return Point C to its original location at (0, 0).

Left-click on one of the other three points.

Hold down **Shift** and use the arrow keys on the keyboard to move the point around.

Observe the effect it has on the equation of the circle.

Predict

Where would you have to move Points C, D, E and F so that both circles have a radius of 6 and touch each other one (and only once) at (−4, −4)?

Check: Use Autograph to check your answer.

TASK 10: Work out the centre and the radius of a circle with the equation: $(x − 20)^2 + (y − 9)^2 = 36$.

Enter the equation $(x − 20)^2 + (y − 9)^2 = 36$ to check your answer.

Note: for the squared, use the little 2 button or press "**alt 2**" together.

Check: Did this match your answer for Task 10?

TASK 11: Work out the centre and the radius of a circle with the equation: $x^2 + y^2 + 20x + 16y + 100 = 0$.

Hint: Complete the square!

Enter the equation $x^2 + y^2 + 20x + 16y + 100 = 0$ to check your answer.

Check: Does this match your answer for Task 11?

ACTIVITY 3

Open up a New **2D Graph Page**.

Select **Equal Aspect Mode** from the top toolbar.

This alters the x scale so that the axes are square.

Edit the axes as follows:

x: Minimum: −30   Maximum: 30   Numbers: 2   Pips: 1
y: Minimum: −16   Maximum: 16   Numbers: 2   Pips: 1
Remove all of the green ticks underneath **Auto**.

TASK 12: Without using Autograph, show that the point (9, −7) lies on the circle with equation: $x^2 + y^2 − 8x + 4y = 30$

Enter the equation: $x^2 + y^2 − 8x + 4y = 30$

Enter the point (4, −2).

Select **Text Box** from the menu.
Change the word "Point" to "C" and click **OK**.

The **Text Box** should now display the co-ordinates of this point which we have called C, which is also the centre of the circle.

Left-click on the bottom half of the circle (it should turn black).

Enter the point with an x value of 9.

Select **Text Box** from the menu.
Change the word "Point" to "P" and click **OK**.

TASK 13: What would be the gradient of a line segment connecting Points C and P?

TASK 14: Use your answer to Task 13 to work out the equation of the tangent to the circle which passes through Point P.

Enter the equation: y = x + k
Still on the **Enter Equation** screen, click on **Edit Constants**.
Change the value of k to 2.

Predict *What will this line look like? Which direction will it slope? Where will it cross the axes?*

Click **OK** twice.

Left-click on an unoccupied part of the graph to *de-select everything*.

Left-click on the line (it should turn black).

Select **Text Box** from the toolbar.
Change "Equation 2" to "Line 1".
Tick the box which says **Show Detailed Object Text** and click **OK**.

This should now display the equation of the line and the current value of k (1).

Click on **Constant Controller** on the toolbar
The **up-down** buttons alter the value of the constant.
The **left-right** buttons alter the size of the step.

Adjust the value, change the value of k until the equation of Line 1 is the same as the answer you got in Task 14 for the equation of the tangent.

Check: Does this match your answer for Task 14?

TASK 15: Using the **Constant Controller** to adjust the value of k, find the equation of another line which is a tangent to the circle. Write down the equation of the line, and the co-ordinates of the point of intersection of the line and the circle.

[box]

TASK 16: Use algebra to *prove* that this second line is a tangent to the circle.

Hint: Think about the solution to simultaneous equations when tangents are involved.

[box]

Use the **Constant Controller** to **Shift** Line 1 so it is once again tangential to the circle at the Point P (9, −7).

Enter the point (16, 0).

Right-click and select **Text Box** from the menu.

Change the word "Point" to "T" and click **OK**.

Left-click on an unoccupied part of the graph area to *deselect everything.*

Hold down the **Shift** button to select more than one object:

Left-click on Points C, P and T (they should all have squares around them).

Right-click and select **Group to Shape** from the menu.

You should now see a triangle.

TASK 17: Calculate the area of triangle CPT, using surds in your working and simplifying your answer as much as possible.

[box]

SUMMARY

Use the space below to summarise what you have learnt during this activity:

[box]

ANSWERS

ACTIVITY 1

TASK 1: Distance between two points AB:

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(10 - 5)^2 + (4 - 6)^2} = \sqrt{29} = 5.385$ (4 sig fig)

TASK 2: Gradient of line segment AB:

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 6}{10 - 5} = -\frac{2}{5} = -0.4$

TASK 3: Equation of straight line through AB:

$y - y_1 = m(x - x_1)$

$y - 6 = -0.4(x - 5)$

$y = -0.4x + 8$
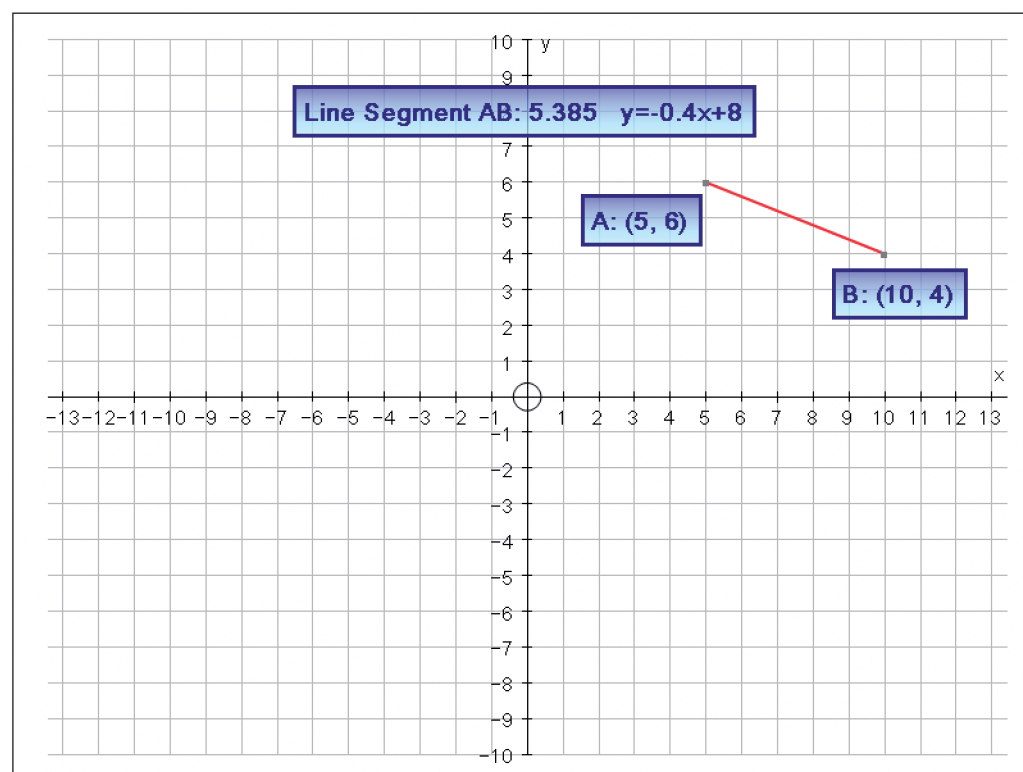
TASK 4: Mid-point of line segment AB:

$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{5 + 10}{2}, \frac{6 + 4}{2} \right) = (7.5, 5)$

TASK 5: Gradient of line perpendicular to AB:

$m_1 \times m_2 = -1$

$m_2 = \frac{-1}{-\frac{2}{5}} = \frac{5}{2} = 2.5$

TASK 6: Equation of perpendicular bisector to AB:

$y - y_1 = m(x - x_1)$

$y - 5 = 2.5(x - 7.5)$

$y = 2.5x - 13.75$

TASK 7: Point where perpendicular bisector crosses x-axis:

$2.5x - 13.75 = 0$

$x = \frac{13.75}{2.5} = 5.5$

Co-ordinates = (5.5, 0)
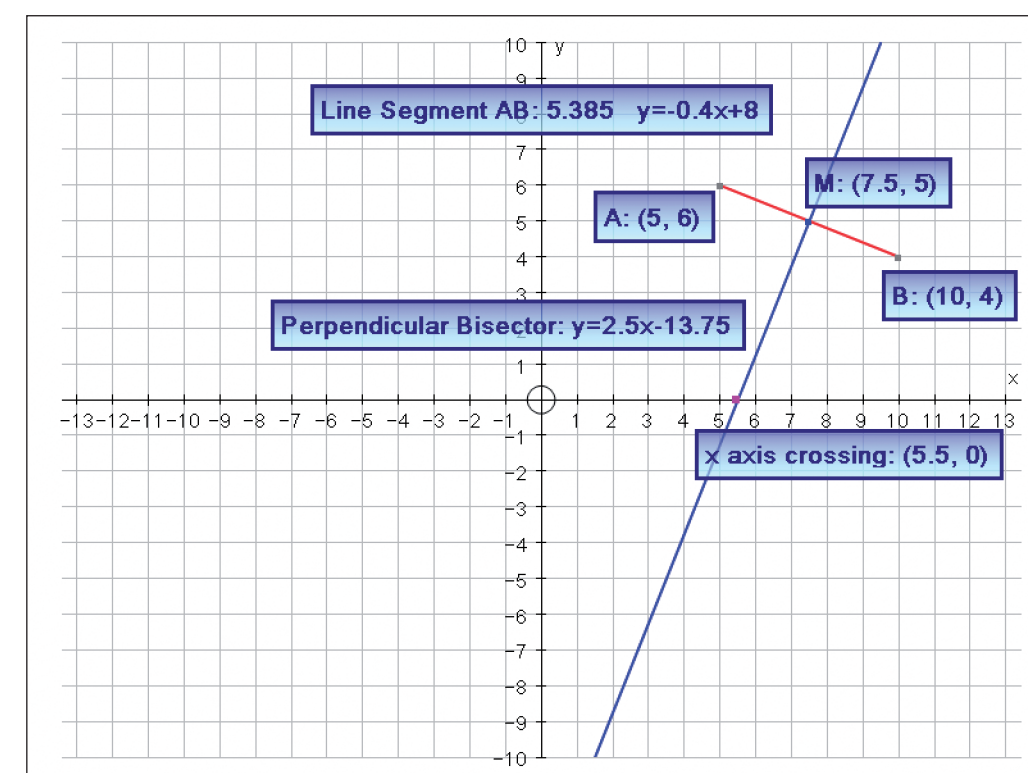
Predict: Perpendicular Bisector is x-axis:

Line Segment AB: 12   x=5

A: (5, 6)

Perpendicular Bisector: y=0

M: (5, 0)

x axis crossing: No Solutions

B: (5, -6)

Predict: Perpendicular Bisector is y-axis:

Line Segment AB: 10   y=6

M: (0, 6)

B: (-5, 6)

A: (5, 6)
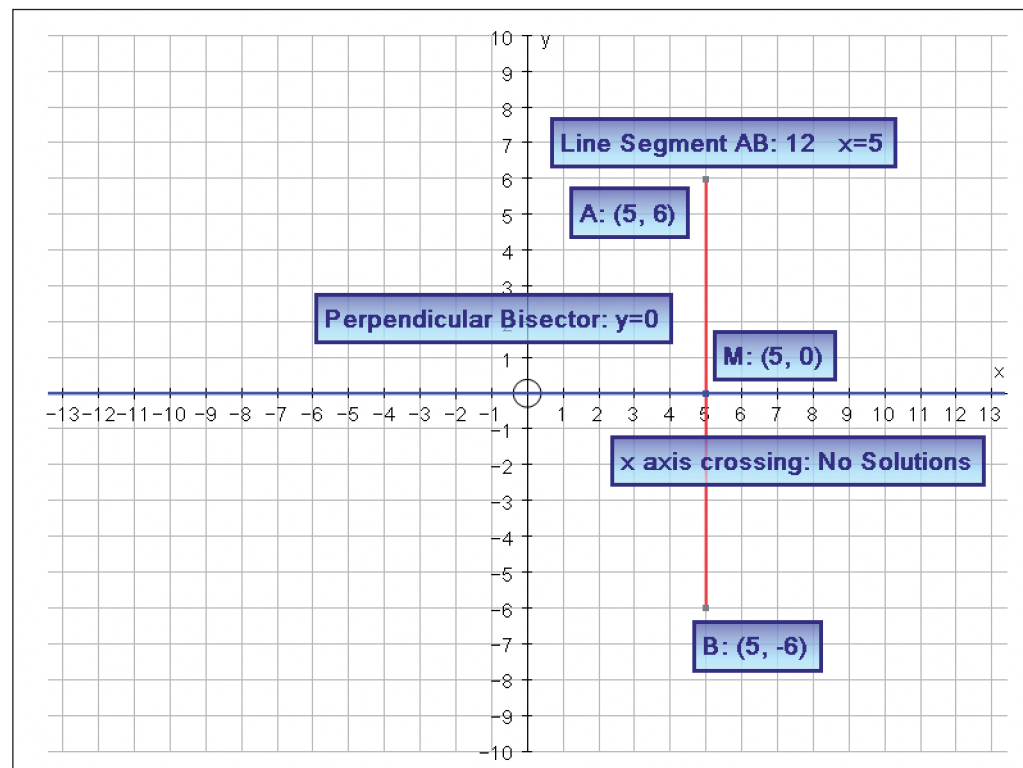
Perpendicular Bisector: x=0

x axis crossing: No Solutions
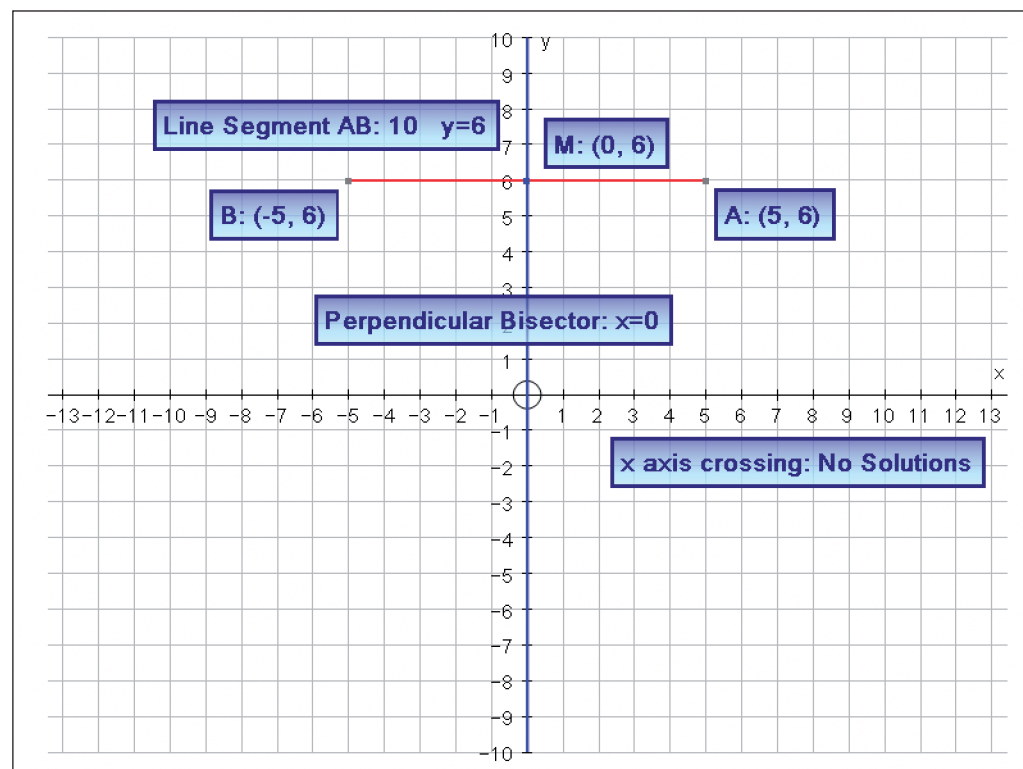
Predict: Perpendicular Bisector passes through (1, 0) - many possible answers, of which two are:

Line Segment AB: 10.2   y=-5x+31

A: (5, 6)

Perpendicular Bisector: y=0.2x-0.2

M: (6, 1)

x axis crossing: (1, 0)

B: (7, -4)

Line Segment AB: 14.14   y=x+1

A: (5, 6)

Perpendicular Bisector: y=-x+1

M: (0, 1)

x axis crossing: (1, 0)

B: (-5, -4)
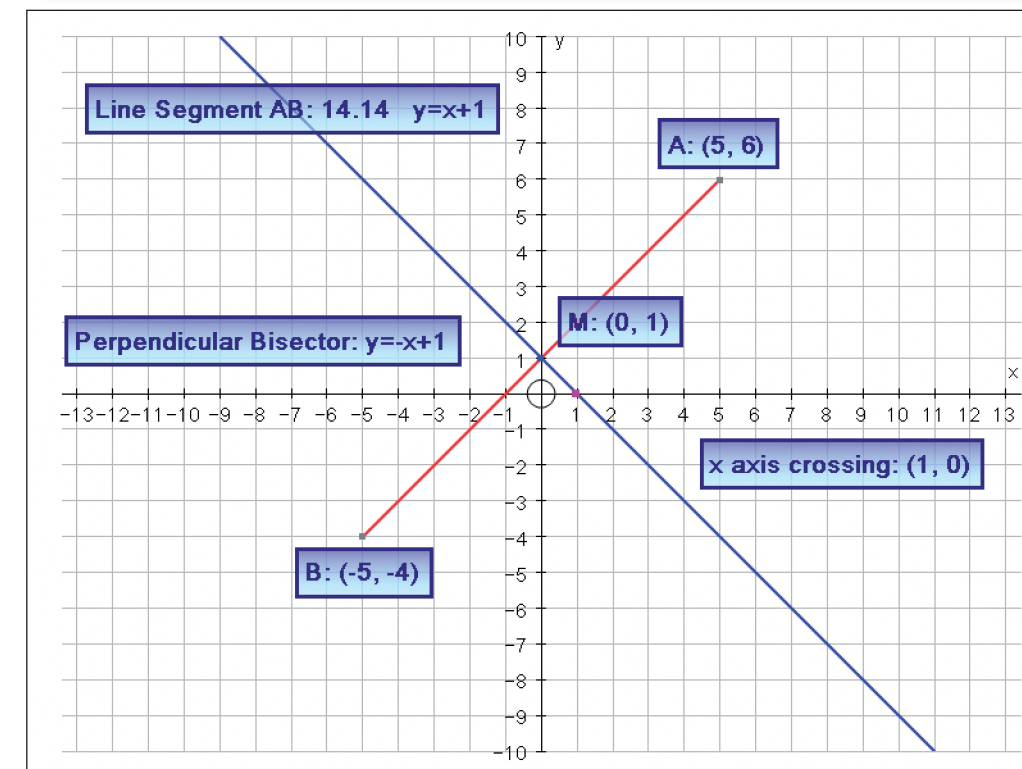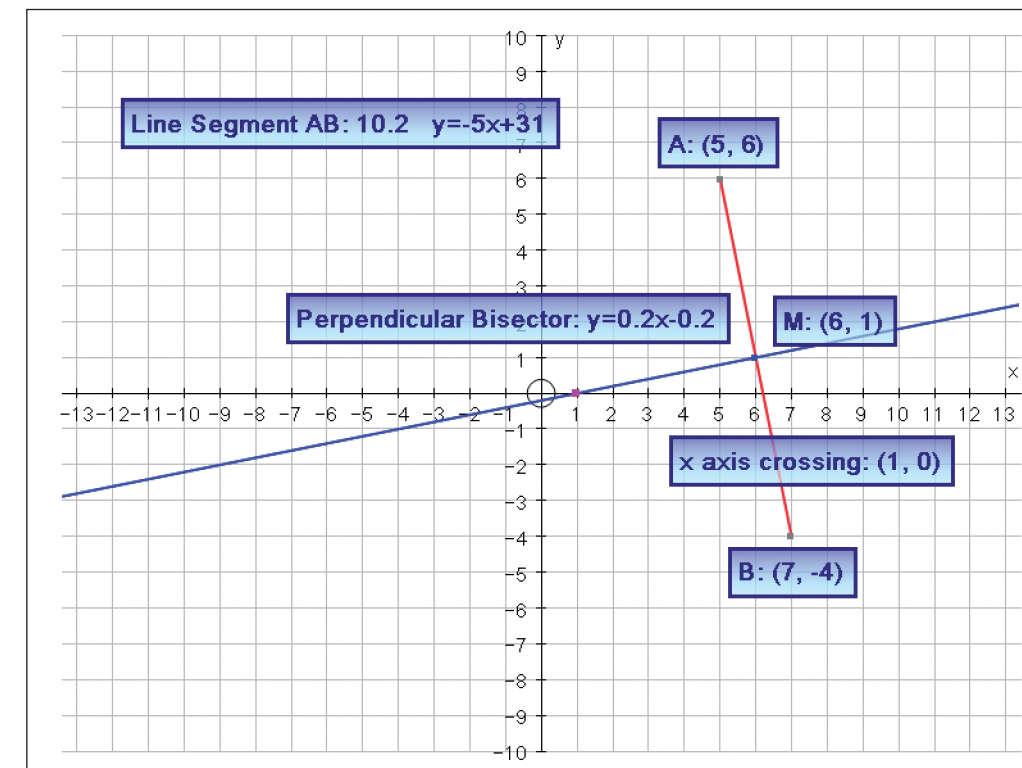
ACTIVITY 2

TASK 8: Equation of a circle with C at centre and D on the circumference:

(1) Find radius:

$= \sqrt{(4-0)^2 + (3-0)^2} = \sqrt{25} = 5$

(2) Find equation of circle:

$(x − a)^2 + (y − b)^2 = r^2$

$x^2 + y^2 = 5^2$

TASK 9: Equation of a circle with E at centre and F on the circumference:

(1) Find radius:

$\sqrt{(−24 + 20)^2 + (−2 −4)^2} = \sqrt{52}$

(2) Find equation of circle:

$(x − a)^2 + (y − b)^2 = r^2$

$(x + 20)^2 + (y − 4)^4 = 52 = 7.211^2$ (4 sig fig)

Predict: Radius twice as big:
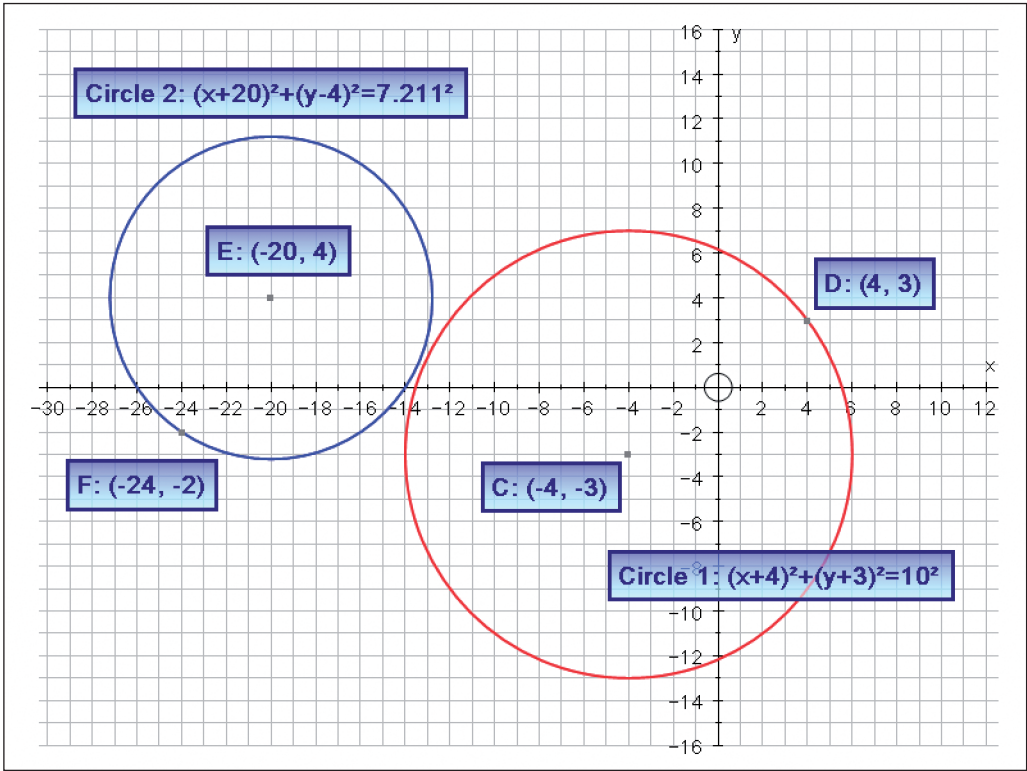
Predict: Circles have radius of 6 and touch at (−4, −4):
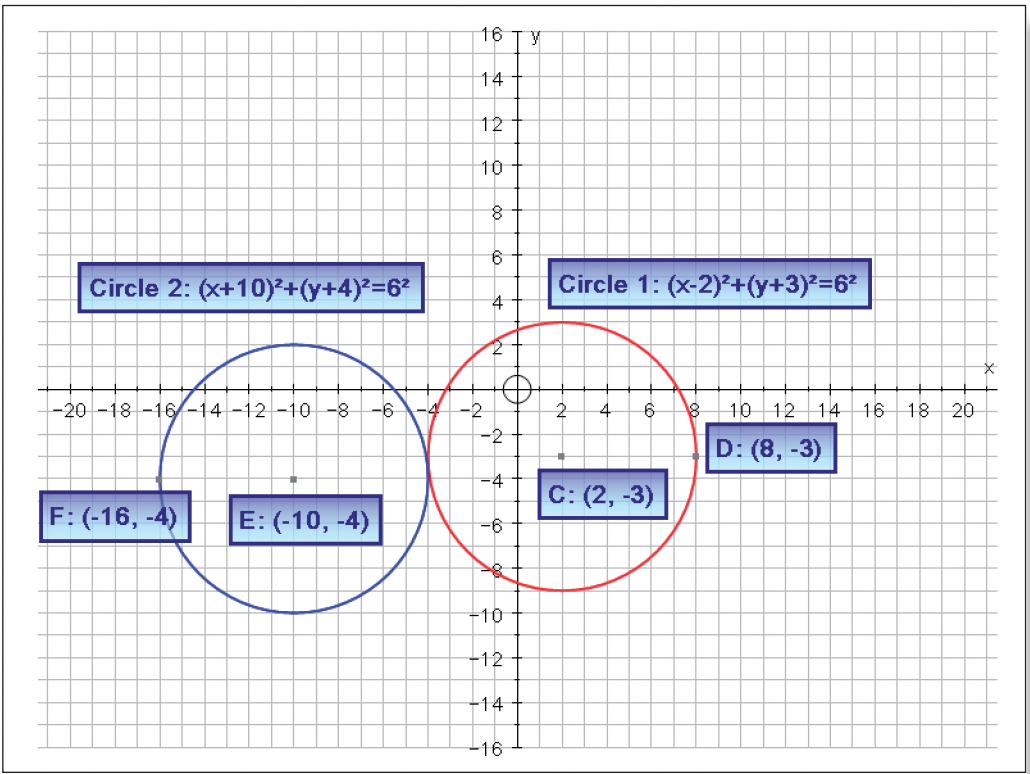
TASK 10: Radius = $\sqrt{36}$ = 6 and the centre = (20, 9).

TASK 11: Centre and radius of circle:

$x^2 + y^2 + 20x + 16y + 100 = 0$

$(x + 10)^2 − 100 + (y + 8)^2 − 64 + 100 = 0$

$(x + 10)^2 + (y + 8)^2 = 64$

Radius = $\sqrt{64}$ = 8 and the centre = (−10, −8).
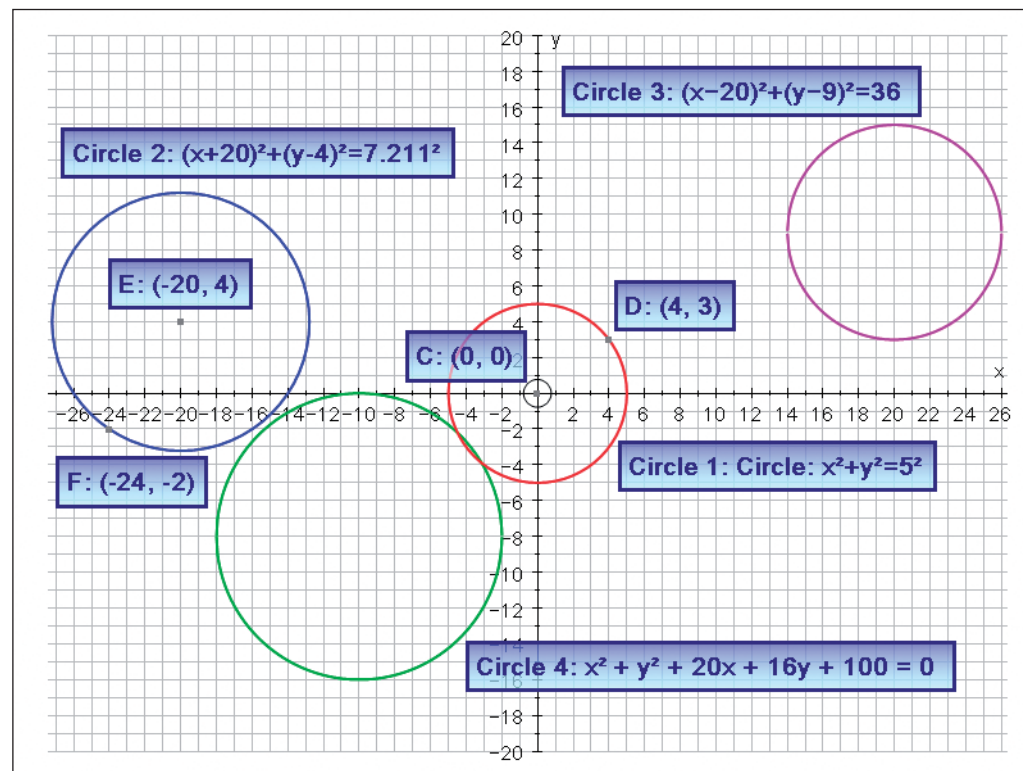
Activity 3

Task 12: Showing that point (9, −7) lies on the circle:

$x^2 + y^2 - 8x + 4y = 30$
$x = 9, y = -7$
$9^2 + (-7)^2 - 8(9) + 4(-7) = 30$

Task 13: Gradient of line segment CP:

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 + 2}{9 - 4} = -1$$

Task 14: Equation of tangent to circle passing through point P:

Gradient of tangent = 1

$y - y_1 = m(x - x_1)$
$y + 7 = 1(x - 9)$
$y = x - 16$
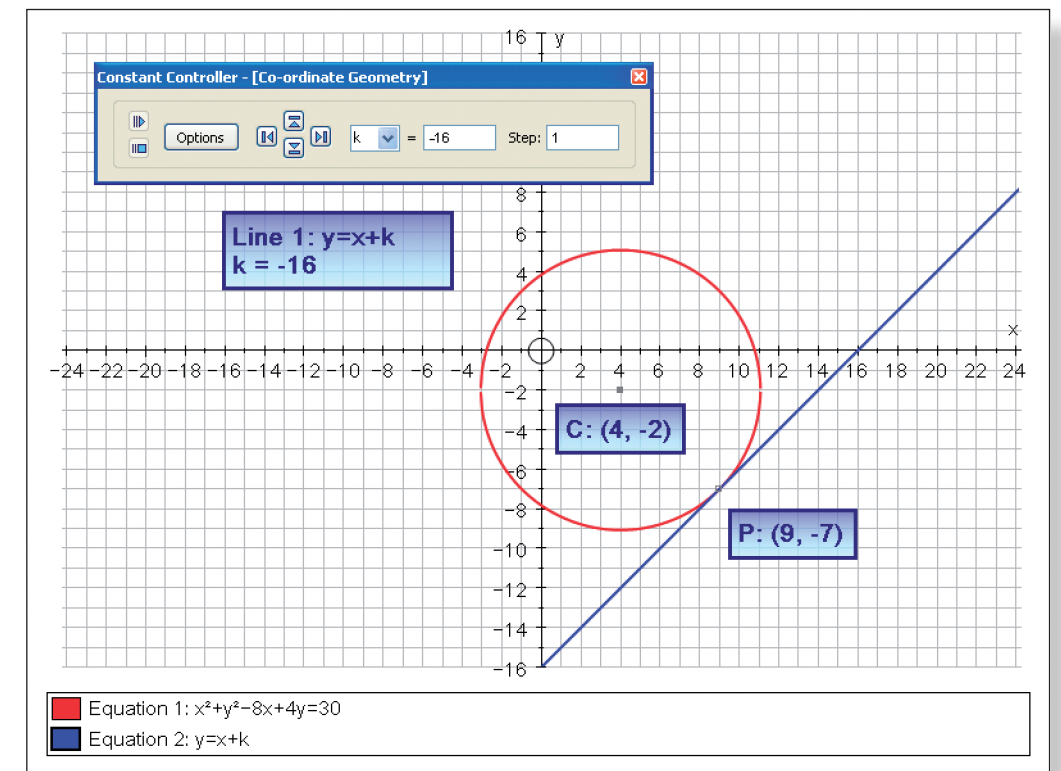
Task 15: Equation of other tangent: y = x + 4

Co-ordinates of intersection: (−1, 3)
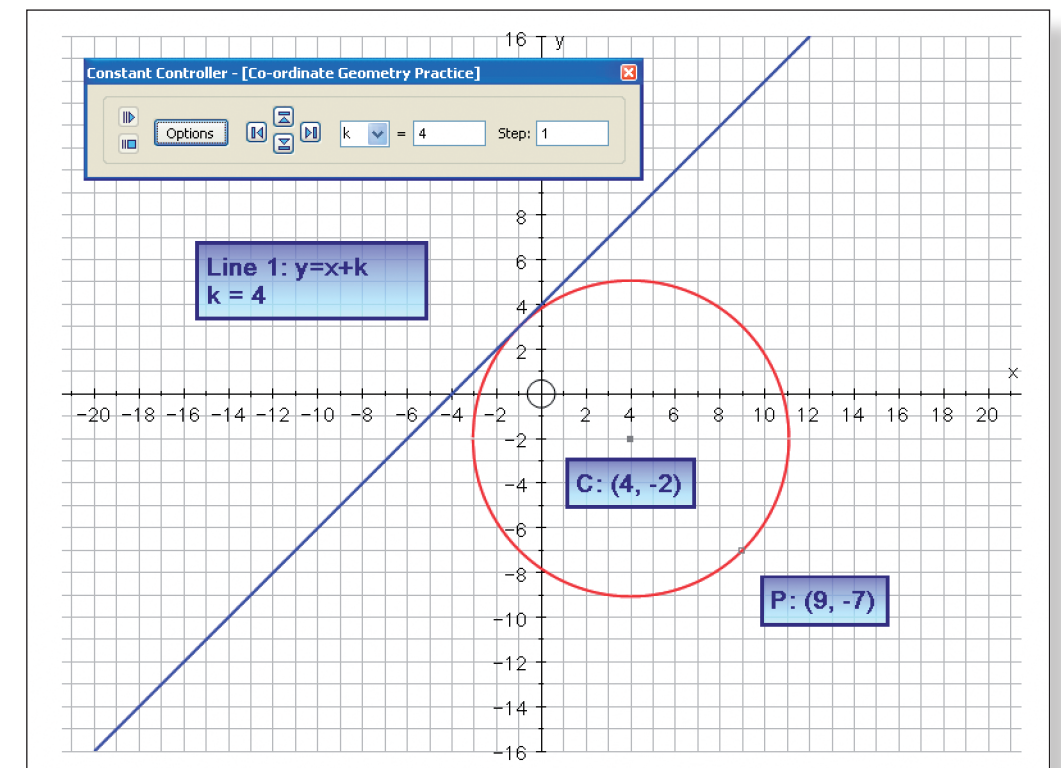
Task 16: Proving a line is a tangent to a circle:

Equation of line: y = x + 4

Substitute into equation of circle:

$$x^2 + (x + 4)^2 - 8x + 4(x + 4) = 30$$
$$x^2 + x^2 + 8x + 16 - 8x + 4x + 16 = 30$$
$$2x^2 + 4x + 2 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$

There is a repeated root ($x = 1$), so the line only intersects the curve at one point, therefore it must be a tangent.

TASK 17: Calculating the area of the triangle CPT:

Length of line segment PT:

$$\sqrt{(16 - 9)^2 + (0 + 7)^2} = \sqrt{98}$$

Length of line segment CP:

$$= \sqrt{(9 - 4)^2 + (-7 + 2)^2} = \sqrt{50}$$

Line segments PT and CP must be perpendicular as PT is part of a tangent.

Therefore, triangle CPT is a right angled triangle.

Area = ½ (base × height)

$$= \frac{\sqrt{50} \times \sqrt{98}}{2} = \frac{5\sqrt{2} \times 7\sqrt{2}}{2} = 35$$
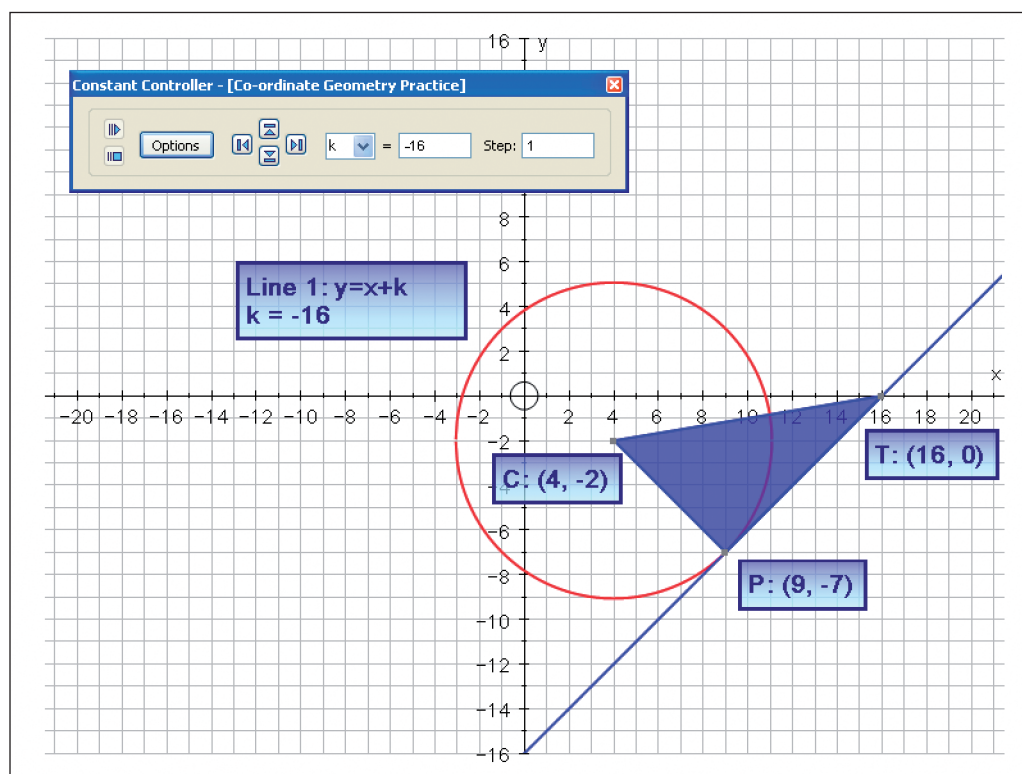
TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT

- To investigate the relationship between the roots of a cubic equation in the form $f(x) = (x − a)(x − b)(x − c)$, and the tangents to the mid-points of pairs of these roots.

- To practice and consolidate important mathematical skills, including algebraic manipulation, substitution and differentiation, in the context of co-ordinate geometry.

- To appreciate the difference between specific examples and general cases.

- To understand the process involved in testing a theorem, generalising, and then proving it.

EXTENSION MATERIAL

- To be able to use algebraic skills to form a proof for a specific type of cubic equation.

- To be able to understand why this proof is actually a proof for all types of cubic equations.

REQUIRED PRIOR KNOWLEDGE

To be able to carry out the following tasks:

- Use the Factor Theorem (or another suitable method) to factorise a cubic into the form $f(x) = (x − a)(x − b)(x − c)$.

- To understand the relationship between cubic functions in the form $f(x) = (x − a)(x − b)(x − c)$ and their graphical representation.

- To substitute both numbers and letters into formulae.

- To be comfortable manipulating and simplifying algebraic expressions, including those involving powers and fractions.

- To be able to differentiate polynomials.

- To be able to use differentiation to find the gradient of a tangent at a given point.

- To have a general appreciation of the concept of proof.

Description of the Activity

- This activity is designed to investigate an extremely interesting property of all cubic graphs that contain three real roots – namely, that a tangent drawn to the mid-point of any two of these roots will always pass through the third root. It is also designed as a means to practice and consolidate important mathematical skills such as differentiation, algebraic manipulation, and work general work on co-ordinate geometry.

- In Activity 1, students are encouraged to discover this property of cubic functionss through a specific example. They are called upon to use such skills as differentiation, substitution into formulae, and key co-ordinate geometry techniques, including finding the equations of tangents.

- In Activity 2, students have the opportunity to use Autograph's **Constant Controller** function to dynamically test whether this property works for other cubic functionss. This is the first important step on the road towards generalisation and proof.

- Activity 3 contains the extension material. Here students are carefully guided through a proof of the property, using Autograph as a visual stimulus.

Pre-Activity Setup

None required.

Suggested Introduction

"Today we are going to use Autograph to help us investigate an extremely interesting property of cubic functions and their graphs. Once you have discovered this property, you then have the opportunity to prove that it works for every single cubic equation."

Suggested Prompts

- The Autograph functions called upon during this investigation are relatively straight forward, and most students should sail through the technical aspects of this investigation without any trouble. The most important Autograph feature, in terms of the success of the investigation, is the correct use of the **Constant Controller**. Just be on hand to help out any students who may struggle at first with this.

- Mathematically, perhaps the most difficult concept is the move from the specific numerical example of Activity 1, to the general cases of Activities 2 and 3. Again, some students may need prompting in the right direction.

- The later stages of Activity 3 (the Extension Material) contains a "show that" style question so that pupils can check their calculations are on track. If they cannot get this answer, encourage them to go back a couple

of tasks to check through their work.

Suggested Plenary

- Can anybody summarise the key points they have learnt from this investigation?

- What other questions does this investigation throw up?

  » Does this property hold for other types of polynomials, for example quartic functionss?

  » What about when a cubic does not have three real roots? What about imaginary roots? Would this property still hold on an Argan Diagram?

Ideas for Further Work

- Revise any areas of this investigation that the pupils struggled on.

- Exam style questions on co-ordinate geometry.

STUDENT WORKSHEET   NAME:...

ACTIVITY 1 – AN INTERESTING CUBIC

TASK 1: Using the Factor Theorem (or any other method) express $f(x) = x^3 - 4x^2 - 7x + 10$ in the form:

$f(x) = (x - a)(x - b)(x - c)$

Hint: Work out f(1) to get you going!

TASK 2: Use your answer to Task 1 to make a quick sketch of the graph of f(x) in the space below, marking on the points where it crosses both the x-axis and the y-axis.

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Edit the axes as follows:
   x: Minimum: −4   Maximum: 6   Numbers: 1   Pips: 0.5

   y: Minimum: −24   Maximum: 20   Numbers: 2   Pips: 2

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Click on **Slow Plot** mode.

Enter the equation: $y = x^3 - 4x^2 - 7x + 10$
Note: To enter x³ use the little 3 button, or press "**Alt 3**" together, or type "**xxx**".

Click **OK**.

The curve should begin to be drawn on the screen.

Press **Pause Plotting** both to stop the process and to resume it.
Note: Pressing the **Spacebar** also works!

Check that the position of the graph and the points where it crosses the axes match your prediction in Task 2

Important Point: The three points where the cubic graph crosses the x-axis are known as the "roots" of the cubic equation.

TASK 3: Work out the x value of the mid-point of the first two roots (x = −2 and x = 1).

TASK 4: Use your answer to Task 3 to work out the y value of this mid-point on the curve.

Enter your answers to Task 3 and Task 4 as a co-ordinate.

TASK 5: Use differentiation to work out the Gradient Function of $f(x) = x^3 -$

$4x^2 - 7x + 10.$

TASK 6: Use your answer to Task 5 to work out the gradient of the curve at the mid-point of the first two roots.

TASK 7: Use your answer to Tasks 3, 4 and 6 to work out the equation of the tangent to the curve at the mid-point between the first two roots. Give your answer in the form $y = mx + c$.

Enter your answer to Task 7 as an equation and click **OK**.

Check: Does your answer form a tangent to the curve at the mid-point of roots a and b? If not, don't worry, but you had best go back and check your work!

TASK 8: What do you notice about where the tangent crosses the x-axis?

TASK 9: Using a similar method, find the equation of the tangent to the curve at the mid-point of the second and third roots ($x = 1$ and $x = 5$).

Enter your answer to Task 8 as an equation and click **OK**.

Important Point: As far as we know, this interesting relationship between the roots and the tangents may just be particular to this cubic. We are now going to see if it is true of all cubics.

Activity 2: Let's Generalise!

Clear the screen by selecting **Edit** > **Select All** and pressing **Delete** on the keyboard.

Note: You can do the same thing by **right-clicking** on an unoccupied part of the graph area.

You should now be left with just a set of axes.

Enter the equation: $y = (x - a)(x - b)(x - c)$

Still in the **Add Equation** window, click on **Edit Constants** and adjust the values as follows: $a = -2$, $b = 1$ and $c = 5$.
Click **OK**.

The graph of the same cubic curve as in Activity 1 should now appear on the screen.

Left-click on the curve (it should turn black).

Enter a co-ordinate with an **x** value of **a**.

Click **OK**.

Left-click to make sure point a is selected (it should have a square around it).

Select **Text Box** from the toolbar.
Click **Remove Object Text**.
Change the word "Point" to "**a**" and click **OK**.

Point a should now be labelled.

Repeat the above instructions to create and label points **b** and **c**.

Left-click again on the curve to make sure nothing else is selected (it should be black).

Select **Text Box** from the toolbar.
Tick the box next to **Show Detailed Object Text** and click **OK**.

The **Text Box** should now display the equation of the curve, together with the current values of the constants **a**, **b** and **c**.

Click on the **Constant Controller** on the top toolbar.
The **drop-down** menu allows you to select different constants.
The **up-down** arrows alter the value of the selected constant.
The **left-right** arrows adjust the size of the step.

Use the **Constant Controller** to adjust the values of all three constants and observe the effect on the graph of the cubic.

Return the constants to their original values: **a** = −2, **b** = 1 and **c** = 5.

Select **No Axes** from the toolbar.

You should now be left with your curve and the three labelled roots.

TASK 10: How would you write the mid-point of the first two roots (**x** = **a** and **x** = **b**) in terms of **a** and **b**?

Left-click on the curve (it should turn black).

Enter a co-ordinate with an **x** value equal to your answer to Task 9.
Note: You may need to use brackets, and to divide use the / symbol.

Click **OK**.

Left-click to ensure that just this new point is selected (it should have a square around it).

Right-click and select **Tangent** from the menu.

The tangent should go through root **c** as before.

Left-click on the tangent (it should turn black).

Right-click and select **Edit Draw Options** from the menu.

Change the **Dash Style** to one of the dashed line options, and the **Colour** to whatever you like.

Click **OK**.

TASK 11: Repeat Task 10 and the instructions that followed to create a tangent to the mid-point of roots **b** and **c**, and then one to the mid-point of roots **a** and **c**.

Use the **Constant Controller** to adjust the values of all three constants and observe the effect on the tangents and the roots.

TASK 12: Use your Autograph page to briefly explain the relationship between the roots of a cubic equation and the tangents to the mid-points of these roots.

TASK 13: Adjust the values of constants a and b so that they are the same. Briefly explain what happens to the tangent at the mid-point of these two roots. Is the relationship between the roots and the tangents still the same?

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TASK 14: Adjust the values of constants so that two of the tangents are parallel to each other. What do you notice about the values of the roots of the cubic equation?

Hint: Look carefully at the shape of the graph. Find another set of values where the tangents are parallel if you don't spot it first time!

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Important Point: Even though we have used Autograph to test out lots of different cubics, we cannot be sure that the relationship between roots and tangents is true for every cubic. But we can if we can prove it…

Activity 3: Getting to the Root of the Problem

Adjust the values of the constants to the following values: a = −3, b = 0 and c = 3.5.

Close the **Constant Controller**.

Left-click on the tangent that passes through root a (it should turn black).

Right-click and select **Delete** from the menu, or simply press **Delete** on the keyboard.

Left-click on the tangent that passes through root b (it should turn black).

Right-click and select **Delete** from the menu, or simply press **Delete** on the

keyboard.

You should now be left with the curve and one tangent passing through root c.

Select **Cartesian Grid** from the toolbar to see the axes again.

Important Point: To make this proof a little more straight-forward, we are going to assume that the middle root passes through the origin, and hence the value of b is 0. Therefore, the equation of the curve is given by: f(x) = (x − a)(x − 0)(x − c), or more simply: f(x) = x(x − a)(x − c).

TASK 15: What is the x value of the mid-point of roots a and b in terms of a?

Hint: Remember, b = 0.

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TASK 16: Task 16: Use your answer to Task 15, together with the equation of the function: f(x) = x(x − a)(x − c) to work out the y value of the curve at this mid-point, writing your answer as simply as possible.

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TASK 17: Use differentiation to work out the gradient function of f(x) = x(x − a)(x − c).

Hint: Expand the brackets first!

TASK 18: Use your answers to Task 15 and Task 17 to work out the gradient of the tangent to the curve at the mid-point between roots a and b.

TASK 19: Use your answers to Tasks 15, 16 and 18 to show that the equation of the tangent to the curve at the mid-point of roots a and b can be written as:
$4y + a^2x = a^2c$

Hint: Use your algebraic skills to get rid of any nasty looking fractions as early as possible!

TASK 20: Use your answer to Task 19 to show that this tangent passes through root c.

TASK 21: Explain briefly how, even though we assumed that the middle root of the cubic (b) passed through the origin, our answer to Task 20 is actually a proof of the special relationship between tangents and roots for any cubic in the form $f(x) = (x − a)(x − b)(x − c)$, where a, b and c are real numbers.

ANSWERS

ACTIVITY 1 – AN INTERESTING CUBIC

TASK 1: Using the Factor Theorem:

$f(x) = x^3 - 4x^2 - 7x + 10$
$f(1) = (1)3 - 4(1)^2 - 7(1) + 10 = 0$

So $(x - 1)$ must be a factor.

Using long division, equating co-efficients or another method:

$(x - 1)(x^2 - 3x - 10) = (x - 1)(x + 2)(x - 5)$

TASK 2: The roots are: $x = -2$, $x = 1$, and $x = 5$

The y-intercept is 10.

Quick Sketch:

(Need a Sketch!!!!!)

Equation 1: $y = x^3 - 4x^2 - 7x + 10$
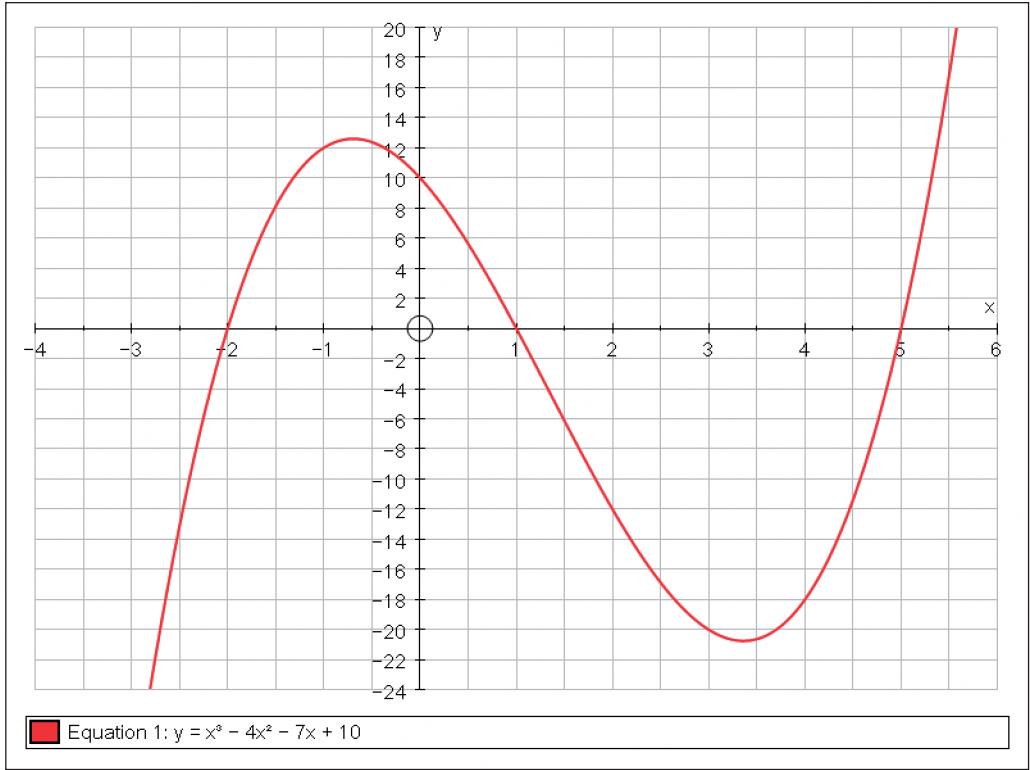
TASK 3: Mid-point (x value) = $(-2 + 1)/2$

$x = -0.5$

TASK 4: y value at mid-point:

$$f(x) = x^3 - 4x^2 - 7x + 10$$
$$f(-0.5) = (-0.5)^3 - 4(-0.5)^2 - 7(-0.5) + 10 = 12.375$$

TASK 5: Gradient Function:

$$f(x) = x^3 - 4x^2 - 7x + 10$$
$$f'(x) = 3x^2 - 8x - 7$$

TASK 6: Gradient at mid-point:

$$f'(x) = 3x^2 - 8x - 7$$
$$f'(-0.5) = 3(-0.5)^2 - 8(-0.5) - 7 = -2.25$$

TASK 7: Equation of tangent:

$$y - y_1 = m(x - x_1)$$
$$y - 12.375 = -2.25(x + 0.5)$$
$$y = -2.25x + 11.25$$

Equation 1: y = x³ − 4x² − 7x + 10
Equation 2: y = −2.25x + 11.25
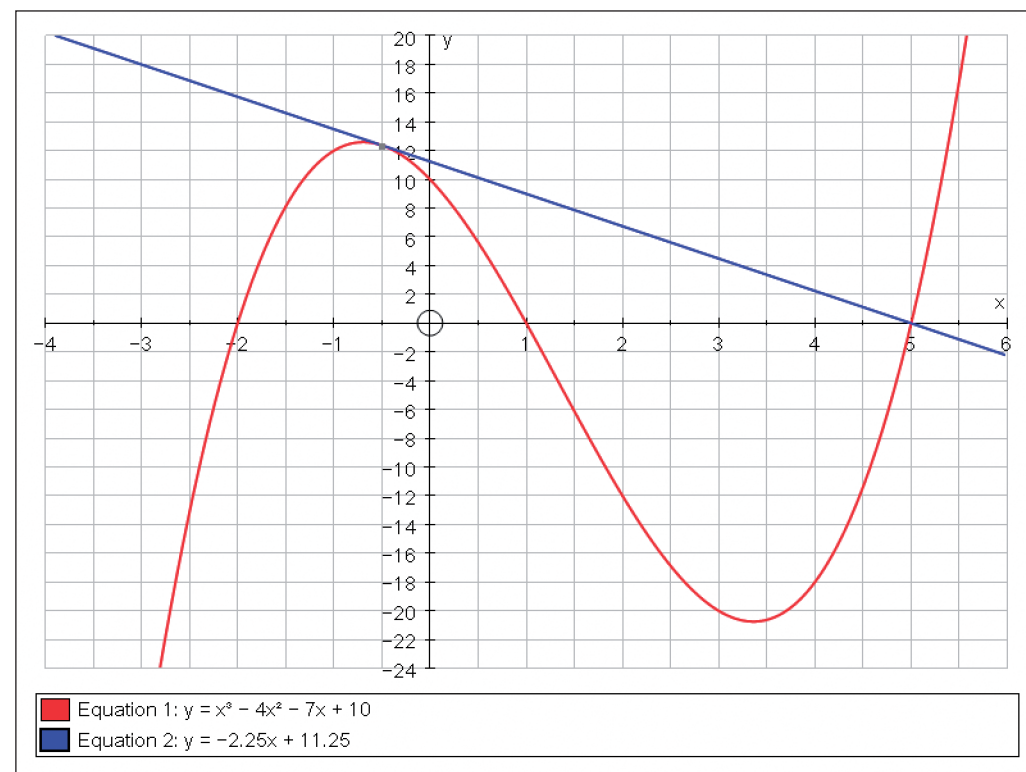
TASK 8: The tangent passes through the other root (x = 5).

TASK 9: x value of mid-point: $\frac{(1 + 5)}{2} = 3$

y value: $f(x) = x^3 - 4x^2 - 7x + 10$
$$f(3) = (3)^3 - 4(3)^2 - 7(3) + 10 = -20$$

Gradient function: $f'(x) = 3x^2 - 8x - 7$

Gradient at mid-point: $f'(x) = 3x^2 - 8x - 7$

$$f'(3) = 3(3)^2 - 8(3) - 7 = -4$$

Equation of tangent:
$$y - y_1 = m(x - x_1)$$
$$y + 20 = -2.25(x - 3)$$
$$y = -4x + 8$$

Activity 2: Let's Generalise!

TASK 10: x value of mid-point: $\frac{(a + b)}{2}$

TASK 11: x value of mid-points: $\frac{(b + c)}{2}$ and $\frac{(a + c)}{2}$

TASK 12: The tangent to the curve at the mid-point of any two roots seems to cross the third root on the x-axis.

TASK 13: When two of the roots are the same (repeated root), the relationships between the roots and the tangents is still true. The roots now lie on a stationary point, and hence the gradient is zero, which means the tangent becomes the x-axis, and thus necessarily passes through the third root.

TASK 14: When the two of the tangents are parallel, it means the roots are equidistant (evenly spaced) along the x-axis.

Activity 3: Getting to the Root of the Problem

TASK 15: Mid-point (x value) = $\frac{a}{2}$

TASK 16: y value at mid-point:

$$f(x) = x(x - a)(x - c)$$

$$f\left(\frac{a}{2}\right) = \frac{a}{2}\left(\frac{a}{2} - a\right)\left(\frac{a}{2} - c\right)$$

$$= -\frac{a^2}{4}\left(\frac{a}{2} - c\right)$$

$$= -\frac{a^3}{8} + \frac{a^2 c}{4}$$

$$= \frac{2a^2 c - a^3}{8}$$

TASK 17: Finding the Gradient Function:

$$f(x) = x(x - a)(x - c)$$
$$f(x) = (x^2 - xa)(x - c)$$
$$f(x) = x^3 - x^2 c - x^2 a + acx$$
$$f(x) = x^3 - x^2(a + c) + acx$$
$$f'(x) = 3x^2 - 2x(a + c) + ac$$

TASK 18: Gradient at the mid-point:

$$f'(x) = 3x^2 - 2x(a + c) + ac$$

$$f'\left(\frac{a}{2}\right) = 3\left(\frac{a}{2}\right)^2 - 2\left(\frac{a}{2}\right)(a + c) + ac = \frac{3a^2}{4} - a^2 - ac + ac = -\frac{a^2}{4}$$

TASK 19: Equation of tangent:

$$y - y_1 = m(x - x_1)$$

$$y - \left(\frac{2a^2c - a^3}{8}\right) = -\frac{a^2}{4}\left(x - \frac{a}{2}\right)$$

$$8y - (2a^2c - a^3) = -2a^2\left(x - \frac{a}{2}\right)$$

$$8y - 2a^2c + a^3 = -2a^2x + a^3$$
$$8y + 2a^2x = 2a^2c$$
$$4y + a^2x = a^2c$$

TASK 20: Equation of tangent: $4y + a^2x = a^2c$

At root c, $x = c$

Which gives: $4y + a^2c = a^2c$

$4y = 0$
$y = 0$

Therefore, the tangent passes through root c $(c, 0)$.

TASK 21: Any cubic with three real roots can be subjected to a horizontal translation to ensure the idle root (b) passes through the origin. Such a transformation would not alter the relative positions of tangents and roots, and hence the above proof would be valid. Hence, the relationship is true for any cubic in the form $f(x) = (x - a)(x - b)(x - c)$.

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

- To understand that the gradient of a curve changes at various points along it, as opposed to straight lines which each have a constant gradient throughout.

- To understand that the gradients of curves follow a pattern and are functions in their own right.

- To understand the relationship between Gradient Functions and tangents.

- To understand that we can use these Gradient Functions to find the gradient at any point along the curve without having to draw the curve or a tangent.

- To understand that quadratic functions have linear Gradient Functions and cubic functions have quadratic Gradient Functions.

- To be able to find the gradient function of standard quadratic and cubic functions and use this to find the gradient of the tangent at a given point.

EXTENSION MATERIAL:

- To be able to find the Gradient Function of polynomials of an order higher than 3.

- To begin to look at negative powers.

- To be able to express the rule for finding Gradient Functions using words and letters standing for constants.

Please Note: The aim of this investigation is *not* for the students to understand the concept of differentiation from first principles (this is covered in Teacher Demonstration 3). The aim is for students to be able to derive the rule for differentiating polynomials themselves and to be able to use this to find the gradient at any point on a curve.

REQUIRED PRIOR KNOWLEDGE

- Be aware of the concept of a tangent.

- Be understand that m represents the gradient when a straight line is expressed as y = mx + c.

- Be able to recognise the shapes of quadratic and cubic functions.

Description of the Activity

- Activity 1 introduces the curve $y = x^2$, and illustrates to the students that the gradient of the curve changes as you move along it. The students are then asked to record the gradient at various points along the curve, and this leads to them deriving the Gradient Function and using it to predict the gradient at various points.

- In Activities 2 to 4, a variety of constants are added to the basic quadratic function and the students investigate what effect this has on the gradient function. These activities illustrate the fact that lines in the form $y = kx$ have a gradient function of k, and lines in the form $y = k$ have a gradient function of 0. Because the students actually discover for themselves where these results come from, this should help reduce any future common errors.

- Activity 5 gives the students an opportunity to use what they have learnt to find the gradient functions of a variety of quadratic functions. They can use the self-checking mechanism on Autograph to confirm and adjust their answers. At the end of this activity, many students will be able to generalise, both in symbols and in words, the rule for finding the gradient function of quadratic functions.

- Activity 6 does the same thing, but this time for cubics, before moving onto the extension material. Here the students are asked to find the gradient functions of polynomials of an order greater than 3, as well as one function with a negative power. Again, they can use Autograph to check and adjust their answers.

Pre–Activity Setup

None required.

Suggested Introduction

"If you think about a curve, such as $y = x^2$, you will hopefully realise that its gradient is constantly changing: negative to positive, left to right. Today we are going to investigate exactly how that gradient changes, and not just for $y = x^2$. Hopefully, by the end of the activity, I will be able to give you the equation of any quadratic function, and a point that lies on it, and you will be able to tell me the gradient of the curve at the point. Some of you may even get onto doing them same for cubics, and quartics, and all sorts of complicated functions. And it's not just about finding the gradients of curves, because the things you learn today will form the basis of possibly the most important branch of mathematics ever discovered."

Suggested Prompts

- This activity makes use of several tools on Autograph. All of these are fairly simple to grasp… This is the answer to common complaints such as "my text box has labeled the wrong thing", and "my point won't animate".

- Once the students realise that they can check their answers using the Gradient Function on Autograph, their confidence and willingness to try out different answers and methods should increase, and the number of questions they direct at you should go down.

- Perhaps the trickiest part of this investigation is writing down the rules for finding the Gradient Function using both words and symbols. Drawing the students' attention to the examples they have already completed and asking them to explain what the question and answers have got in common should help steer them along the right path.

Suggested Plenary

- If I gave you the following function… and asked you to tell me that gradient when $x = \ldots$, how would you do it?

- Can anybody summarise in words how to find the Gradient Functions of quadratics, cubics, any function?

- Does anybody have any tips or techniques that they found useful when finding gradient functions?

- Can anybody think of any uses of this technique outside of finding the gradients of curves?

Ideas for Further Work

- Introduction of the "dy by dx" notation.

- Formalise the rule for differentiation in words and symbols.

- Consolidation by trying other examples.

- Exam style questions, perhaps in the form: "find two points on the curve which have a gradient of 3".

- Co-ordinate geometry work, such as finding the equation of tangents using the gradient function.

- Introduction of how to deal with negative powers.

- Moving on to look at how differentiation can help us with some co-ordinate geometry style questions, such as finding the co-ordinates of stationary points.

STUDENT WORKSHEET

NAME:...

ACTIVITY 1

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Edit the axes as follows:

    x: Minimum: −5   Maximum: 5   Numbers: 1   Pips: 0.5
    y: Minimum: −10   Maximum: 25   Numbers: 2   Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Click on **Slow Plot** mode (there should be a blue square around the button).

Enter the equation: y = x²
Note: To enter x², either use the little 2 button or press "**xx**".

Predict

> *What will this curve look like? Where will it cross the axes?*

Click **OK**.

The curve should begin to plot on the screen.

Left-click on the curve (it should turn black).

Enter the point with an **x** value of **2**.

This should display the point (2, 4) on the curve.

Select **Text Box** from the toolbar.
Change the word "Point" to "A" and click **OK**.

The point should now be labelled and its co-ordinates displayed.

Predict

> *What a tangent to this point look like? Which direction will it slope? Have a guess at the size of its gradient.*

Left-click to select Point A (it should have a square around it).

Right-click and select Tangent from the menu.

Left-click on tangent (it should turn black).

Select **Text Box** from the toolbar and click **OK**.

The equation of the tangent should now be displayed.

TASK 1: Using the equation of the tangent, what is the gradient of the curve y = x² at the point x = 2?

Predict

> *How will the equation of the tangent, and specifically the size of the gradient, alter as you move along the curve?*

Left-click on an unoccupied part of the graph to *de-select everything*.

Left-click on Point A (it should have a square around it).

Select **Animate Object** from the top toolbar.

You can now control the position of Point A using the **left-right** arrows.

Use the **Scribble Tool** to mark the points on the curve with the following x values: −4, −3, −2, −1, 0, 1, 3, 4.

TASK 2: Using the equation of the tangent and altering the position of Point A with the **Animation Controller**, find the gradient of the function at the x values marked on the curve. Each time you get an answer, write it in the table below.

Equation y = x²								
x	−4	−3	−2	−1	0	1	2	3	4
Gradient									
Gradient Function =									

TASK 3: Look at your completed table. What is the rule (in terms of x) that gets you from each x value to its gradient? This rule is called the Gradient Function of the original function y = x². Fill in your answer at the bottom of the above table.

TASK 4: Use your Gradient Function to predict the gradient of the curve when x = 2.7. Check your answer by moving Point A into the correct position using the **Animation Controller**.

Ensure **Slow Plot** mode is still turned on.

Make sure the curve is selected (it should be black).

Click on **Gradient Function**, and click **OK**.

This will plot the Gradient Function of the line y = x² as a dashed line.

Autograph will pause the plotting at important parts of the function. Press **Pause Plotting** (or the space bar on the keyboard) both to stop the process and to resume it. As you work through this investigation, think about why Autograph considers these particular points important.

Check: Think about the equation of this dashed line. Does it match the equation of the Gradient Function you have written in the table above?

Enter the equation of the Gradient Function you found (in the form y =) and check it matches.

Note: You will know if it matches if they appear as one line.

Crucial Point: By substituting x values into our Gradient Function, we can now use it to find the gradient at any point on the curve y = x².

We are now going to use the same method to find the Gradient Function of other quadratic functions, with the overall aim of finding a rule that can find us the Gradient Function of any function.

Activity 2

Open up a New **2D Graph Page**.

Edit the axes as follows:
x: Minimum: −8   Maximum: 2   Numbers: 1   Pips: 0.5
y: Minimum: −8   Maximum: 10   Numbers: 2   Pips: 1
Remove all of the green ticks underneath **Auto**.

Enter the equation: y = x² + 5x

Predict | *What will this curve look like? Where will it cross the axes?*

Click **OK**.

TASK 5: Using the same method described in Activity 1, use the equation of the tangent to find the gradient of the function at the following x values: −6, −5, −4, −3, −2, −1, 0 and 1. Each time you get an answer, write it in the table below.

Equation y = x2 + 5x								
x	−6	−5	−4	−3	−2	−1	0	1
Gradient								
Gradient Function =								

TASK 6: Look at your completed table. What is the rule (in terms of x) that gets you from each x value to its gradient? Fill in your answer at the bottom of the above table.

TASK 7: Use your Gradient Function to predict the gradient of the curve when x = −0.5. Check your answer by moving Point A into the correct position.

Make sure the curve is selected (it should be black).

Click on **Gradient Function**, and click **OK**.

This will plot the Gradient Function of the line y = x² + 5 as a dashed line.

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it.

Check: Think about the equation of this dashed line. Does it match the equation of the Gradient Function you have written in the table above?

Enter the equation of the Gradient Function you found (in the form y =) and check it matches.

TASK 8: Compare the Gradient Function of y = x² + 5x with that of y = x². What effect did the 5x have on the Gradient Function and can you explain this?

Hint: Think about the gradient of tangents to the line y = 5x.

Activity 3

Open up a New **2D Graph Page**.

Edit the axes as follows:
x: Minimum: −5   Maximum: 5   Numbers: 1   Pips: 0.5

y: Minimum: −10   Maximum: 20   Numbers: 2   Pips: 1
Remove all of the green ticks underneath **Auto**.

Enter the equation: y = x² − 3

Predict

What will this curve look like? Where will it cross the axes? How does it compare to the graph of y = x²?

Click **OK**.

TASK 9: Using the same method described in Activity 1, use the equation of the tangent to find the gradient of the function at the following x values: −4, −3, −2, −1, 0, 1, 2, 3 and 4. Each time you get an answer, write it in the table below.

Equation y = x² − 3									
x	−4	−3	−2	−1	0	1	2	3	4
Gradient									
Gradient Function =									

TASK 10: Look at your completed table. What is the rule (in terms of x) that gets you from each x value to its gradient? Fill in your answer at the bottom of the above table.

TASK 11: Use your **Gradient Function** to predict the gradient of the curve when x = −3.9. Check your answer by moving Point A into the correct position.

Make sure the curve is selected (it should be black).

Click on **Gradient Function**, and click **OK**.

This will plot the Gradient Function of the line y = x² − 3 as a dashed line.

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it.

Check: Think about the equation of this dashed line. Does it match the equation of the Gradient Function you have written in the table above?

Enter the equation of the Gradient Function you found (in the form y =) and check it matches.

TASK 12: Compare the Gradient Function of y = x² − 3 with that of y = x². What effect did the −3 have on the Gradient Function and can you explain this?

Hint: Think about the gradient of tangents to the line y = −3.

ACTIVITY 4

Open up a New **2D Graph Page**.

Edit the axes as follows:
   x: Minimum: −3   Maximum: 3   Numbers: 1   Pips: 0.5
   y: Minimum: −10   Maximum: 20   Numbers: 2   Pips: 1
Remove all of the green ticks underneath **Auto**.

Enter the equation: y = 3x²

Predict

What will this curve look like? Where will it cross the axes? How does it compare to the graph of y = x²?

TASK 13: Using the same method described in Activity 1, use the equation of the tangent to find the gradient of the function at the following x values : −2.5, −2, −1, 0, 1, 2 and 2.5. Each time you get an answer, write it in the table below.

Equation y = 3x2							
x	−2	−1	0	1	2	−2.5	2.5
Gradient							
Gradient Function =							

TASK 14: Look at your completed table. What is the rule (in terms of x) that gets you from each x value to its gradient? Fill in your answer at the bottom of the above table.

TASK 15: Use your **Gradient Function** to predict the gradient of the curve when x = 1.3. Check your answer by moving Point A into the correct position.

Make sure the curve is selected (it should be black).

Click on **Gradient Function**, and click **OK**.

This will plot the Gradient Function of the line y = 3x² as a dashed line.

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it.

Check: Think about the equation of this dashed line. Does it match the equation of the Gradient Function you have written in the table above?

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it.

Enter the equation of the Gradient Function you found (in the form y =) and check it matches.

TASK 16: Compare the Gradient Function of y = 3x² with that of y = x². What effect did the 3 in front of the x² have on the Gradient Function?

ACTIVITY 5

TASK 17: Using what you have learnt in Activities 1 to 4, can you *predict* what the Gradient Function would be to the function: y = 4x² + 6x − 5

Hint: Think about each term in turn and what effect it will have on the **Gradient Function**.

Open up a New **2D Graph Page**.

Edit the axes as follows:
x: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.5
y: Minimum: −10   Maximum: 20   Numbers: 2   Pips: 1
Remove all of the green ticks underneath **Auto**.

Enter the equation: y = 4x² + 6x − 5

Predict   *What will this curve look like? Where will it cross the axes?*

Make sure the curve is selected (it should be black).

Click on **Gradient Function**, and click **OK**.

This will plot the Gradient Function of the line y = 4x² + 6x − 5 as a dashed line.

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it.

Check: Think about the equation of this line. Does this line match the Gradient Function you predicted in Task 16?

Enter the answer you got (in the form y =) and check it matches. If not, don't worry, just have another think and try another one!

TASK 18: *Predict* the Gradient Function for the following three functions. Each time use the above method (each on a New **2D Graph Page** and editing the axes where necessary) to check your answer is correct.

Function	Gradient Function
y = 5x² + 7x + 6	
y = 8x² − 2x − 9	
y = −3x² + x + 12	

TASK 19: Each of the functions we have looked so far at in this investigation have been Quadratic Functions. Can you summarise in words how to find the Gradient Function of any quadratic function?

TASK 20: What about expressing the rules mathematically. If you were given the function: y = ax² + bx + c where a, b and c could be any number, what would be the Gradient Function?

TASK 21: Each of the functions we have looked so far at in this investigation have been Quadratic Functions. What have all the Gradient Functions had in common?

We are now going to use have a look at other types of functions, still with the overall aim of finding a rule that can find us the Gradient Function of *any* function involving powers of x.

ACTIVITY 6

Open up a **New 2D Graph Page**.

Edit the axes as follows:

    x: Minimum: −3   Maximum: 3   Numbers: 1   Pips: 0.5
    y: Minimum: −20 Maximum: 20 Numbers: 2 Pips: 1

Remove all of the green ticks underneath **Auto**.

Enter the equation: $y = x^3$
Note: To enter x^3, either use the little 3 button or press "**xxx**".

Predict

What will this curve look like? Where will it cross the axes?

Make sure the curve is selected (it should be black).

Click on **Gradient Function**, and click **OK**.

This will plot the Gradient Function of the line $y = x^3$.

Press **Pause Plotting** (or the **Spacebar** on the keyboard) both to stop the process and to resume it. Again, think about why Autograph is pausing the plotting at these points.

TASK 22: Look at the Gradient Function of $y = x^3$. What type of function is it? (Linear, quadratic, cubic or something else...)

TASK 23: Thinking about the rules for finding the gradient function of quadratic functions, and the answer to Task 22, can you predict what the Gradient Function of $y = x^3$ is?

Enter your answer as an equation (in the form y =) and keep trying until you get it!

TASK 24: Using the answers to Task 23 and Task 18, predict the Gradient Function for the following three cubic functions. Each time use the above method (each on a **New 2D Graph Page** and editing the axes where necessary) to check your answer is correct.

Function	Gradient Function
$y = 4x^3$	
$y = 3x^3 + 2x^2 − 5x − 10$	
$y = −2x^3 − 5x^2 + 3x + 4$	

TASK 25: Can you express the rule for finding the Gradient Function of a cubic function mathematically? If you were given the function: $y = ax^3 + bx^2 + cx + d$, where a, b, c and d could be any number, what would be the Gradient Function?

TASK 26: Using what you have learnt in this investigation, can you predict what the Gradient Function will be for the following functions? Each time use the above method (each on a **New 2D Graph Page** and editing the axes where necessary) to check your answer is correct.

Function	Gradient Function
$y = 6x^5$	
$y = 4x^4 − 2x^3 + 8x − 2$	
$y = x^{-1}$	

TASK 27: Can you express in words a general rule of how to find the Gradient Function of any of the functions which we have looked at in this investigation?

Summary

Use the space below to summarise what you have learnt during this activity:

ANSWERS

ACTIVITY 1

TASK 1: The gradient at (2, 4) is 4.

TASKS 2 AND 3: Completed table:

Equation $y = x^2$									
x	−4	−3	−2	−1	0	1	2	3	4
Gradient	−8	−6	−4	−2	0	2	4	6	8
Gradient Function = 2x									

TASK 4: The gradient when x = 2.7 is 5.4.

Adjust Parameter — Animate — Position: 2.7 — Step: 0.1
Tangent: y=5.4x-7.29
A: (2.7, 7.29)
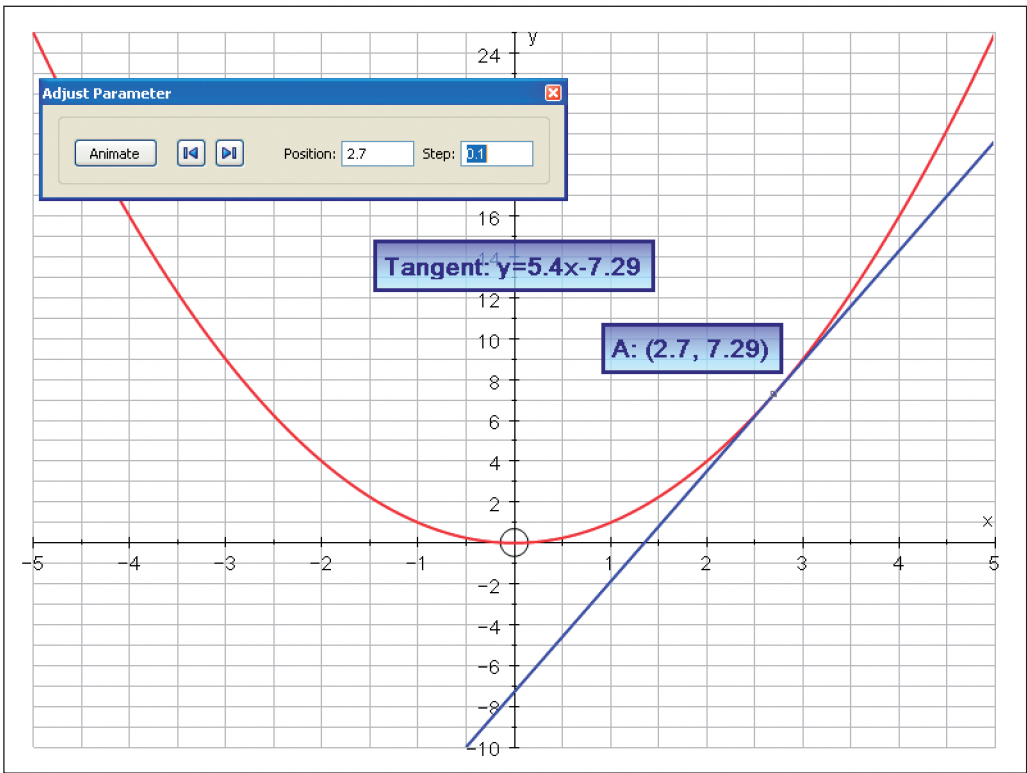
ACTIVITY 2

TASKS 5 AND 6: Completed table:

Equation $y = x^2 + 5x$						
x	−6	−5	−4	−3	−2	−1	0	1
Gradient Function = 2x + 5						

Equation $y = x^2 + 5x$							
Gradient	−7	−5	−3	−1	1	3	5	7
Gradient Function = 2x + 5							

TASK 7: The gradient when x = −0.5 is 4.

TASK 8: The 5x component of the function simply added a constant value of 5 to our Gradient Function. This is because the gradient at any point on the line y = 5x is always 5. This is the intuition behind why lines in the form y = kx give a Gradient Function of k.

Tangent: y=4x-0.25
Adjust Position — Animate — Position: -0.5 — Step: 0.1
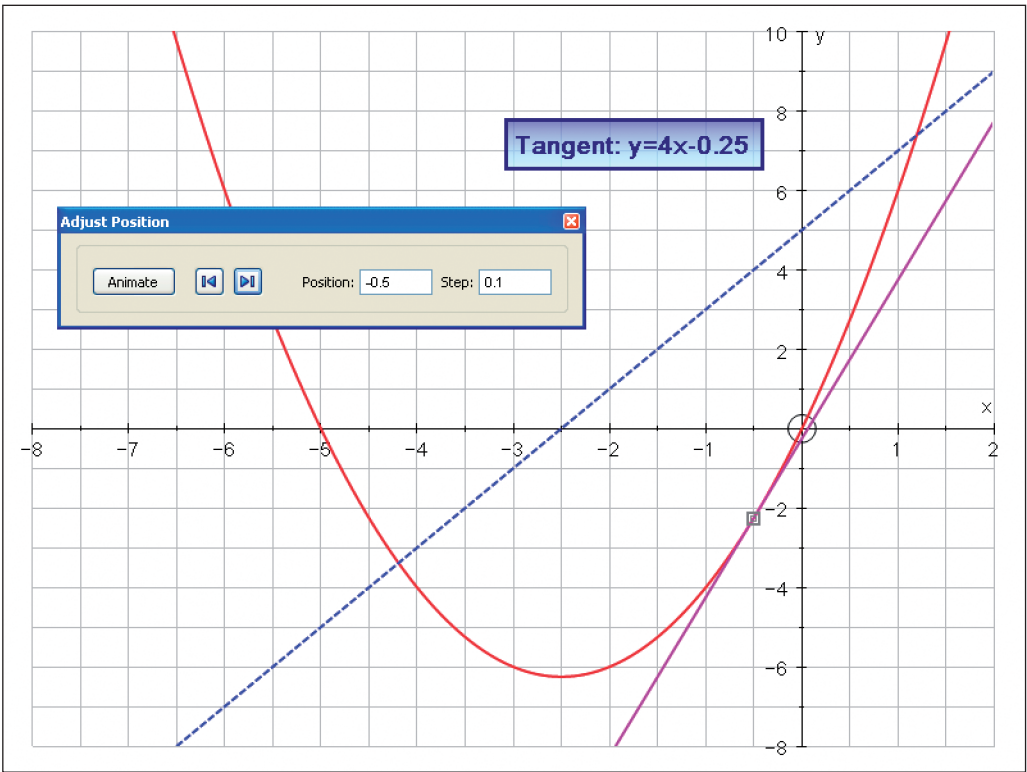
ACTIVITY 3

TASKS 9 AND 10: Completed table:

Equation $y = x^2 − 3$									
x	−4	−3	−2	−1	0	1	2	3	4
Gradient	−8	−6	−4	−2	0	2	4	6	8
Gradient Function = 2x									

TASK 11: The gradient when x = −3.9 is −7.8.

TASK 12: The −3 component of the function had no effect on the gradient at each x value. This is because the gradient at the any point in the line y = −3 is always 0. Hence, all that has happened is that is y = x² has been translated down three units to give y = x² − 3, and so the gradient at any given x value remains

the same. This is the intuition behind why lines in the form y = k give a Gradient Function of 0.
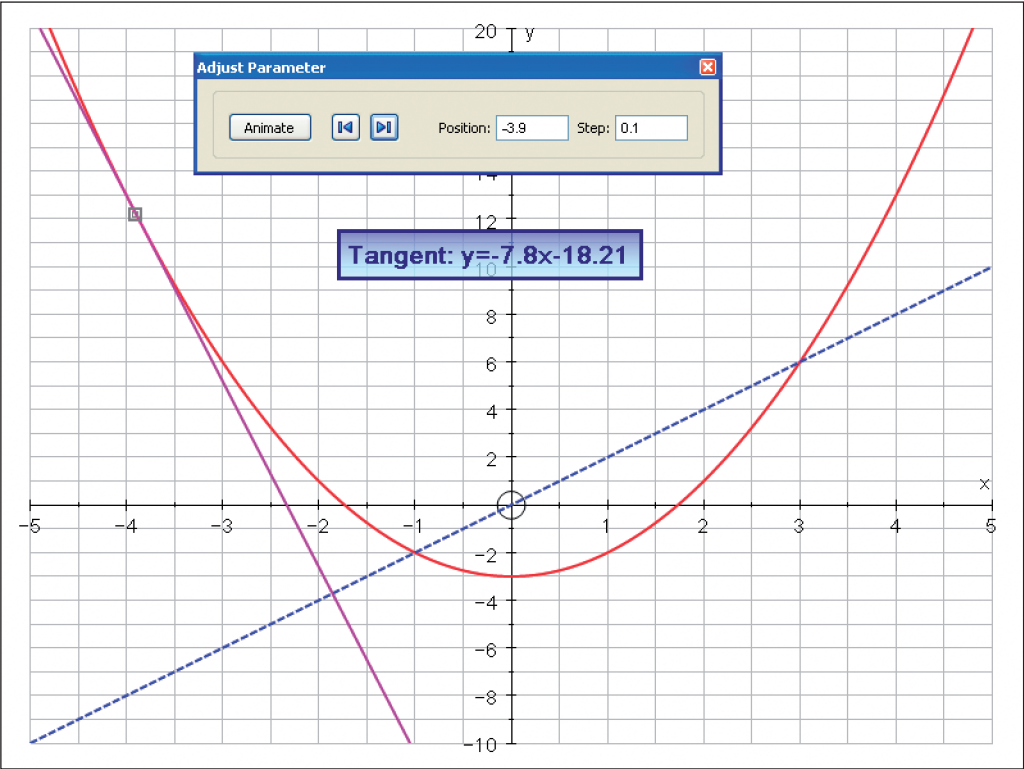
Activity 4

Tasks 13 and 14: Completed table:

Equation y = 3x²							
x	−2	−1	0	1	2	−2.5	2.5
Gradient	−12	−6	0	6	12	−15	15
Gradient Function = 6x							

Task 15: The gradient when x = 1.3 is 7.8.

Task 16: The 3 has the effect of multiplying the whole gradient function by 3.
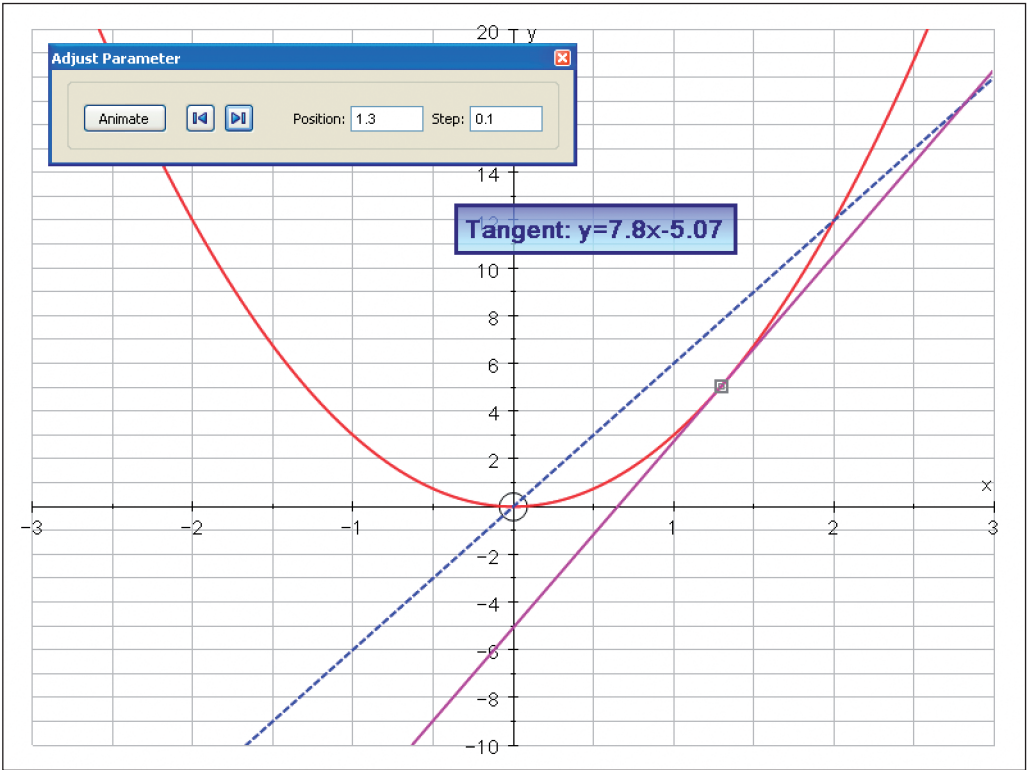
Activity 5

Task 17: The Gradient Function of y = 4x² + 6x − 5 is 8x + 6.
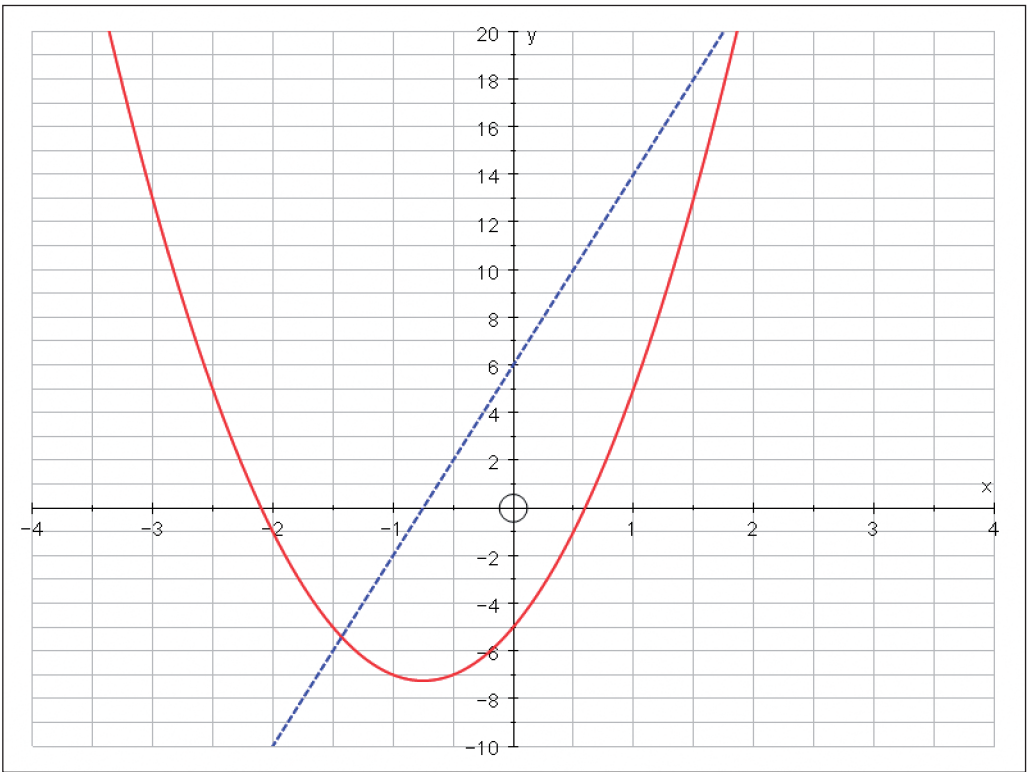
Task 18: Function: y = 5x² + 7x + 6. Gradient Function: 10x + 7.
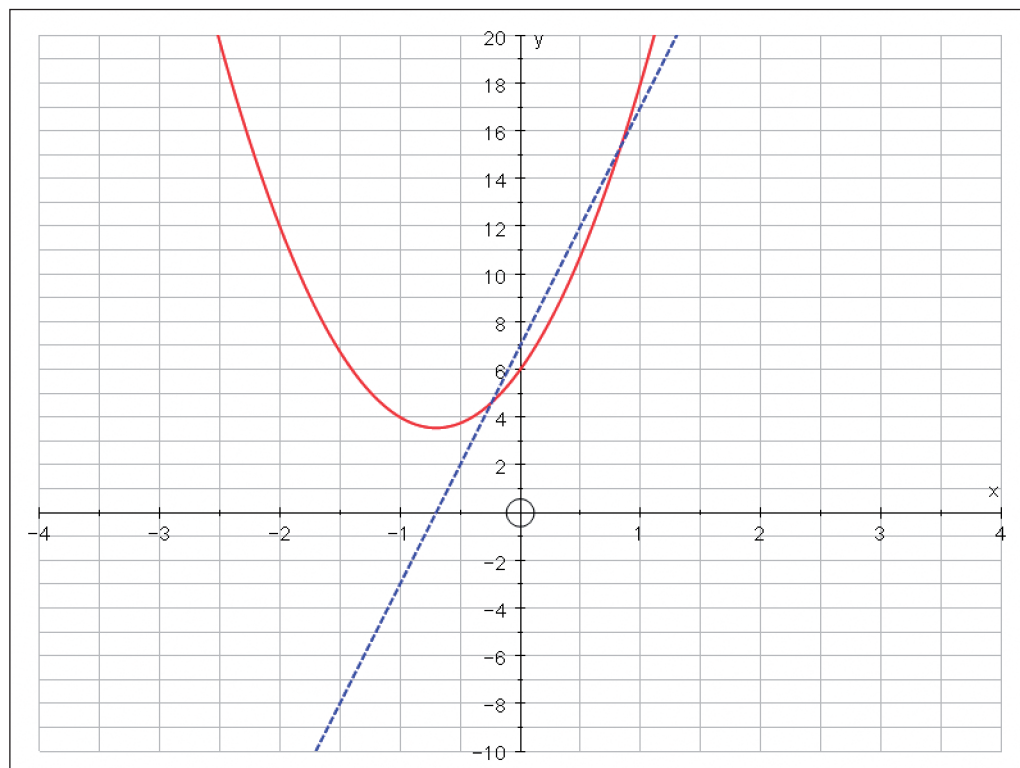
Function: y = 8x² − 2x − 9. Gradient Function: 16x − 2.
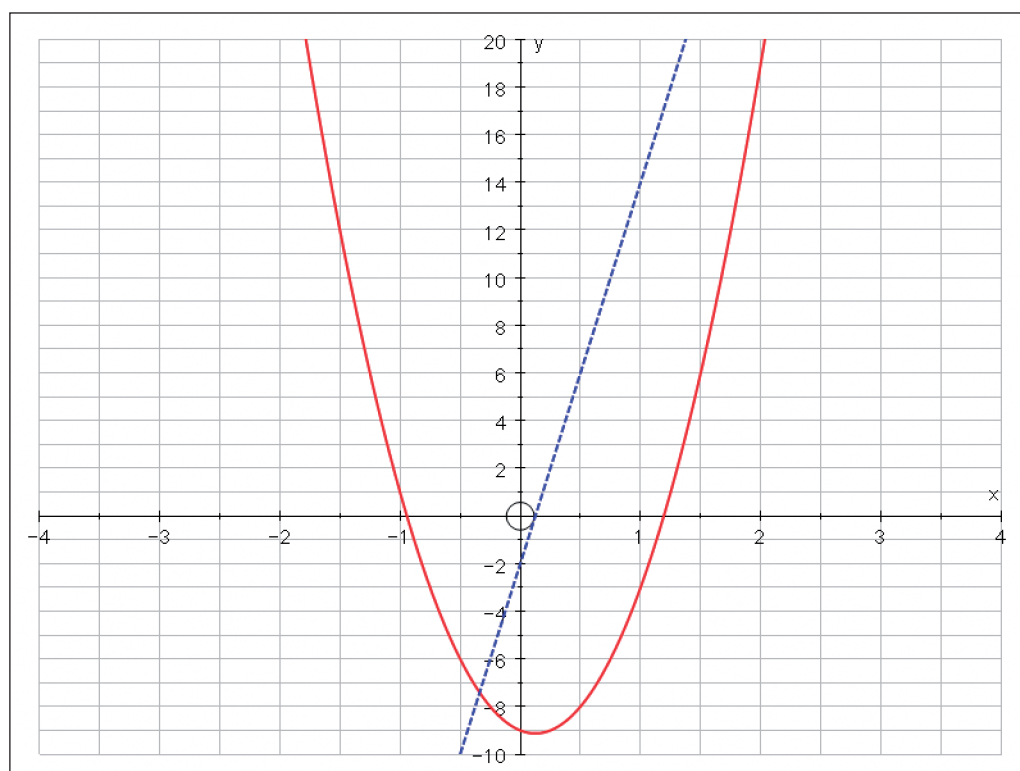
Function: y = −3x² + x + 12. Gradient Function: −6x + 1.
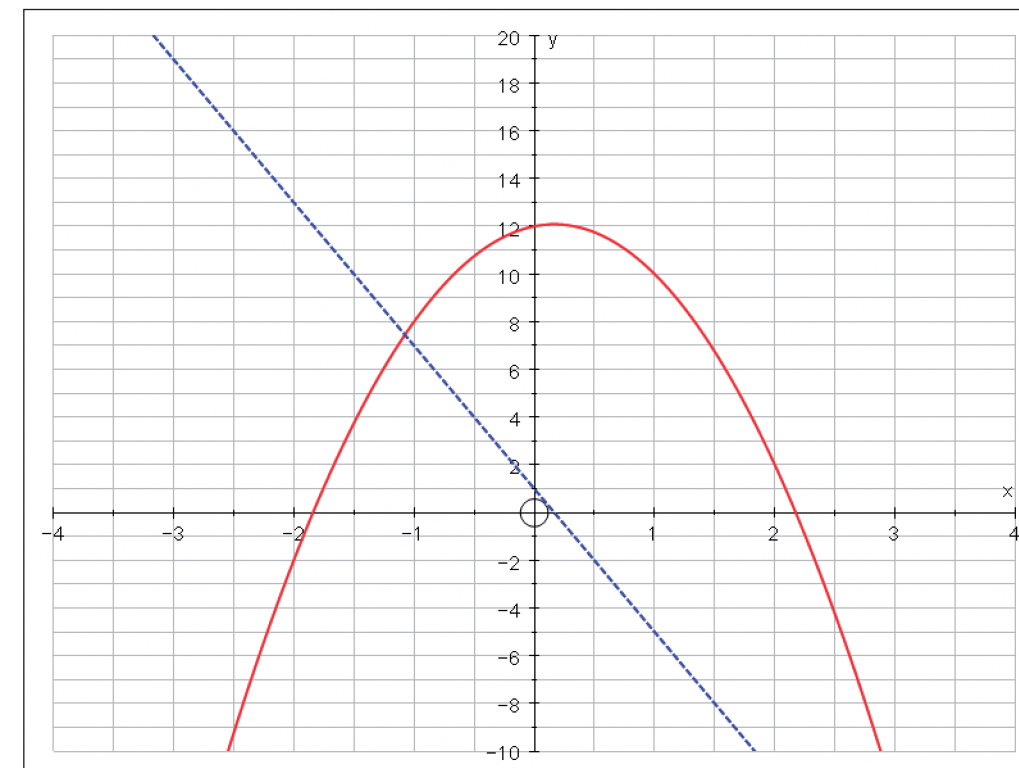
TASK 19: To find the Gradient Function of a Quadratic Function, multiply the co−efficient of the x-squared term by 2 and change the x-squared to an x, and get rid of the x term but keep the co-efficient.

TASK 20: Function: y = ax² + bx + c. Gradient Function: 2ax + b.

TASK 21: For each quadratic function, the Gradient Function has been *linear*.

ACTIVITY 6

TASK 22: For the function y = x³, the Gradient Function is *quadratic*.

TASK 23: For the function y = x³, the Gradient Function is 3x².
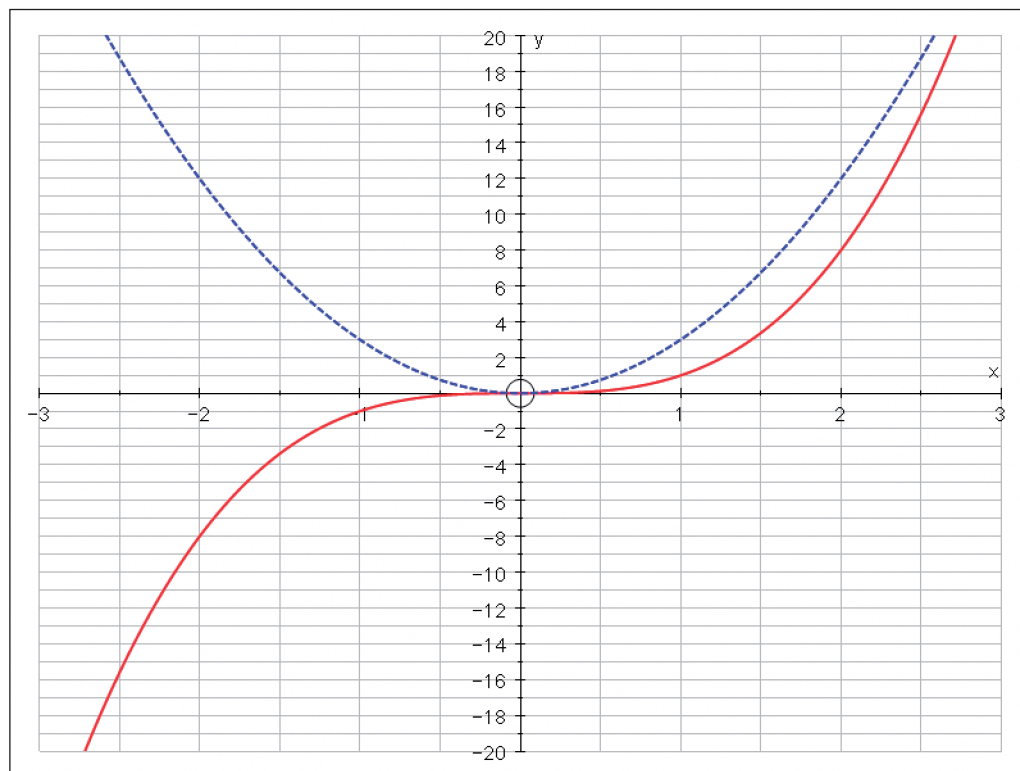
TASK 24: Function: y = 4x³. Gradient Function: 8x².

Function: y = −2x³ − 5x² + 3x + 4. Gradient Function: −6x² − 10x + 3.

Function: y = 3x³ + 2x² − 5x − 10. Gradient Function: 9x² + 4x − 5.
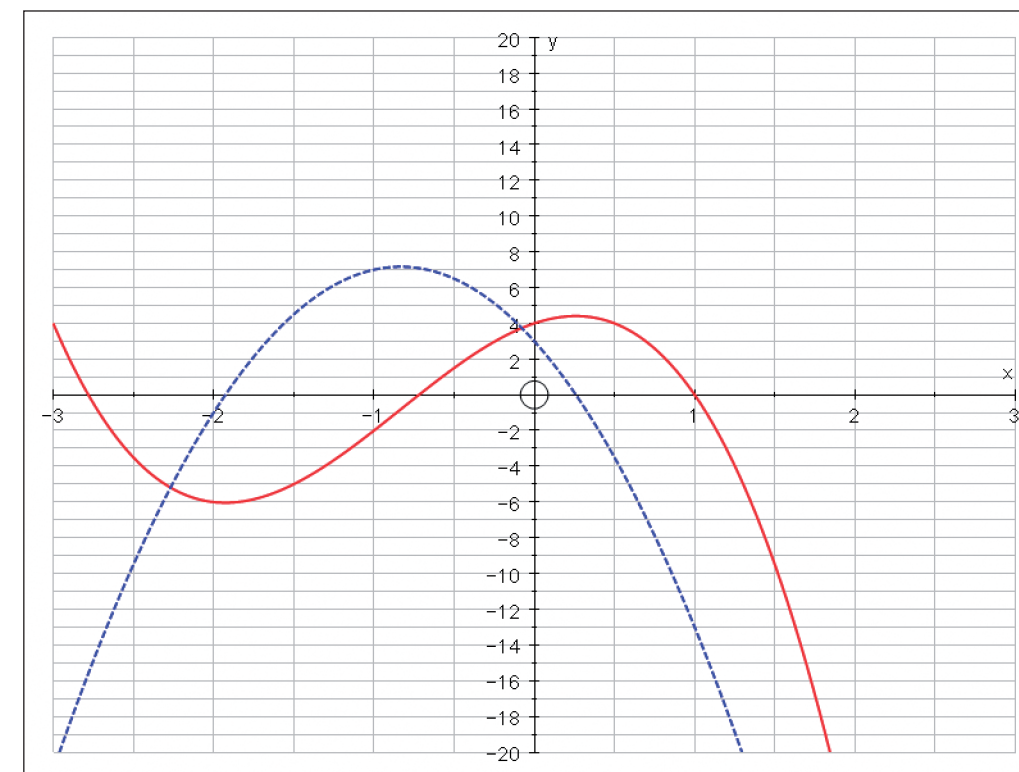
TASK 25: Function: y = ax³ + bx² + cx + d. Gradient Function: 3ax² + 2bx + x.

TASK 26: Function: y =6x⁵. Gradient Function: 30x⁴.
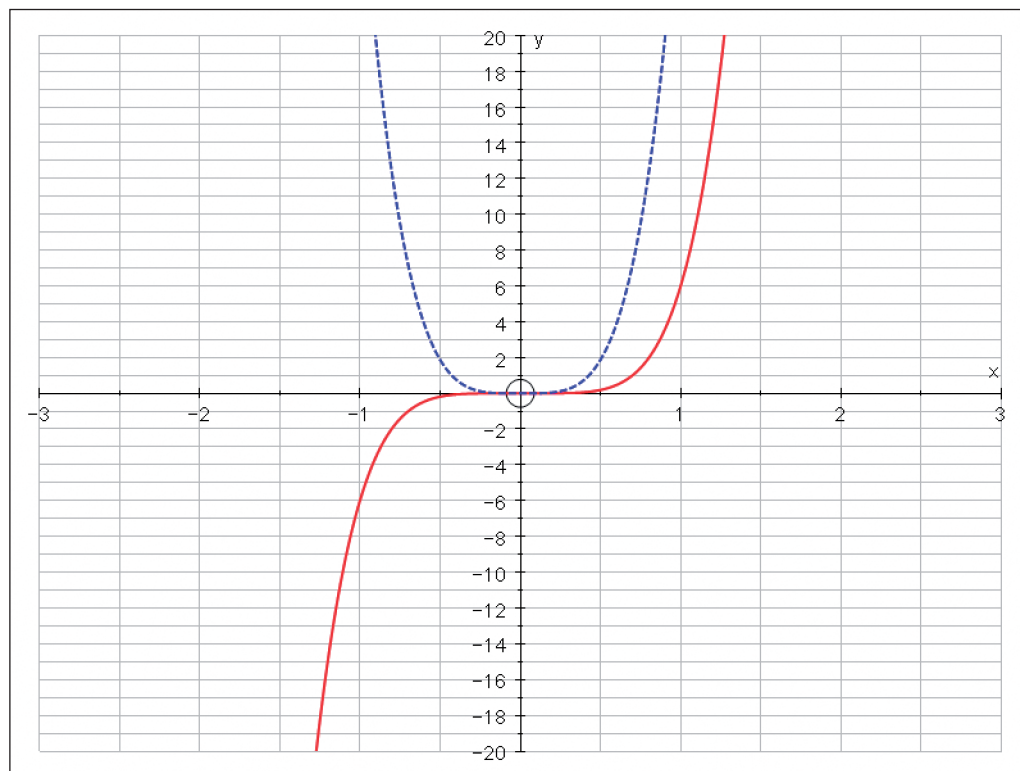
Function: y = 4x⁴ − 2x³ + 8x − 2. Gradient Function: 16x³ − 6x² + 8.
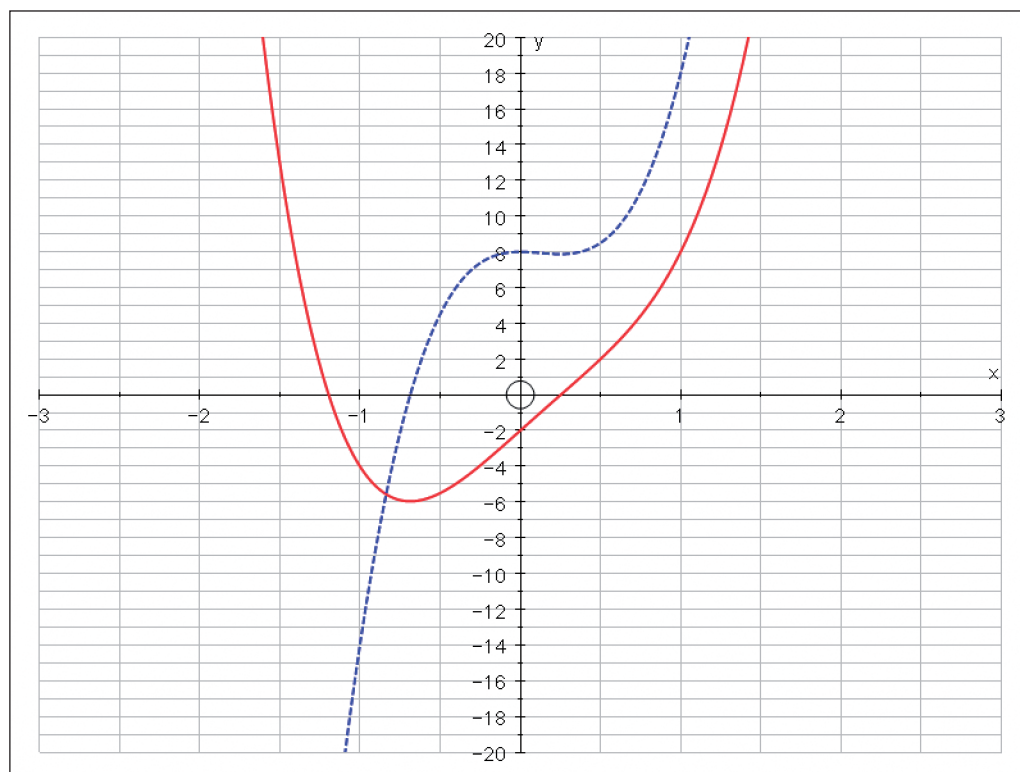
Function: y = x⁻¹. Gradient Function: −x⁻².
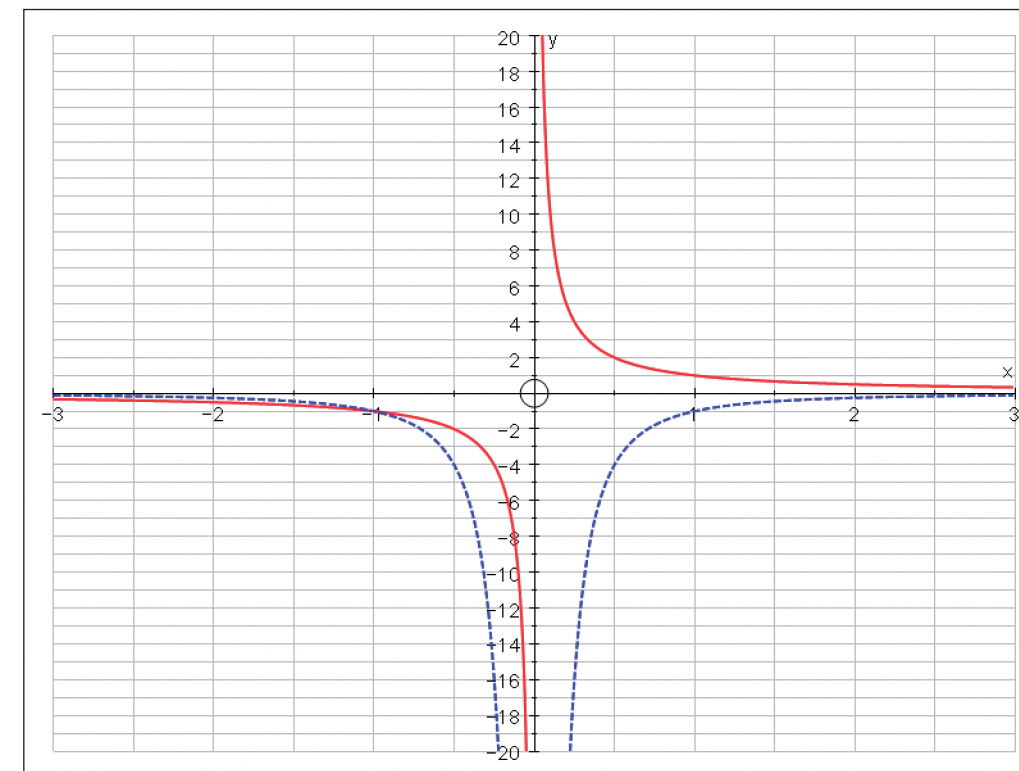
TASK 27: Multiply the power of the x term by the co–efficient in front of the x term (remembering to watch out for minuses), and subtract one from the power.

TEACHER NOTES

LEARNING OBJECTIVES
....

CORE CONTENT:

- To be able to understand the following relationships between variables: One-One, Many-One, One-Many and Many-Many, and how they relate to shapes of graphs.

- To understand that only the first two relationships fit the definition of a function.

- To understand the concepts of domain and range.

- To understand the symbols $\in$ (belongs to) and $\mathbb{R}$ (all the real numbers) and how they relate to the domain and range.

- To be able to use both the equation of functions, and their graphs, to find the domain and range.

- To understand how the completed square form of an equation $y = ((x - q)^2 + r)$ can help us find the range.

- To understand the concept of an asymptote.

- To be able to construct examples of functions given their domain and range.

EXTENSION MATERIAL:

- To understand the graphical and algebraic relationships between functions and their inverse.

- To understand that functions must be One-One in order to have an inverse (or be Many-One but have their domain restricted).

- To understand the graphical and algebraic nature of self-inverse functions.

REQUIRED PRIOR KNOWLEDGE
....

- To be comfortable with $f(x)$ and $g(x)$ notation.

- To be able express quadratic expressions in completed square form: $y = (x - q)^2 + r$.

- To be aware of the equation of circles.

- To be aware of the problem of dividing something by zero.

- To be able to re-arrange quadratic equations to make x the subject.
- Some knowledge of transformations would be useful, but is not essential.

Description of the Activity

This activity is designed to provide a nice introduction to Functions, giving students a solid grounding in both the algebraic and graphical representation of functions, and how the two forms relate to each other.

In Activity 1, students are introduced to four types of relationship between variables: One-One, Many-One, One-Many and Many-Many, and they are either shown, or asked to construct, examples of each.

In Activity 2, students are introduced to the concepts of the domain and the range and how they relate to the shape and nature of the function. Asymptotes and undefined areas of graphs are also tackled here.

Activity 3 contains the extension material. Here students are introduced to the concept of Inverse Functions and how they are represented algebraically and graphically. Students are asked probing questions about restricted domains and self-inverse functions which links directly to the work covered in Activities 1 and 2.

Pre–Activity Setup

It might be a good idea to have the graphs of some of the functions covered in this investigation available on your workstation or via an overhead projector to aid the students with some of the more difficult questions.

Suggested Introduction

"Today we are going to look at the topic of Functions. Now, this is very closely related to pretty much everything you have ever done in mathematics which involved drawing graphs. However, not all graphs are functions, and some functions have some very strange properties indeed…"

Suggested Prompts

- The Autograph functions employed during this investigation are relatively straight forward, so long as the students follow the instructions carefully. Once they have got the hang of the **Constant Controller** and entering functions and equations in Activity 1, that should see them right for the rest of the investigation.
- The students can check the answers to most of the tasks by using Autograph when prompted, and it is often worthwhile reminding them of this fact.
- However, this investigation does ask a lot of rather difficult, thought

provoking questions, especially in the extension material at the end. For this reason it might be a good idea to have the graphs in Activity 3 already available on your own workstation / projector, to help illustrate some of the concepts involved.

Suggested Plenary

- Can anybody summarise the key points they have learnt from this investigation?
- What are some of the things we need to look out for when determining the domain and range of a function?
- What are Inverse Functions and how do we work them out graphically and algebraically?
- Has anybody got any top tips, or way they have found successful, to help us avoid tripping up with work on functions?

Ideas for Further Work

- Further questions and examples to consolidate understanding of domain and range
- Further coverage of inverse functions
- This investigation would lead nicely onto some work on composite functions f(g(x)) possibly leading to the introduction of the modulus function.

STUDENT WORKSHEET

NAME:...

ACTIVITY 1: MAPPINGS

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Edit the axes as follows:
    x: Minimum: −11   Maximum: 11   Numbers: 1   Pips: 1
    y: Minimum: −6   Maximum: 6   Numbers: 1   Pips: 1
Remove all of the green ticks underneath **Auto**.

Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Select **Equal Aspect Mode** from the top toolbar.

This alters the **x** scale so that the axes are square.

Enter the equation: y = a
Still on the **Add Equation** screen click on:
Draw Options, and from the **drop-down** menu change the style of the line from solid to dashed.
Edit Constants, and change the value of **a** to anything between −5 and 5.

Predict | *What will this line look like?*

Click **OK** twice.

Enter the equation: x = b

Again, use **Draw Options** to make the line dashed.

Click on **Edit Constants**, and change the value of **b** to anything between −10 and 10.

Predict | *What will this line look like?*

Click **OK** twice.

Click on the **Constant Controller** on the top toolbar.

The **drop-down** menu allows you to select different constants.

The **up-down** arrows alter the value of the selected constant.
The **left-right** arrows adjust the size of the step.

Use the **Constant Controller** to adjust the position of both lines.

(a) One-One Relationships:

Select **Function Definitions** from the top toolbar.

Define f(x) as 2x − 1, and leave g(x) as 1.

Click on **Slow Plot** mode (there should be a blue square around the button).

Enter the equation: y = f(x).

Predict | *What will the graph of this function look like?*

Click **OK**.

TASK 1: Use the **Constant Controller** to adjust the value of **a**. In all locations, how many times does the horizontal line cut the graph of y = f(x): is it at most once, or sometimes more than once? Repeat for the vertical line, and write your answer below.

Definition: Relationships like this between variables are called one-one relationships. For each value of **x**, there is a unique value of **y**, and vice-versa.

(b) Many-One Relationships:

Select **Function Definitions** from the top toolbar.

This time, define g(x) as 0.2x³ − 2x + 1.

Note: To enter x³, press x followed by "**Alt 3**" together, or simply type "**xxx**"

Enter the equation: y = g(x).

Predict | *What will the graph of this function look like? Where will it cross the axes? What direction will it slope?*

Click **OK**.

TASK 2: Use the **Constant Controller** to adjust the value of **a**. In all locations,

how many times does the horizontal line cut the graph of y = g(x): is it at most once, or sometimes more than once? Repeat for the vertical line, and write your answer below.

<div style="border:1px solid blue; height:150px"></div>

Definition: Relationships like this between variables are called many-one relationships. Many (more than one) value of x, may map to one value of y.

TASK 3: Explain briefly why all quadratic functions must have a Many-One relationship.

Try entering a few quadratic equations into Autograph to help you.

<div style="border:1px solid blue; height:150px"></div>

TASK 4: Can you think of a cubic equation which gives us a One-One relationship?

Enter your answer as an equation in Autograph to check. Is this the only one?

<div style="border:1px solid blue; height:150px"></div>

(c) One-Many Relationships:

Enter the equation: x = y²

Predict

What will the graph of this function look like? Where will it cross the axes? What direction will it slope?

Click **OK**.

TASK 5: Use the **Constant Controller** to adjust the value of a. In all locations, how many times does the horizontal line cut the graph of x = y²: is it at most

once, or sometimes more than once? Repeat for the vertical line, and write your answer below.

<div style="border:1px solid blue; height:150px"></div>

Definition: Relationships like this between variables are called one-many relationships. One value of x may map to many (more than one) value of y.

(d) Many-Many Relationships:

TASK 6: The final type of relationship is Many-Many. Here, many (more than one) values of x may map to many (more than one) values of y. How would the horizontal and vertical lines cross the graph in this case?

<div style="border:1px solid blue; height:150px"></div>

TASK 7: Can you write down an equation which will give us a Many-Many relationship?

Enter your answer as an equation in Autograph to check.

<div style="border:1px solid blue; height:150px"></div>

Crucial Point: Only the first two cases (One-One and Many-One) are Functions. For a relationship to be a function, each input must give one and only one output. Or, in other words, the function must be uniquely defined for all the values of the domain.

ACTIVITY 2: DOMAIN AND RANGE

Open up a **New 2D Graph Page**.

Edit the axes as follows:
x: Minimum: −6   Maximum: 6   Numbers: 1   Pips: 0.5
y: Minimum: −12   Maximum: 12   Numbers: 2   Pips: 1

Remove all of the green ticks underneath **Auto**.

As in Activity 1, enter the dashed lines, y = a and x = b.

Click on the **Constant Controller** on the top toolbar.

Select **Function Definitions** from the top toolbar.

Define f(x) as x³ − 5x² + 2x + 2, and leave g(x) as 1.

Ensure **Slow Plot** mode is still turned on.

Enter the equation: y = f(x)

What will the graph of this function look like? Where will it cross the axes? What direction will it slope?

Click **OK**.

Observe:

Adjusting the value of a and b, notice that every possible x value, when entered into the function f(x), gives you a y value, and likewise that you can obtain any possible y value with an x value.

Use the **Zoom Out x**, **Zoom Out y**, and **Drag** functions to convince yourself that this is true, even for x and y values outside of your current **View**, such as x = 1,000, and y = −2,380.
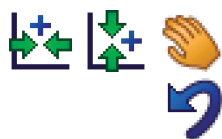
Use the **Zoom In** and **Drag** to return to our original **View** of the graph.

Note: Pressing **Undo** several times is often a quicker way of achieving this.

Definitions

The **Domain** of a function is all the possible inputs (values of x) that you can put into a function and get an answer out.

Note: It is, of course, possible to input any value into a function, but if such a value would give you an error on your calculator, then this value is *not* part of the domain.

The **Range** of a function is all the possible outputs (values of y) that can be obtained using the function.

For f(x) above, we saw that every x value gave a y value, and every y value could be obtained. We write this as follows:

Domain: x ∈ ℝ Range: f(x) ∈ ℝ

Meaning that both the domain and range belong to (∈) the set of all real numbers (ℝ).

Choose **Select Mode**.

Select **Function Definitions** from the top toolbar.

Now define f(x) as x² − 4, and leave g(x) as 1.

Note: To write x², either press x followed by "**alt 2**" together, or simply write "**xx**".

What will the graph of this function look like? Where will it cross the axes? What direction will it slope?

Click **OK** and the graph of y = f(x) should appear on the screen.

TASK 8: Using the vertical and horizontal lines (if they help you), together with the zoom functions, write down the domain and range of f(x).

Hint: Are there any values of y that cannot be obtained no matter what the value of x? How would you write this using signs such as ≥ ≤ > < ≠?

TASK 9: f(x) = x² − 3x − 6. Express f(x) in the form (x − q)² + r, where q and r are constants to be found (i.e. complete the square).

TASK 10: How would your answer to Task 9 help you find out the domain and range of f(x) without having to sketch it?

Check your answer in Autograph by entering f(x) = x² − 3x − 6 into the **Function Definitions**.

[blank rounded box]

TASK 11: Can you think of a function which has the following characteristics:

Domain: $x \in \mathbb{R}$ Range: $f(x) \leq 5$

Again, use Autograph to check your answer.

[blank rounded box]

TASK 12: Think about the function: $f(x) = \dfrac{1}{x-2}$

Can you think of a value of x which would not be part of the domain of the function? Briefly explain your answer.

[blank rounded box]

TASK 13: Still thinking about the function: $f(x) = \dfrac{1}{x-2}$

Can you think of a value of y which can never be obtained no matter what the value of x? Briefly explain your answer.

[blank rounded box]

TASK 14: Use your answers to Task 12 and Task 13 to state the domain and range of $f(x)$, and try to predict what the graph will look like.

[blank rounded box]

Checking your answer:

Choose **Select Mode**.

Select **Function Definitions** from the top toolbar.

Define $f(x)$ as $1/(x-2)$ and leave $g(x)$ as 1.

Make sure your graph confirms your answers to Tasks 12, 13 and 14.

Use the **Drag** and **Zoom** tools to explore what is happening to the graph at the critical values given in Tasks 12 and 13.

Use the **Drag** and **Zoom In** functions to have a closer look to confirm that the graph never actually crosses the x-axis ($y = 0$), or the line $x = 2$.

Use **Undo** if you ever lose the graph!

Definition: The lines $x = 2$ and $y = 0$ are known as asymptotes.

This is known as an asymptote.

Left-click on the curve (it should turn black)

Right-click and select **Delete** from the menu, or simply press **delete** on the keyboard.

Enter the equation: $y = \sqrt{(x + c)} + d$

Still on the **Add Equation** screen, click on **Edit Constants**, and alter both the values of c and d to 0.

Predict — *What function will this give you, and what will the graph look like?*

Hint: Think about what would happen with negative x values.

Click **OK** twice.

Use the **Zoom Out x**, **Zoom Out y**, and **Drag** functions to investigate the graph for large (positive) values of x.

TASK 15: Will the graph ever reach a maximum value of y? Explain how you know, giving an example.

TASK 16: State the domain and range of the graph
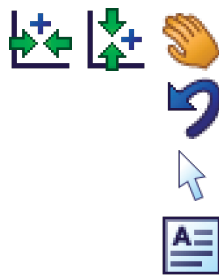
Use the **Zoom In** and **Drag** to return to our original **View** of the graph.

Note: Pressing **Undo** several times is often a quicker way of achieving this.

Left-click on the curve (it should turn black).

Select **Text Box** from the toolbar.
Tick the box next to **Show Detailed Object Text** and click **OK**.

The **Text Box** should now display the equation of the curve, together with the current values of the constants c and d.

Predict

What effect would altering the values of c and d have on the shape and position of the graph?

Use the **Constant Controller** to alter the values of c and d and observe the effect it has on the position of the graph.

TASK 17: Use the **Constant Controller** to find a function which has the following characteristics:

Domain: x ≥ −3 Range: f(x) ≥ −2

Write the equation of the function below.

TASK 18: What would you need to do to the equation given in Task 17 in order for the function to have the following characteristics:

Domain: x ≥ −3 Range: f(x) ≤ −2

Write the equation of the function below, and check your answer by entering the equation in Autograph.

Activity 3: Inverse Functions

Open up a **New 2D Graph Page**.

Edit the axes as follows:
    x: Minimum: −4   Maximum: 15   Numbers: 1   Pips: 1
    y: Minimum: −2   Maximum: 8   Numbers: 1   Pips: 1
Remove all of the green ticks underneath **Auto**.

Select **Equal Aspect Mode** from the top toolbar.

This alters the x scale so that the axes are square.

TASK 19: Express $y = x^2 − 4x + 7$ in the form $y = (x − q)^2 + r$, where q and r are constants to be found (i.e. complete the square).

TASK 20: Use your answer to Task 19 to state the domain and range of $y = x^2 − 4x + 7$.

[blank box]

Checking your answer:

Select **Function Definitions** from the top toolbar.

Define f(x) as x² − 4x + 7, and leave g(x) as 1.

Ensure **Slow Plot** mode is still turned on.

Enter the equation: y = f(x). This gives you the graph of y = x² − 4x + 7.

TASK 21: Use your answer from Task 19 to re-arrange the equation to give x in terms of y (i.e. x = …).

[blank box]

TASK 22: Now simply replace y with x (and vice-versa), so you are left with a function which gives y in terms of x (i.e. y =).

[blank box]

Predict

What will the graph of this function look like? Where will it cross the axes? What direction will it slope?

Enter your answer to Task 22 as an equation.

Note: Remember to be careful with any brackets!

TASK 23: Can you spot a graphical relationship between the two curves?

Hint: Try adding the line y = x (make it dashed using **Draw Options**).

[blank box]

TASK 24: Thinking about what you did in Task 21 and Task 22, can you explain why this is the case?

[blank box]

Definition: Functions that share this special relationship are known as Inverse Functions.

TASK 25: A portion of this second curve seems to be missing. Can you explain why this is the case?

Hint: Think back to Tasks 1 to 7 where we discussed what type of relationships could be defined as Functions.

[blank box]

TASK 26: What must we restrict the domain to in y = f(x) in order to get our reflected second curve?

[blank box]

TASK 27: What does this tell you about the type of relationship variables must have in order to have inverse functions? Can you explain this?

Hint: Picture the restricted portion of y = f(x). What type of relationship is this?

TASK 28: State the domain and range of our reflected second curve.

TASK 29: How does this compare to the domain and range of the restricted y = f(x)? Can you explain why this is the case?

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on the two curves (they should both turn black).

Right-click and select **Delete** from the menu, or press **delete** on the keyboard.

You should now be left with the dashed line y = x.

Enter the equation: $y = \frac{x}{x-1}$

Note: enter it as: y = x/(x − 1)

What will the graph of this function look like? Where will it cross the axes? What direction will it slope? Will it have any asymptotes?

Use the **Zoom Out** and **Drag** functions to investigate a larger portion of the graph.

TASK 30: Looking at the graph of $y = \frac{x}{x-1}$, the line y = x, and thinking about your answer to Task 23 and Task 24, what do you think the equation of the inverse function would be? Explain your answer.

TASK 31: Do the following steps to find the equation of the inverse function of $y = \frac{x}{x-1}$.

1. Re-arrange the function to give x in terms of y (i.e. x =).
2. Replace x with y and vice-versa, so the function gives y in terms of x (i.e. y =).

Does this match your answer to Task 30?

Summary

Use the space below to summarise what you have learnt during this investigation:

ANSWERS

ACTIVITY 1: MAPPINGS

TASK 1: The horizontal and vertical lines always cut the graph *at most once*.

Equation 1: y=a
Equation 2: x=b
Equation 3: y=f(x)
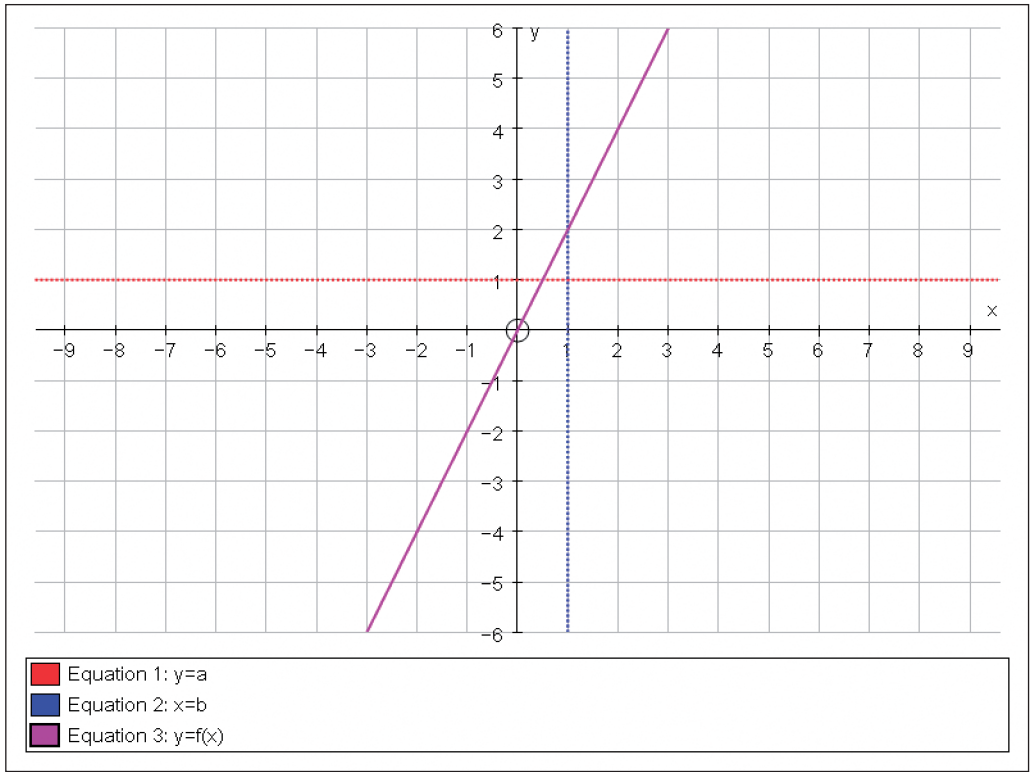
TASK 2: The horizontal line cuts the graph sometimes more than once, and the vertical line cuts the graph at most once.

Equation 1: y=a
Equation 2: x=b
Equation 4: y = g(x)
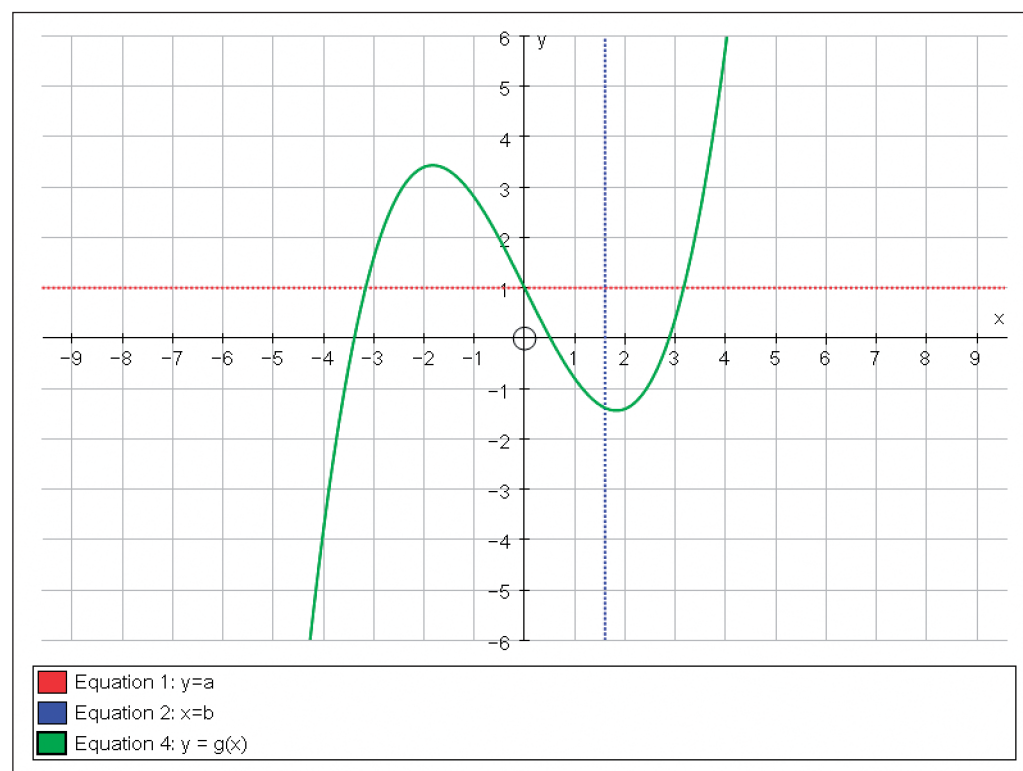
TASK 3: All quadratic functions must have a Many-One relationship, as they are all either U-shaped or upside down U-shaped. Hence, you will always be able to achieve a given y value from two x values (apart from at the minimum or maximum point).

TASK 4: Two possible cubic functions which have One-One relationships are $y = x^3$ and $y = -x^3$. The students could use the **Zoom In** functionto convince themselves that the curves are not horizontal around the point $(0, 0)$. Curves in the form $y = x^3 + a$ and $y = x^3 - a$ also share this relationship.

Equation 1: y=a
Equation 2: x=b
Equation 3: y=x³
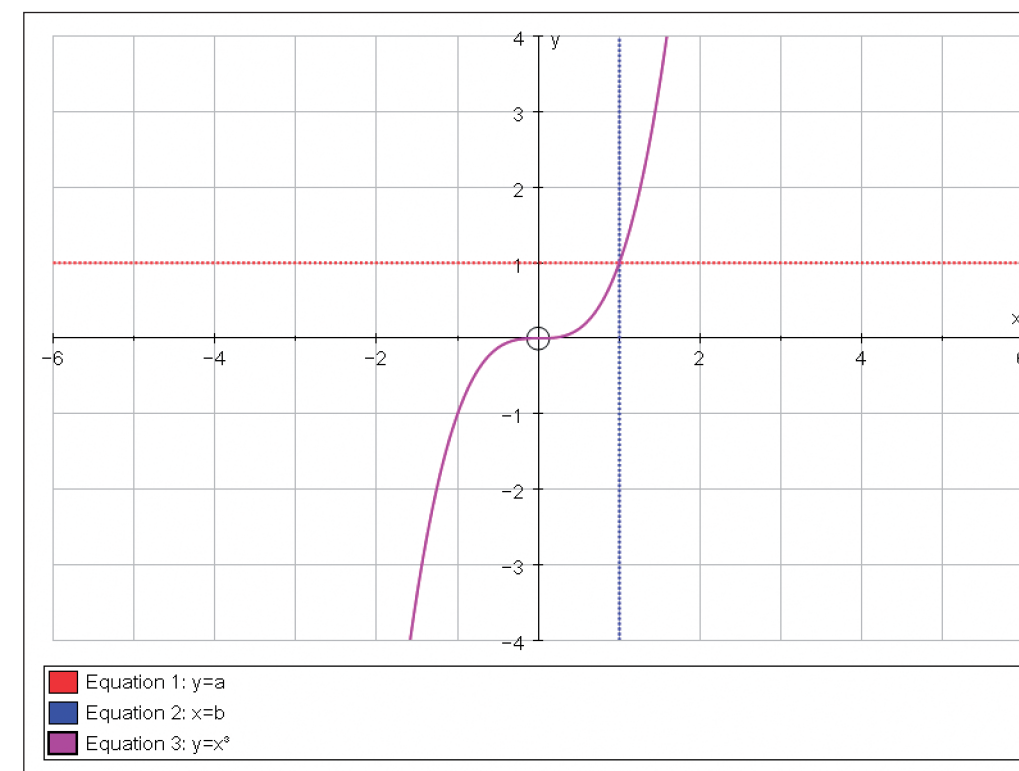
TASK 5: The horizontal line cuts the graph at most once, and the vertical line cuts the graph sometimes more than once.

Equation 1: y=a
Equation 2: x=b
Equation 5: x=y²
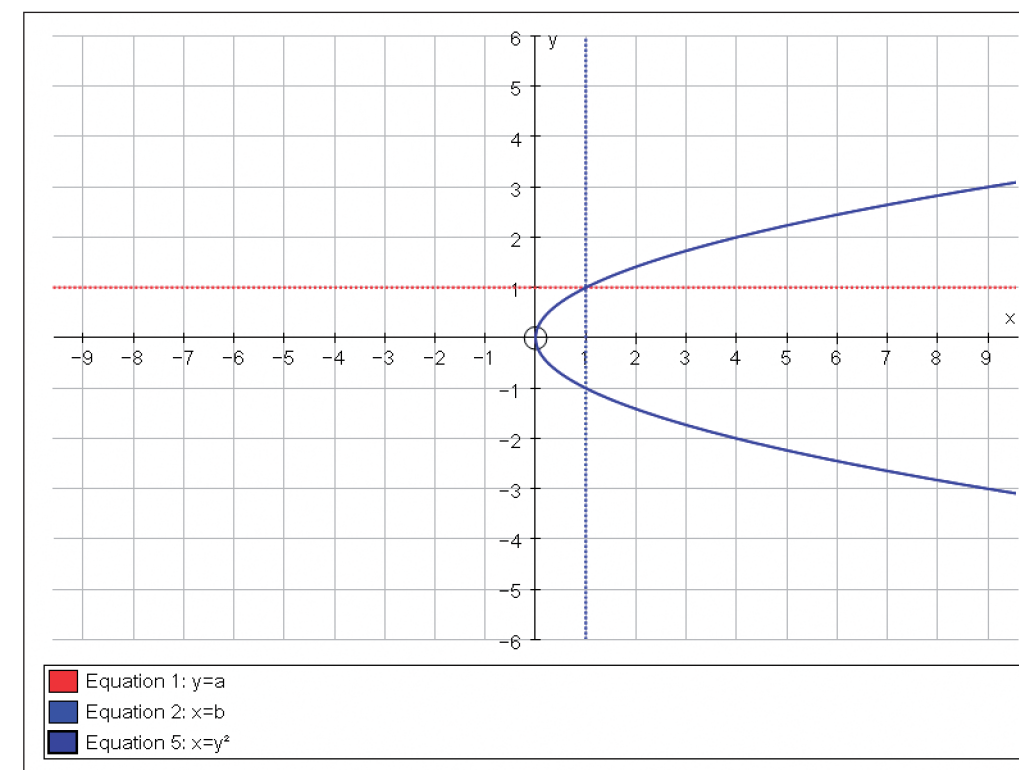
TASK 6: In a Many-Many relationship, both the horizontal and vertical lines would cut the graph sometimes more than once.

TASK 7: Any equation of a circle is an example of a Many-Many relationship,

for example x² + y² = 16:

Equation 1: y=a
Equation 2: x=b
Equation 6: x²+y²=16
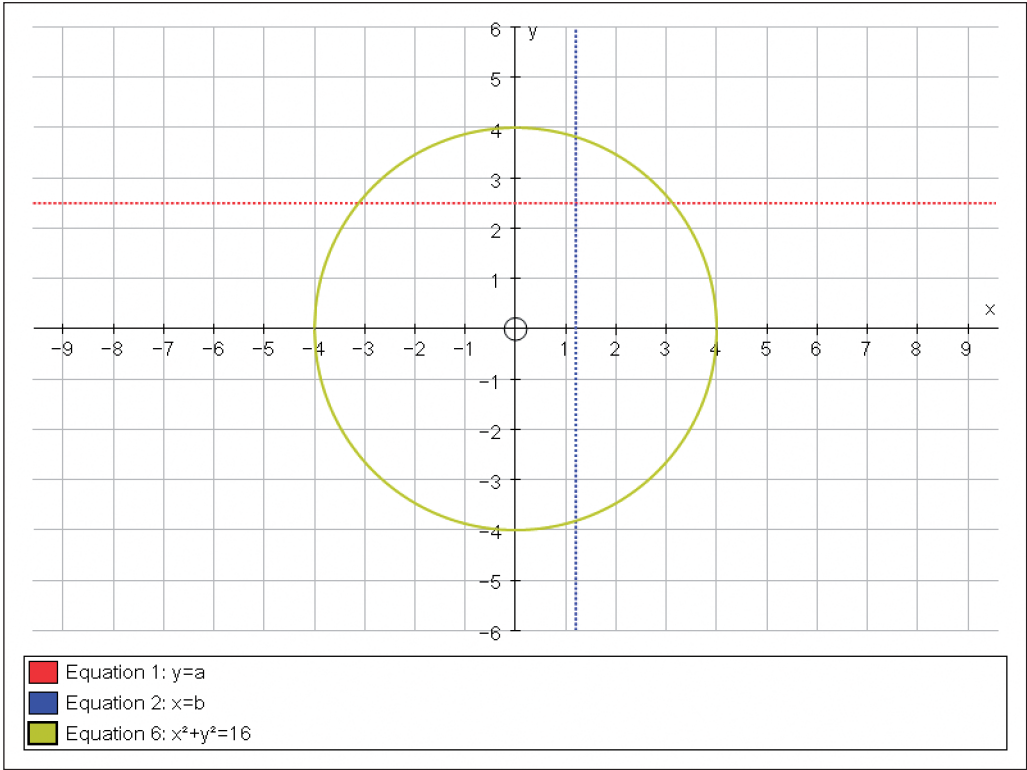
Activity 2: Domain and Range

TASK 8: f(x) = x² − 4 Domain: x ∈ ℝ Range: f(x) ≥ − 4

Equation 1: y=f(x)
Equation 2: y=a
Equation 3: x=b
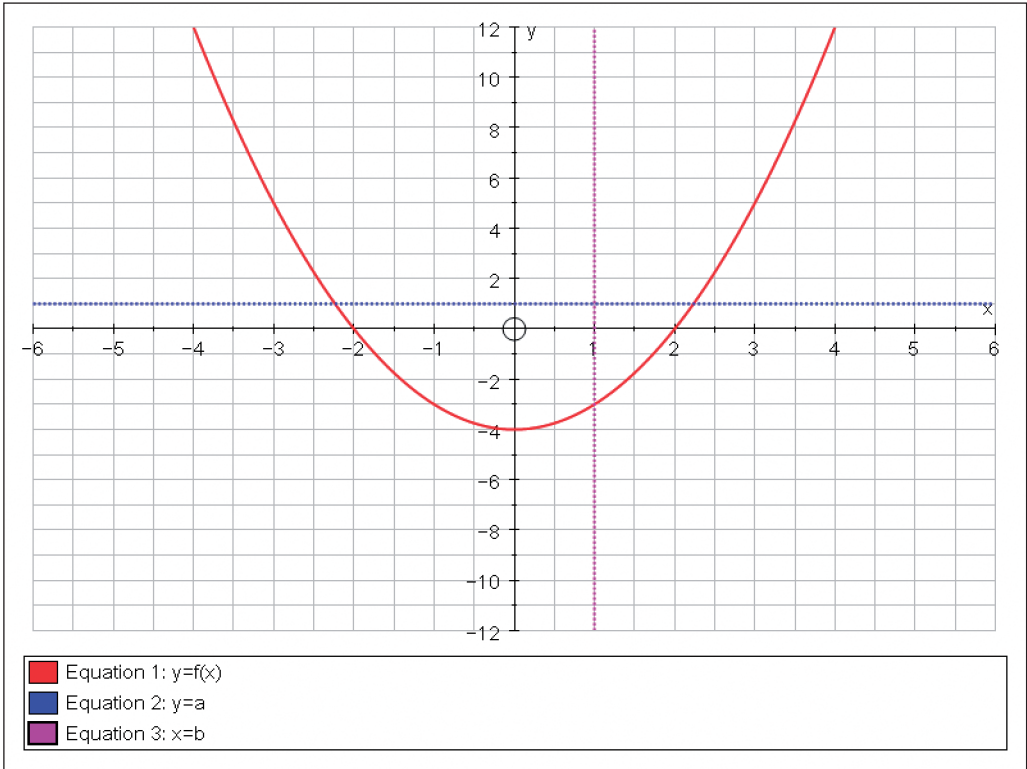
TASK 9: f(x) = x² − 3x − 6

f(x) = (x − 1.5)² − 2.25 − 6

f(x) = (x − 1.5)² − 8.25

TASK 10: The minimum point of f(x) must be at (1.5, −8.25), and so that gives us:

f(x) = x² − 3x − 6 Domain: x ∈ ℝ   Range: f(x) ≥ −8.25

Equation 1: y=f(x)
Equation 2: y=a
Equation 3: x=b
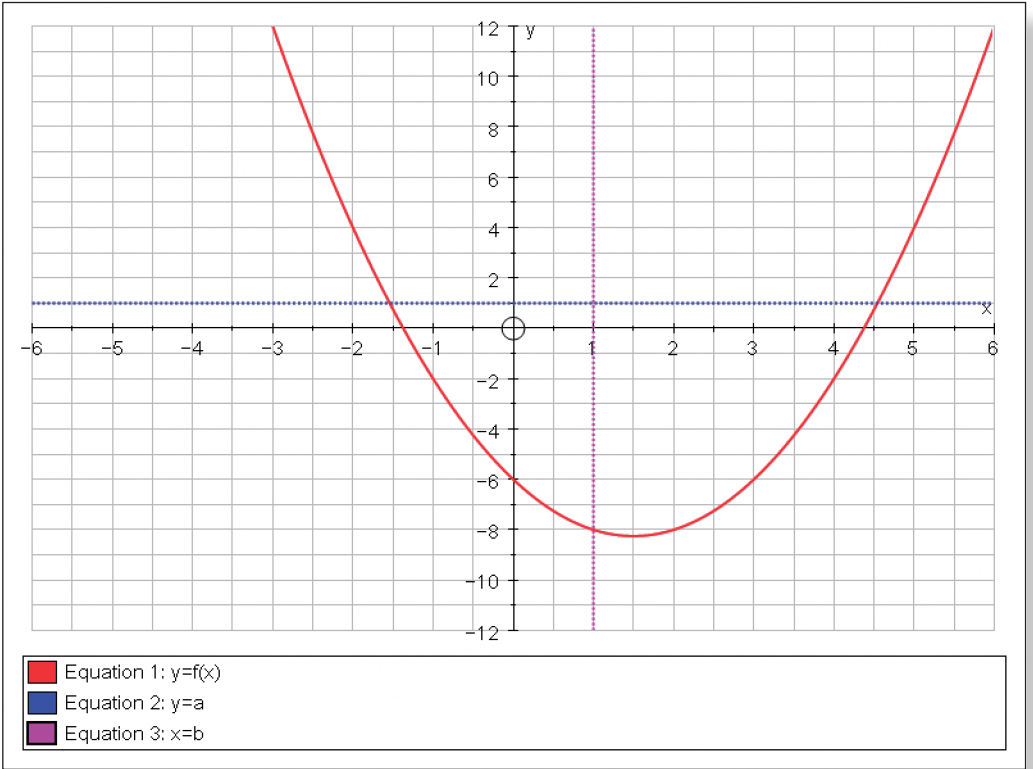
TASK 11: One possible answer is f(x) = −x² + 5
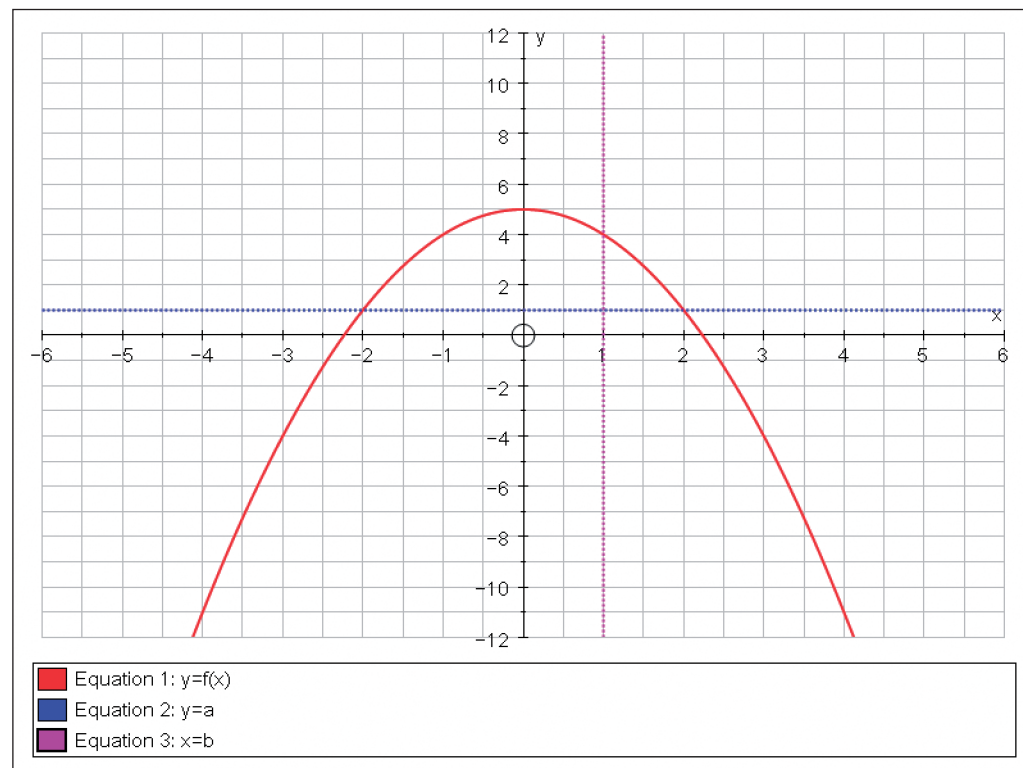
TASK 12: $f(x) = \frac{1}{x-2}$

$x = 2$ would not be part of the domain, as that would make the denominator of the fraction equal to zero, which would make the value of our function undefined.

TASK 13: $f(x) = \frac{1}{x-2}$

No matter what value of x you input, you can never get a y value of 0. Perhaps the easiest pay to see this is to set the function equal to zero in order to try and find the appropriate x value:

$0 = \frac{1}{x-2}$ which leads to the rather worrying conclusion that $0 = 1$.

TASK 14: $f(x) = \frac{1}{x-2}$

Domain: $x \in \mathbb{R}$, $x \neq 2$ Range: $f(x) \in \mathbb{R}$, $f(x) \neq 0$
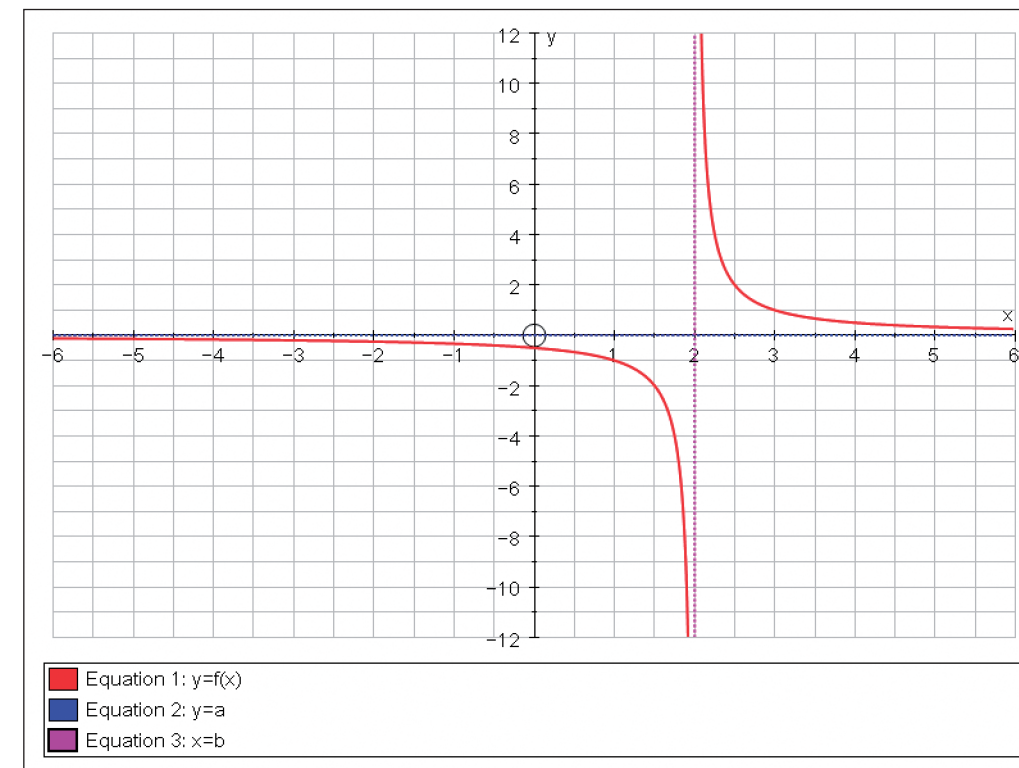
TASK 15: $f(x) = \sqrt{x}$

The function will never reach a maximum value of y. Perhaps the easiest way to think about this is to realise that no matter how big a y value you think of, you can always square it to give you the corresponding x value.

TASK 16: $f(x) = \sqrt{x}$ Domain: $x \geq 0$ Range: $f(x) \geq 0$

TASK 17: $f(x) = \sqrt{(x + 3)} - 2$ Domain: $x \geq -3$ Range: $f(x) \geq -2$

Graph 1 (top left):

Equation 4: y=√(x+c) + d
c = 3
d = -2

Equation 2: y=a
Equation 3: x=b
Equation 4: y=√(x+c) + d
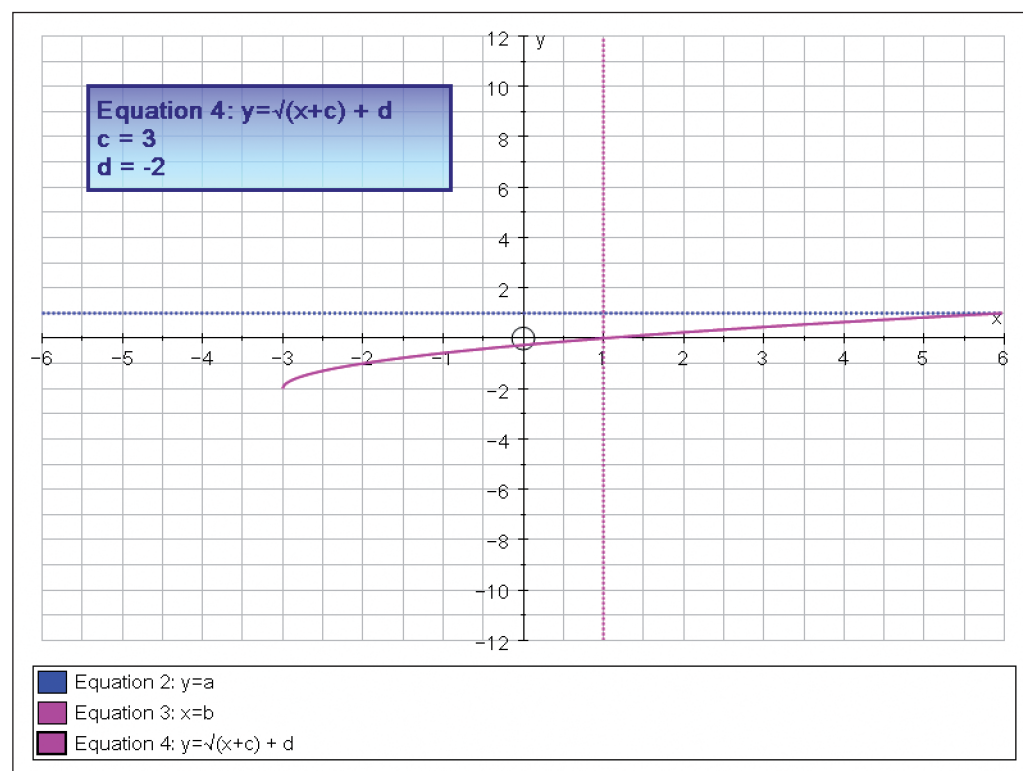
TASK 18: You must reflect the curve about the line y = 0, which is achieved by multiplying the whole function by −1.

f(x) = −(√(x + 3) − 2)   Domain: x ≥ −3   Range: f(x) ≤ −2

Graph 2 (left):

Equation 4: y=−(√(x+c) + d)
c = 3
d = -2

Equation 2: y=a
Equation 3: x=b
Equation 4: y=−(√(x+c) + d)
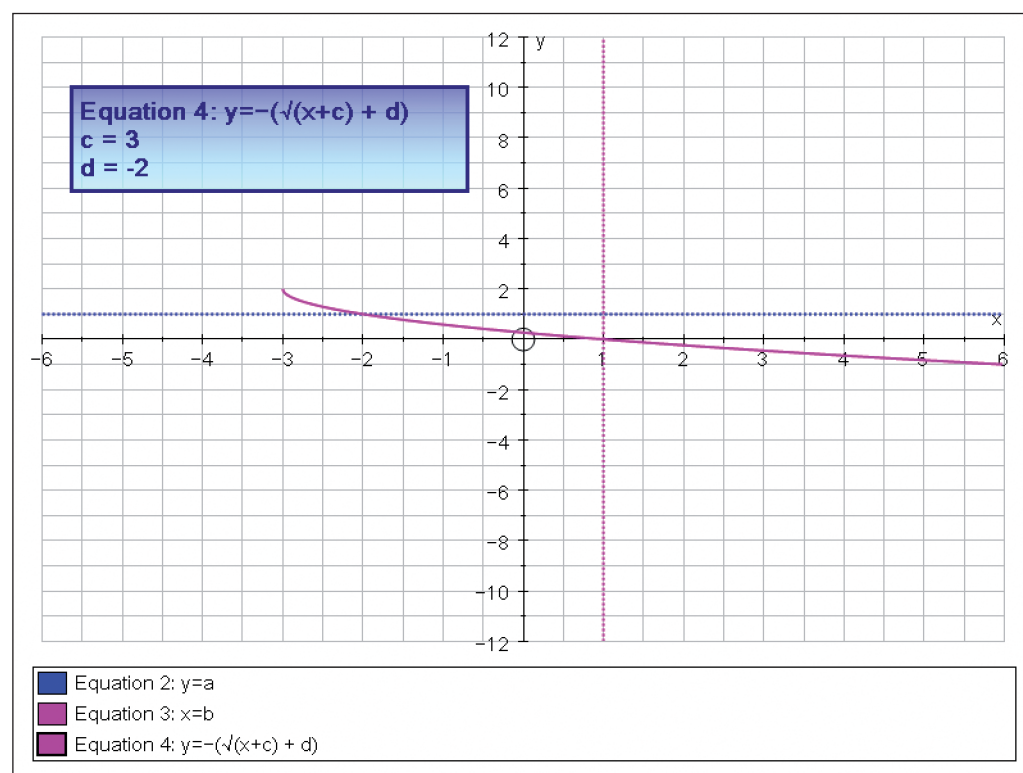
Activity 3: Inverse Functions

TASK 19: y = x² − 4x + 7

y = (x − 2)² − 4 + 7

y = (x − 2)² + 3

TASK 20: y = x² − 4x + 7   Domain: x ∈ R   Range: f(x) ≥ 3

TASK 21: y = (x − 2)² + 3

y − 3 = (x − 2)²

√(y − 3) = x − 2

x = √(y − 3) + 2

TASK 22: y = √(x − 3) + 2

TASK 23: The two curves are reflections of each other in the line y = x.

Graph 3 (right):

Equation 1: y=f(x)
Equation 2: y=√(x−3)+2
Equation 3: y=x
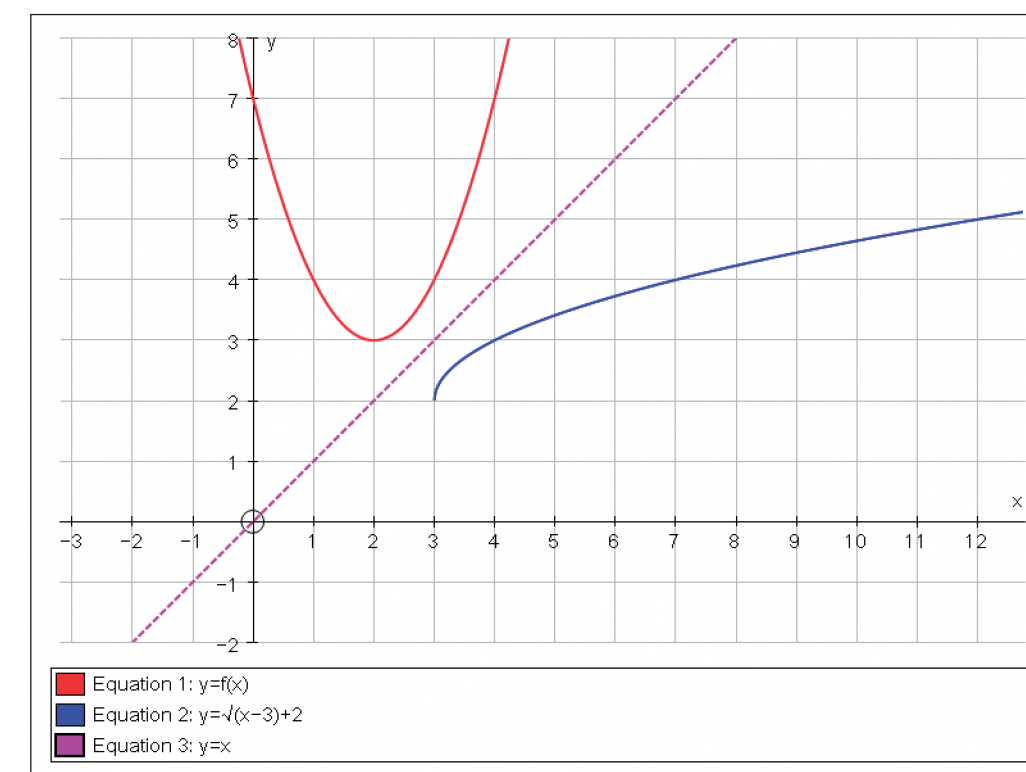
TASK 24: In Tasks 21 and 22, we effectively swapped the x and y variables around. Hence, what was y on the first curve, becomes x on the second curve, and vice-versa. Therefore, the curves are reflections of each other in the line y = x.

TASK 25: The portion of the second curve which is missing is the left hand portion of y = f(x). If this portion of the curve had been included, then we would have had more than one x value mapping to each y value. That would have made the relationship a One-Many, which would not have constituted a function.

TASK 26: We must restrict our domain to: x ≥ 2

TASK 27: Relationships must be One-one in order to have an inverse (or, if they are Many-One, they must have their domain restricted to make them One-One). If they are not, then when we reflect them we will get more than one value of x mapping to each value of y, making them One-Many relationships, which are not functions!

TASK 28: Domain: x ≥ 3   Range: f(x) ≥ 2

TASK 29: The range and domain have switched around. Again, this is all to do with the fact that we effectively swapped out x and y values around when finding the equation of the function.

TASK 30: y = $\frac{x}{x-1}$

If you were to reflect this curve about the line y = x, you would get the exact same curve. Therefore, the equation of the inverse is exactly the same as the equation of the function itself. The function is said to be self-inverse.

Equation 3: y=x
Equation 4: y=x/(x−1)
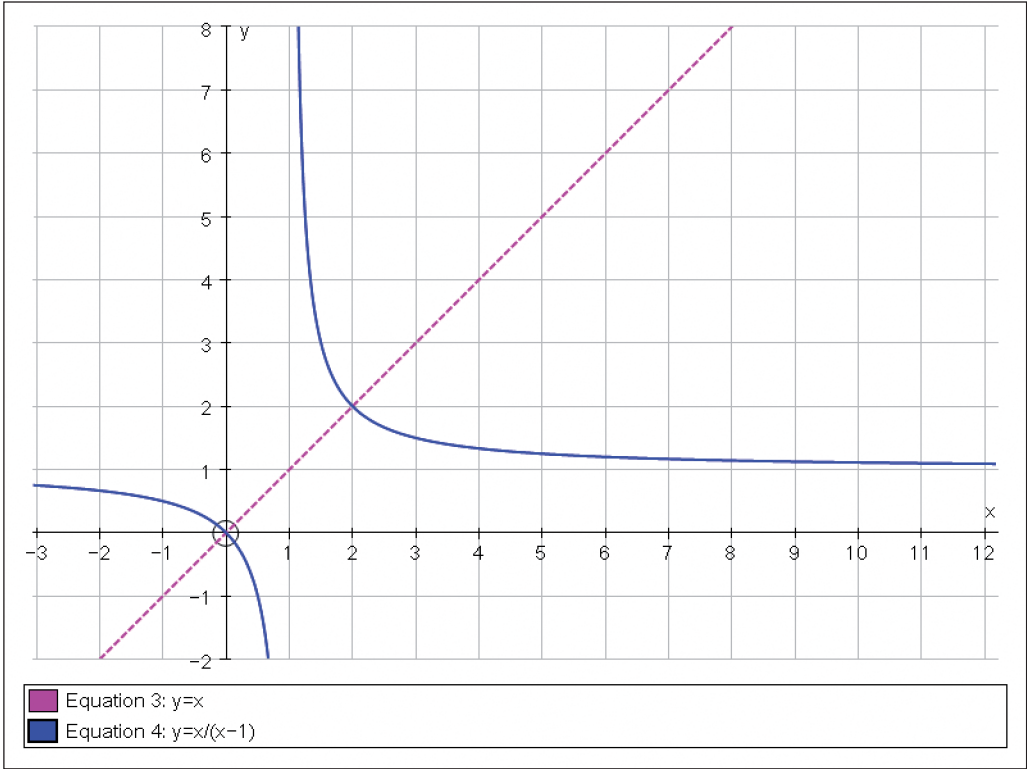
TASK 31:

y = x/x − 1
y(x − 1) = x
xy − y = x
xy − x = y
x(y − 1) = y

x = $\frac{y}{y-1}$

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

- To revise and consolidate the following skills in a visual and interactive manner to help deepen understanding:

  » The different forms of vector notation in 2D.

  » Geometrical applications of positional vectors in 2D, including the addition, subtraction and multiplication of vectors.

  » Working with the vector equation of a line in 2D, including knowing how each of the components effects both the slope and position of the line and how this relates to parallel and perpendicular vectors.

EXTENSION MATERIAL:

- To be able to visualise and manipulate vectors on a 3D set of axes to help deepen understanding.

- To revise and consolidate the following skills:

  » The different forms of vector notation in 3D.

  » Geometrical applications of positional vectors in 3D, including the addition, subtraction and multiplication of vectors.

  » Find the scalar product of two vectors.

  » Finding the magnitude of a vector.

  » Finding the angle between two vectors using the scalar (dot) product.

  » Proving whether or not a vector intersects with a given line.

- To be able to apply knowledge of vectors to solve co-ordinate geometry problems.

REQUIRED PRIOR KNOWLEDGE

FOR THE CORE CONTENT:

- To be comfortable with vector notation in the forms: $\mathbf{q}$, $\overrightarrow{OQ}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

- To be able to add, subtract and multiply vectors in order to obtain the required resultant vector.

- To be comfortable with the notation of the 2D vector equation of a

straight line in the form:

$$\frac{x}{y} = \begin{pmatrix} a \\ b \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \end{pmatrix}$$

- To be comfortable with the concept of a constant.
- To understand the terms parallel and perpendicular, and how they relate to the gradient of a straight line.
- To be aware of the properties of an isosceles triangle.

For the Extension Material:

- To understand the x-y-z set of axes, and how to plot 3D co-ordinates.
- To be comfortable with vector notation in the forms: ai + bj + ck and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
- To be able to find the scalar (dot) product of two vectors −**a**.**b**.
- To be able to find the magnitude of two vectors.
- To be able to use the scalar product and the magnitude to work out the angle between two vectors using: $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|a||b|}$
- To be able to work out the area of a triangle using: ½absin(c)
- To be able to work out the mid-point of a line segment in 3D.
- To be comfortable with the notation of the 3D vector equation of a straight line in the form:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$
- To be able to show whether or not two given lines intersect in 3D using simultaneous equations.

Description of the Activity

- This investigation should provide a nice re**View** of the work the students have completed on vectors. By making use of Autograph's **Constant Controller**, and in particular the 3D interface, the students should hopefully be better able to visualise the important points about vectors, and hence finish the investigation with a much deeper level of understanding.
- In Activity 1, students are asked to combine various quantities of position vectors to find their way around a parallelogram.
- In Activity 2, students are asked to manipulate four different constants in order to find the vector equations of straight lines meeting various sets of criteria.
- Activity 3, students enter the world of 3D, where they are asked to carry out several tasks culminating in finding the area of a triangle and proving that a line bisecting one of the sides will never cross the x-axis.

Pre–Activity Setup

Ensure the files: "Vectors - Activity 1" and "Vectors - Activity 3" are available and accessible to all students.

Suggested Introduction

"Vectors have been around since the middle of the nineteenth century, but more recently they have become an essential tool in the day to day lives of engineers, physicist and other scientists. Today we are going to use Autograph to both consolidate and deepen our understanding of vectors, including venturing into the previously unseen world of 3D…"

Suggested Prompts

- The key to working successfully with vectors in Autograph is selecting the right objects, and selecting them in the correct order. To add two vectors, both must be selected as well as the point which they are to start from, and when subtracting, the order of selection is crucial. This is essential for the whole investigation, and so for this reason, these points are introduced early in Activity 1. Just make sure all students have got to grips with this and then they should be fine for the rest of the investigation.
- Activity 3 uses Autograph's 3D interface, which will be new to most students. Encourage them to play around with it, and think about each point they plot and each line they draw. The world of 3D is often conceptually difficult for students (and teachers!), but hopefully this activity will help alleviate any issues and deepen understanding.

Suggested Plenary

- Can anybody summarise the key points they have learnt from this investigation?
- What are some of the things we need to look out for when dealing with vectors?
- How did you all cope with the 3D world?
- Has anybody got any top tips, or way they have found successful, to help us avoid tripping up with work on vectors?

Ideas for Further Work

- If it falls within the syllabus, the vector (or cross) product, together with further 3D co-ordinate geometry work involving equations of planes would naturally follow on from this investigation.
- Further exam style questions on vectors.

STUDENT WORKSHEET

NAME:..

ACTIVITY 1: POSITION VECTORS

Open up the Autograph file called "Vectors - Activity 1".

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Your screen should look something like this:
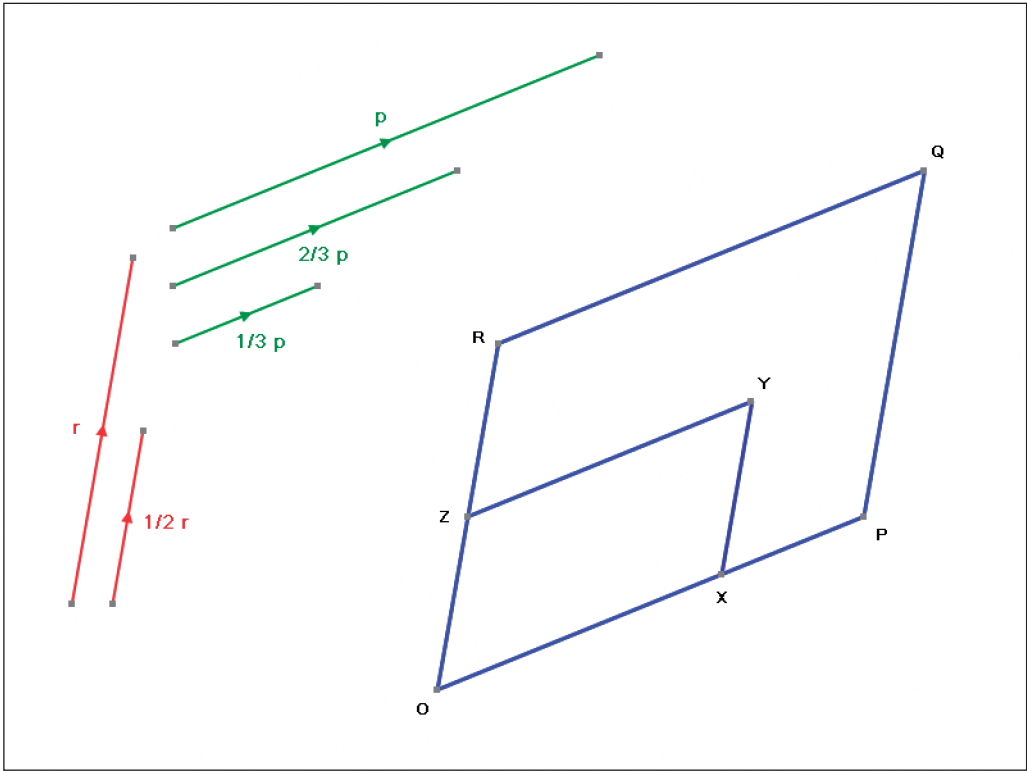
ORQP and OZYX are parallelograms and:

$\overrightarrow{OP}$ = **p** $\overrightarrow{OR}$ = **r** $\overrightarrow{OX}$ = $\frac{2}{3}$**p** $\overrightarrow{OZ}$ = $\frac{1}{2}$**r**

TASK 1: Express the following vectors in terms of **p** and **r** (see below how Autograph can help you):

1. $\overrightarrow{OQ}$ [ ]   6. $\overrightarrow{PY}$ [ ]

2. $\overrightarrow{PR}$ [ ]   7. $\overrightarrow{YR}$ [ ]

3. $\overrightarrow{OY}$ [ ]   8. $\overrightarrow{XQ}$ [ ]

4. $\overrightarrow{XZ}$ [ ]   9. $\overrightarrow{XR}$ [ ]

5. $\overrightarrow{RP}$ [ ]   10. $\overrightarrow{RX}$ [ ]

Using Autograph to help and check your answers:

(a) Adding Vectors

Say you thought that $\overrightarrow{OQ}$ = **p** + $\frac{1}{2}$**r**.

Left-click on an unoccupied part of the graph area to deselect everything.

Hold down the **Shift** button to select more than one object:

Left-click on the vector **p** (it should turn black).

Left-click on the vector $\frac{1}{2}$**r** (it should also turn black).

Left-click on the point O (it should have a little square around it).

Right-click and select **Add Vectors** from the menu.

A vector should now appear from point O, and if it stops at point Q, then you are correct, if not, just have another go.

Always click **Undo** after each attempt to clear the screen ready for another one!

(b) Subtracting Vectors

The method is very similar to adding vectors, but this time the order is crucial!

Say you thought that $\overrightarrow{ZX}$ = $\frac{1}{3}$**p** − $\frac{1}{2}$**r**.

Left-click on an unoccupied part of the graph area to deselect everything.

Hold down the **Shift** button to select more than one object:

Left-click on the vector 1/3**p first** (it should turn black).

Left-click on the vector 1/2**r second** (it should also turn black).

Left-click on the point Z (it should have a little square around it).

Right-click and select **Subtract Vectors** from the menu.

A vector should now appear from point Z, and if it stops at point X, then you are correct, if not, just have another go.

Always click **Undo** after each attempt to clear the screen ready for another one!

TASK 2: What fraction of the area of parallelogram OPQR is not taken up by parallelogram OXYZ?

ACTIVITY 2: THE VECTOR EQUATION OF A LINE

Open up a **New 2D Graph Page**.

Make sure you are not in **Whiteboard Mode**.

Edit the axes as follows:

    x: Minimum: −17   Maximum: 17   Numbers: 2   Pips: 1
    y: Minimum: −9   Maximum: 9   Numbers: 1   Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Select **Equal Aspect Mode** from near the top of the screen.

This alters the x scale so that the axes are square.

Select **Enter Vector Straight Line** from near the top of the screen.

Enter in the following:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \end{pmatrix}$$

The value of each of the constants a, b, p and q is automatically set to 1.

What will this vector straight line look like? Where will it cross the axes? Which direction will it slope? What is its gradient?

Click **OK**.

Left-click on the line (it should turn black).

Select **Text Box** from near the top of the screen.
Tick the box next to **Show Detailed Object Text** and click **OK**.

The **Text Box** should now display the equation of the vector straight line, together with the current values of the constants a, b, p and q.

Predict *Try to think what effect adjusting the values of each of the four constants will have on the vector straight line.*

Click on the **Constant Controller** on the top toolbar.
The **drop-down** menu allows you to select different constants.
The **up-down** arrows alter the value of the selected constant.
The **left-right** arrows adjust the size of the step.

TASK 3: Adjust the values of the constants a and b. What effect does this have on the vector straight line?

TASK 4: Adjust the values of the constants p and q. What effect does this have on the vector straight line?

In this next set of tasks you will be given some information, and you should use the **Constant Controller** to try to find the vector equation of a straight line which fits that information. The following tools might also help you:

Enters the equations of lines and curves.

Enters co-ordinates.

Undoes the last thing to be displayed on the screen.

TASK 5: Find the vector equation of a line which is **parallel** to the line y = 3x + 1 and passes through the point (−3, 1).

TASK 6: Find the vector equation of a line which represents the line y = −0.5x − 3 after it has been **translated** 8 units in the positive y direction (upwards!).

(blank box)

TASK 7: Find the vector equation of a line which is **perpendicular** to the line y = 2 − 4x and passes through the point (−4, −5).

(blank box)

TASK 8: Find the vector equation of a line which represents the line y = 0.2x − 1 after it has been **reflected** in the x-axis.

(blank box)

TASK 9: Explain briefly why there are no unique answers to any of the four tasks above.

(blank box)

TASK 10: Find the vector equation of a line which completes an isosceles triangle bounded by the lines: y = 2x + 2 and y = 7 − 0.5x.

(blank box)

ACTIVITY 3: WELCOME TO THE WORLD OF 3D!

Open up the Autograph file called "Vectors - Activity 3".

Make sure you not are in **Whiteboard Mode**.

Seeing a 3D set of axes like this might not be familiar to you, so take a few moments to have a good look around using the **Drag** tool.

Note: The following functions may also prove useful when working with Autograph in 3D:

+ **Ctrl**   Zooms in and out

+ **Shift**   Shifts the camera left and right

Restores the original view of the graph (**x-y-z Orientation**).

Click on **x-y-z Orientation** to return to your original **View** of the axes.

The triangle ABC has its vertices at the following points:

A (2, −1, 4)   B (3, −2, 5)   C (−1, 6, 2)

The origin O is also marked.

On the top toolbar click on **View** > **Status Box**.

Make sure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on each of the vertices of the triangle, and the origin, to make sure you can identify which point is which. They are colour co-ordinated to try and make this easier.

The co-ordinates of the vertices should appear in the **Status Box**

Note: The **Status Box** will only ever show information about the object or objects selected at that time. If the information ever disappears, just **left-click** to select your desired object again!

TASK 11: Express the vector in the form ai + bj + ck

(blank box)

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on the origin, O (it should have a little cube around it)

Left-click on point A (it should also have a little cube around it)

Right-click and select Create Vector from the menu

The vectorshould now be displayed, with the position vector displayed in the **Status Box**

Check that this matches your answer to Task 11 before carrying on.

TASK 12: Express the vector in the form ai + bj + ck

<div style="border: 2px solid blue; border-radius: 15px; height: 100px;"></div>

Create vectorin the same way as described above and use it to check your answer.

TASK 13: How could you combine vectorsandto give us the vector

Hint: Looking at the directions of the arrows of the vectors displayed on Autograph should help you.

<div style="border: 2px solid blue; border-radius: 15px; height: 80px;"></div>

TASK 14: Use your answer to Task 13 to express the vector in the form ai + bj + ck

Hint: Be very careful with your minus signs!

<div style="border: 2px solid blue; border-radius: 15px; height: 150px;"></div>

Checking your answer:

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on point A (it should have a little cube around it).

Right-click and select Vector from the menu.

Write your answer to Task 14 and click **OK**.

If your answer was correct, the vectorshould now be displayed.

If not, just hit **Undo** and try again!

TASK 15: Use a similar technique (including drawing the relevant vectors in Autograph) to express the vectorin the form ai + bj + ck.

Make sure you check and create your answer using the method described above, as we will need this vector later!

<div style="border: 2px solid blue; border-radius: 15px; height: 220px;"></div>

TASK 16: Calling vector a, and vector c, use your answers to Task 14 and Task 15 to find the scalar (or dot) product a.c.

<div style="border: 2px solid blue; border-radius: 15px; height: 200px;"></div>

Below is a two dimensional sketch of triangle ABC which you may find helpful for the next few tasks:

C (−1, 6, 2)

A (2, −1, 4)

B (3, −2, 5)
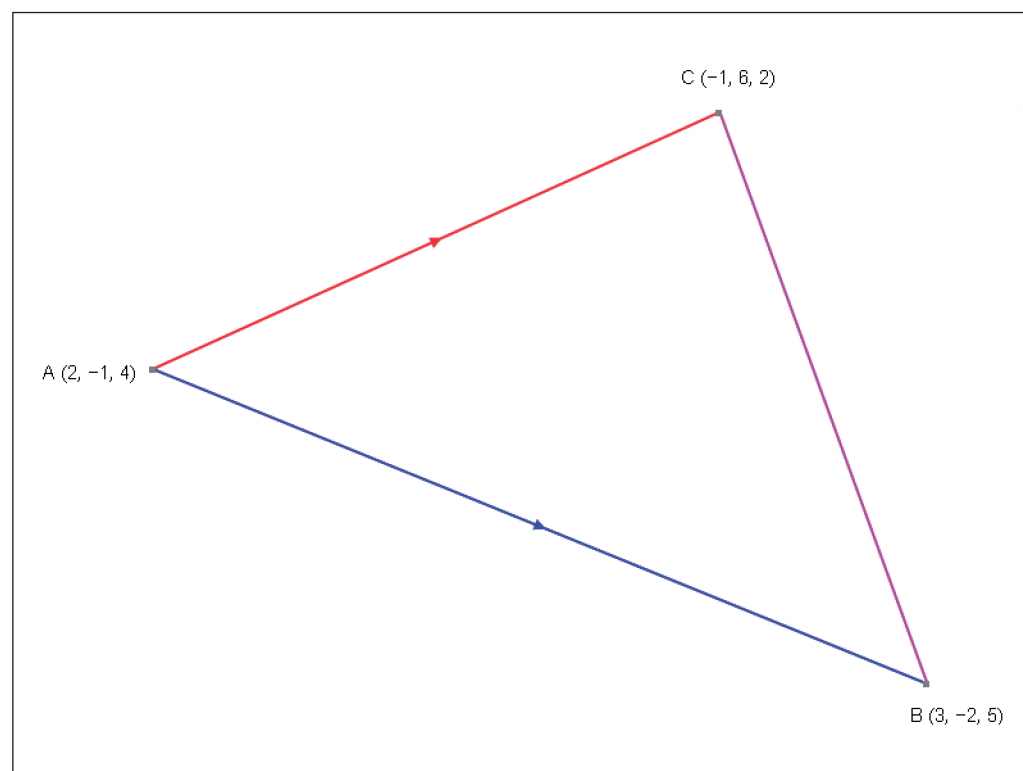
TASK 17: Use your answer to Task 14 to work out the length of the side AB. Leave your answer as a surd and mark it on the diagram.

TASK 18: Use your answer to Task 15 to work out the lengths of the side AC. Leave your answer as a surd and mark it on the diagram.

TASK 19: Use your answers to Tasks 16, 17 and 18 to work out, in **Degrees**, the angle CAB. Round your answer to one decimal place and mark it on the

diagram.

Hint: Think about how to work out the angle between two vectors.

Checking your answer:

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on vector MATHS (it should turn black).

Left-click on vector MATHS (it should turn black).

Right-click and select **Angle Between Vectors** from the menu.

The angle should be displayed in the **Results Box**.

TASK 20: Use the information contained on your diagram to work out the area of triangle ACB. Round your answer to three significant figures.

Hint: How do you work out the area of a triangle when you are given two sides and the included angle?

[blank answer box]

TASK 21: Work out the co-ordinates of the mid-point of the line AC.

[blank answer box]

TASK 22: Write the vector equation of a straight line which is parallel to AB, and which passes through the mid-point of AC.

[blank answer box]

Checking your answer:

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on point A and then point C (they should have cubes around them).

Right-click and select **Mid-Point** from the menu.

The mid-point of the line AC should now be displayed.

Select **Enter Vector Straight Line** from near the top of the screen.

Enter your answer to Task 22.

A vector straight line should now appear which is parallel to AB and which passes through the mid-point of AC.

TASK 23: What is the vector equation of the x-axis?

[blank answer box]

Select **Enter Vector Straight Line** and check your answer.

TASK 24: Will the line from Task 22 ever cross the x-axis? If so, find the co-ordinates of intersection. If not, explain how you know algebraically.

[blank answer box]

SUMMARY

Use the space below to summarise what you have learnt during this investigation:

ANSWERS

ACTIVITY 1: POSITION VECTORS
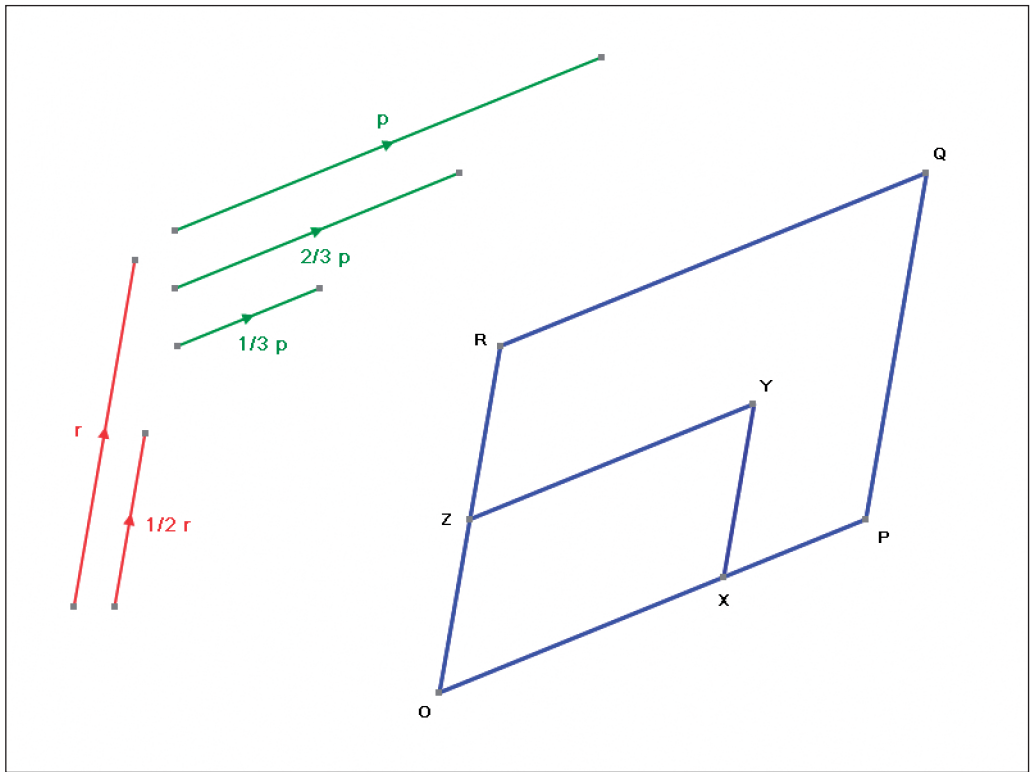
TASK 1: See below for an example of how the students would check Questions 3 and 6 using Autograph.

1.   6.

2.   7.

3.   8.

4.   9.

5.   10.
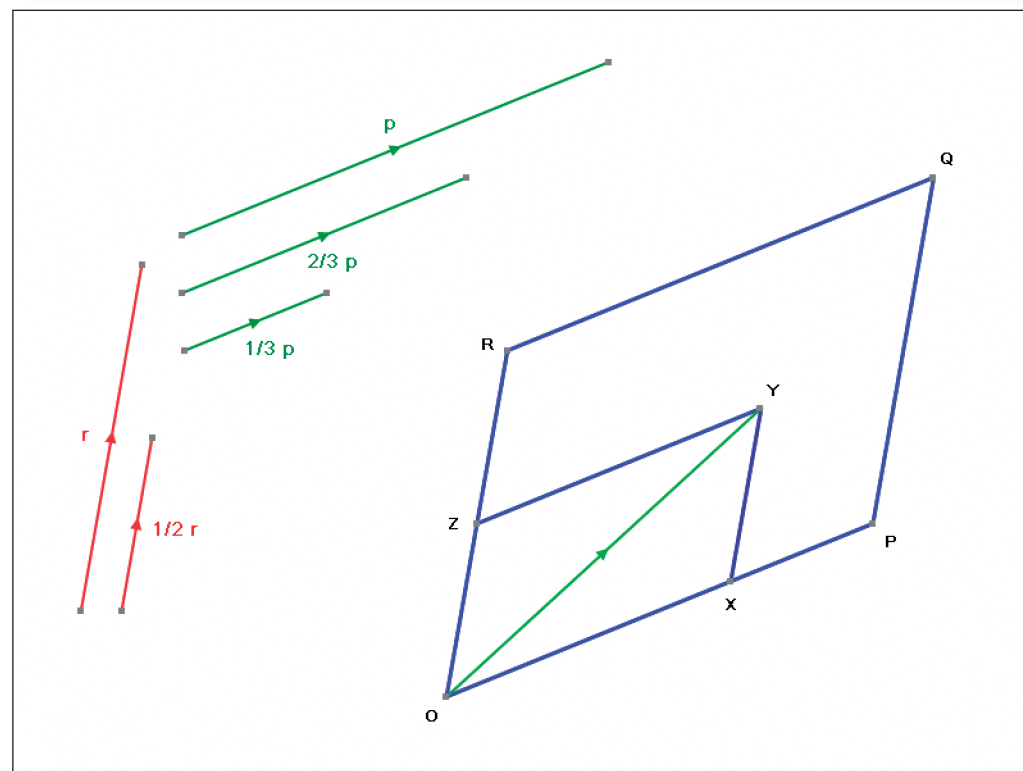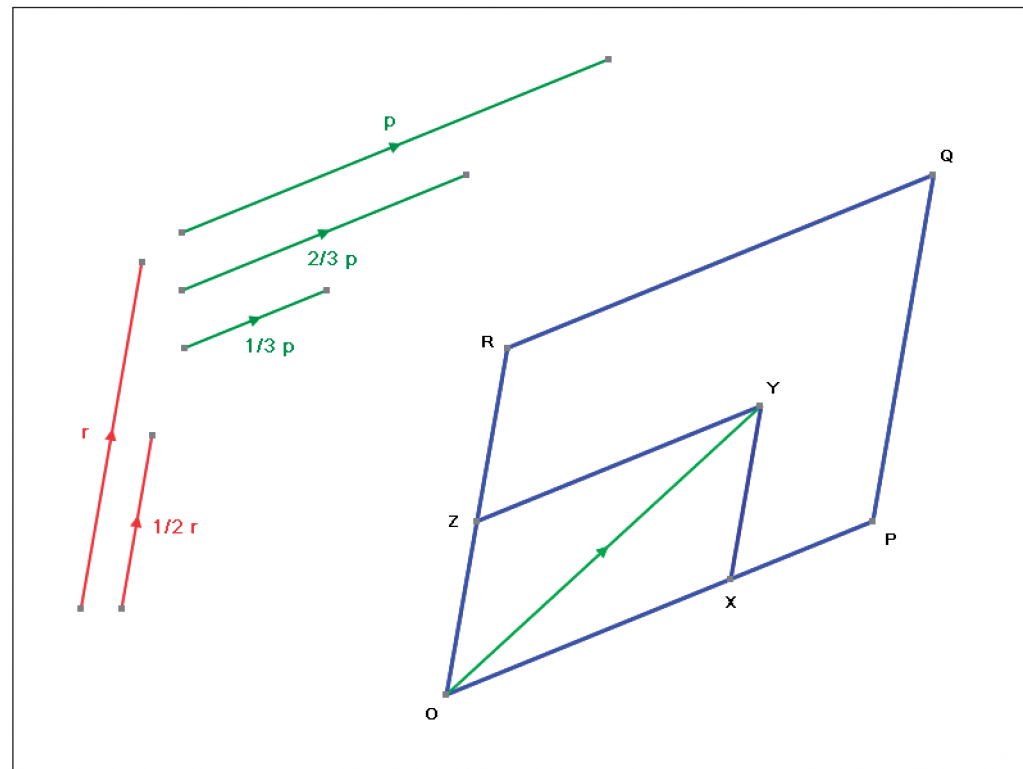
TASK 2: The fraction of the larger parallelogram not taken up by the smaller, is equal to:

MATHS
MATHS

ACTIVITY 2: THE VECTOR EQUATION OF A LINE

TASK 3: Constants **a** and **b** determine a point through which the vector straight line will pass. Hence, adjusting the values of **a** and **b** will serve to translate the position of the vector straight line, keeping the gradient the same.

TASK 4: Constants **p** and **q** determine the vector path along which the line travels. **p** determines how many units to the left or right, and **q** determines how many units up or down, so together they determine the gradient of the vector straight line.

Note: For the following tasks, there are an infinite number of correct answers, due to the number of possible points which lie on a vector straight line, and the different form the gradient may take. My answers should serve as a guide, but the students will have been able to check their own answers during the tasks using Autograph.

TASK 5: The vector equation of a line which is parallel to the line y = 3x + 1 and passes through the point (−3, 1): MATHS
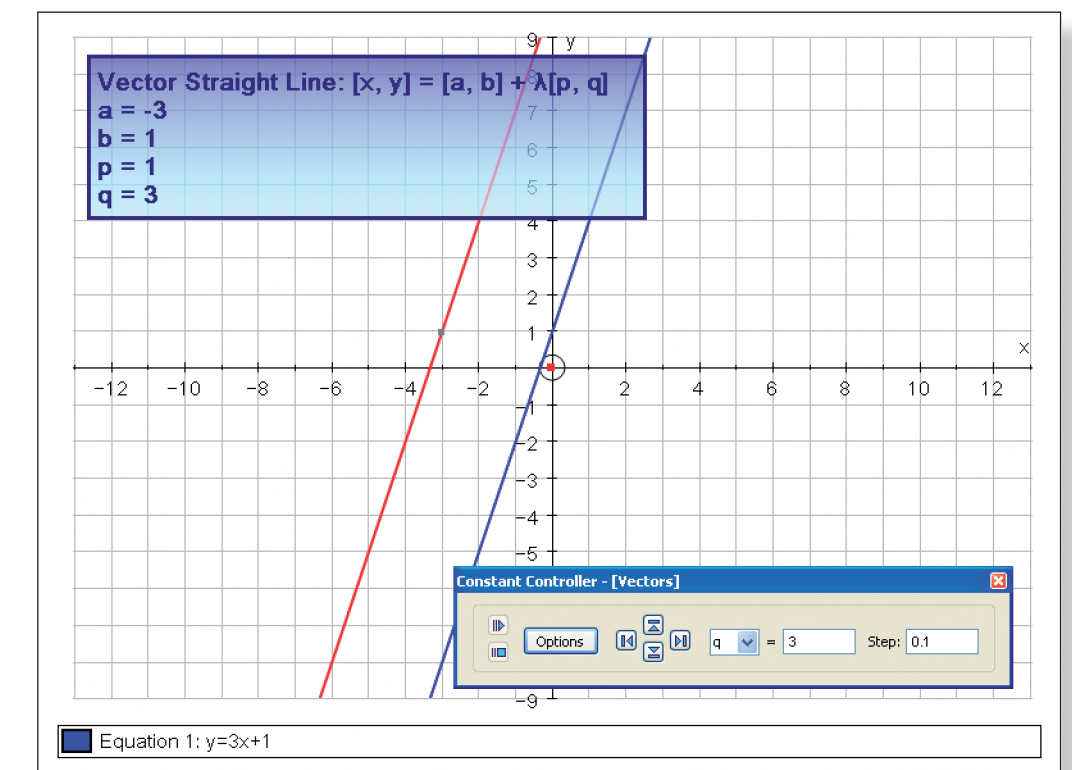
TASK 6: The vector equation of a line which represents the line y = −0.5x − 3 after it has been translated 8 units in the positive y direction: MATHS

Vector Straight Line: [x, y] = [a, b] + λ[p, q]
a = 0
b = 5
p = 2
q = -1

Constant Controller - [Vectors]

Options   q = -1   Step: 0.1

Equation 1: y=−0.5x−3
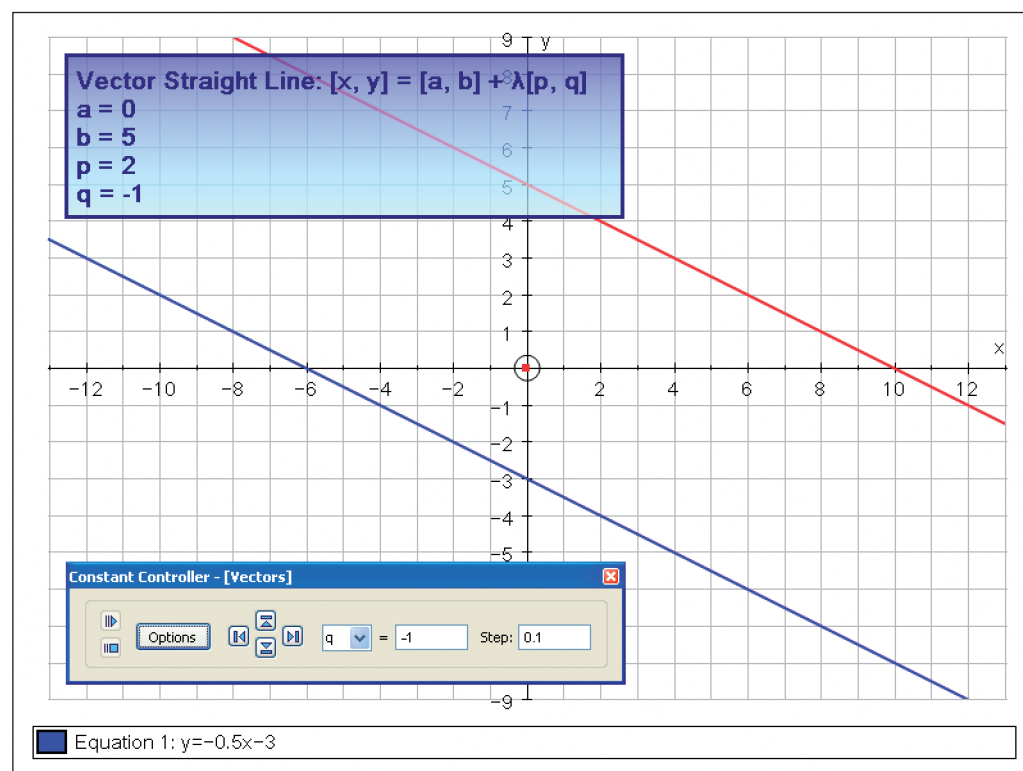
TASK 7: The vector equation of a line which is perpendicular to the line y = 2 − 4x and passes through the point (−4, −5): MATHS
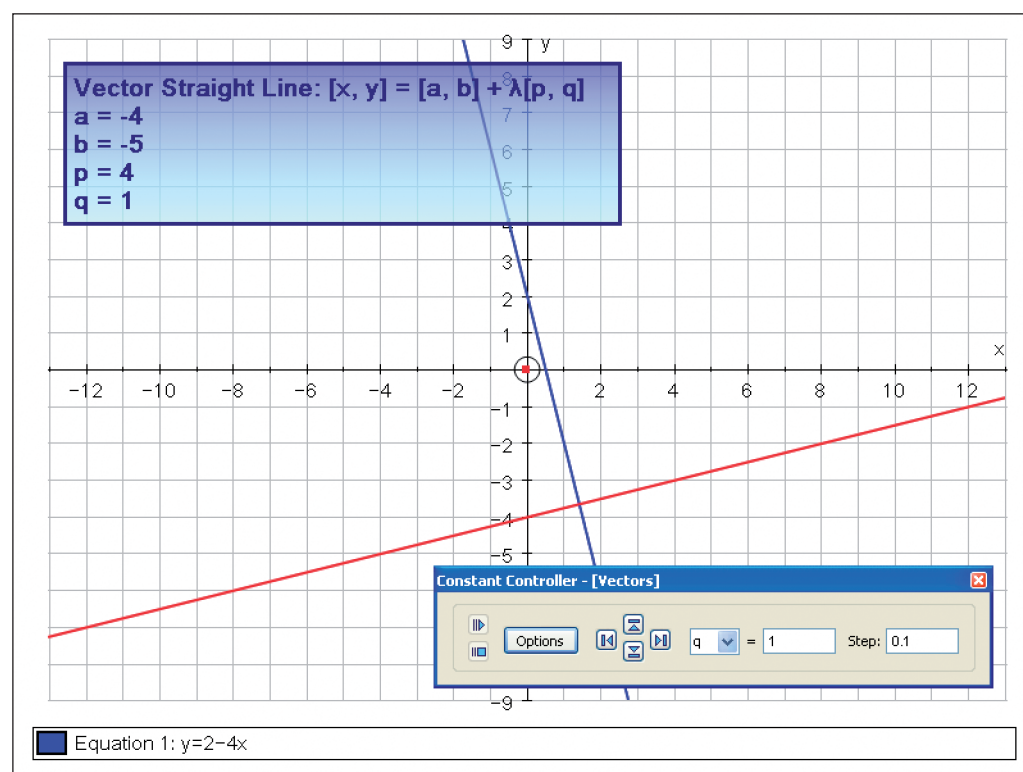
Vector Straight Line: [x, y] = [a, b] + λ[p, q]
a = -4
b = -5
p = 4
q = 1

Constant Controller - [Vectors]

Options   q = 1   Step: 0.1

Equation 1: y=2−4x

TASK 8: Find the vector equation of a line which represents the line y = 0.2x − 1 after it has been reflected in the x-axis: MATHS

Vector Straight Line: [x, y] = [a, b] + λ[p, q]
a = 5
b = 0
p = 5
q = -1

Constant Controller - [Vectors]

Options   q = -1   Step: 0.1

Equation 1: y=0.2x−1
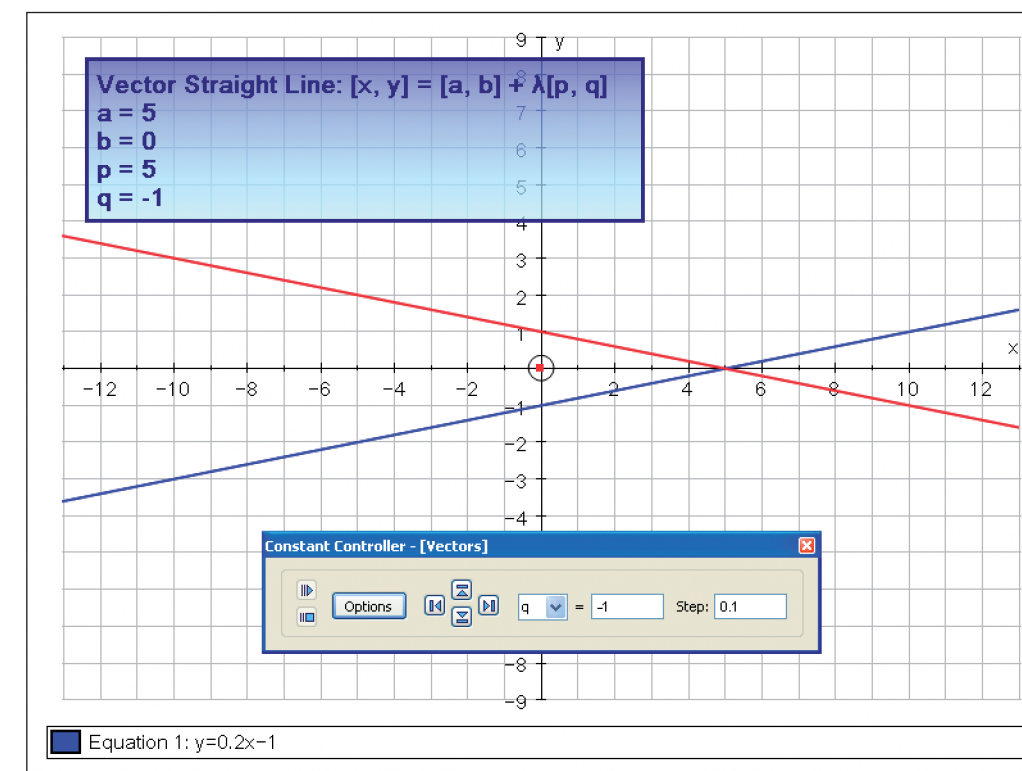
TASK 9: There are no unique correct answers to any of the tasks above for two reasons:

1. There are an infinite number of points which lie along any vector straight line, and hence an infinite number of possible pairs of values for a and b.
2. There are an infinite number of different ways of expressing the direction which a vector takes, and hence an infinite number of possible pairs for p and q. For example, p = 2, q = 6 is exactly the same as p = 8, q = 24.

TASK 10: The vector equation of a line which completes an isosceles triangle bounded by the lines: y = 2x + 2 and y = 7 − 0.5x: MATHS

Vector Straight Line: [x, y] = [a, b] + λ[p, q]
a = 0
b = 2
p = 3
q = 1

Constant Controller - [Vectors]
Options | q | = 1 | Step: 0.1

■ Equation 1: y=2x+2
■ Equation 3: y=7−0.5x
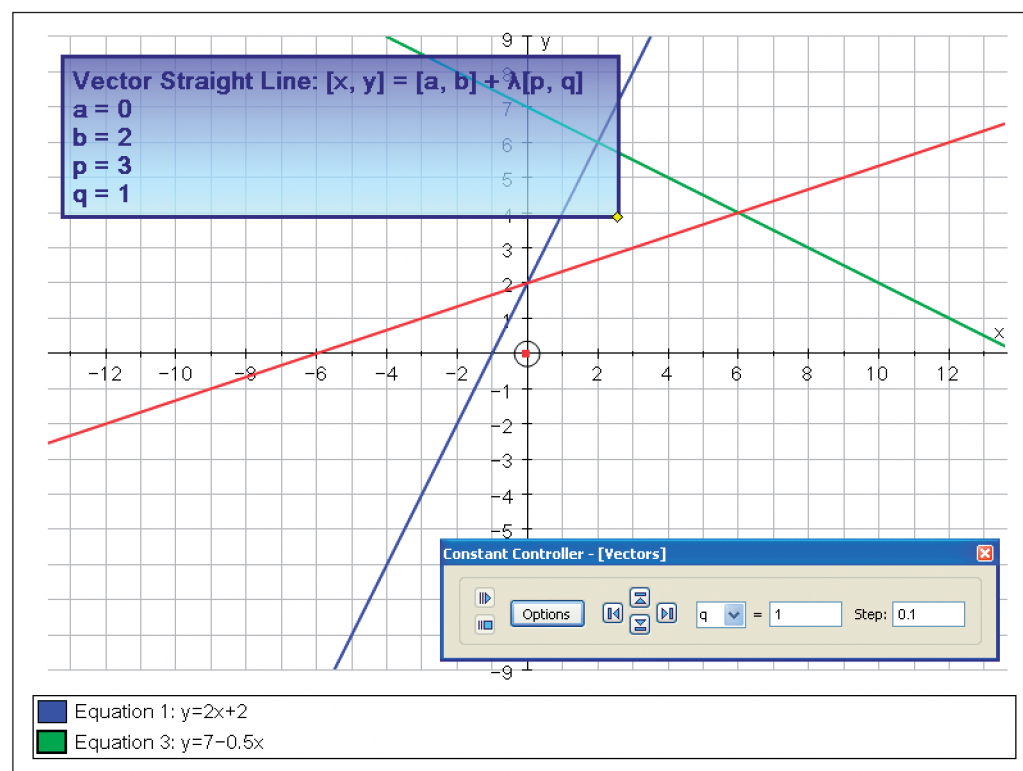
Activity 3: Welcome to the world of 3D!
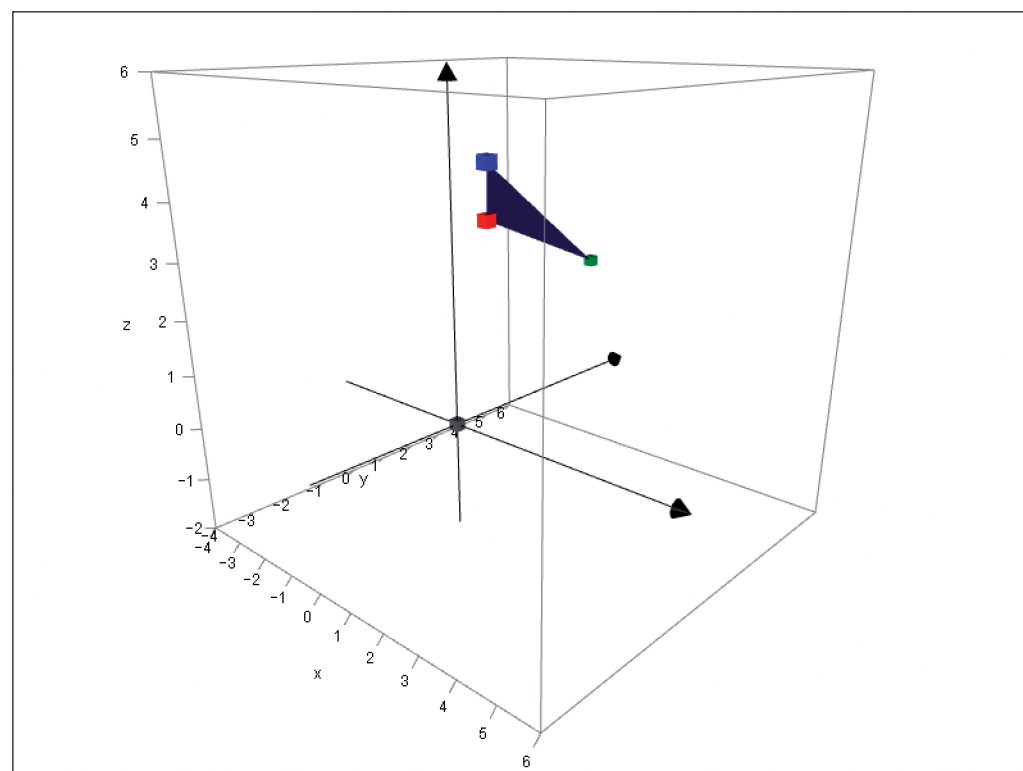
Task 11: = 2i−j + 4k

Task 12: = 3i−2j + 5k

Task 13: MATHS

Task 14: (3i−2j + 5k) − (2i−j + 4k)

= i−j+k

Status Box
x = 2, y = −1, z = 4   Vector: [1, −1, 1]
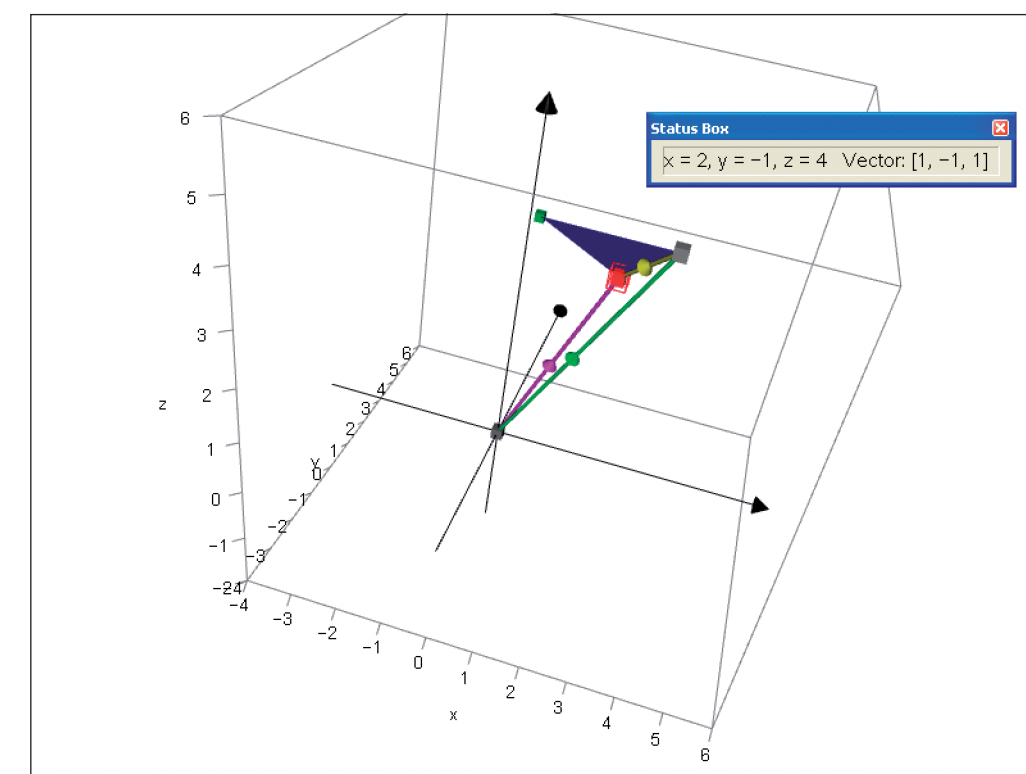
Task 15: −i+ 6j + 2k

(−i+ 6j + 2k) − (2i−j + 4k)

= −3i+7j−2k

Status Box
x = 2, y = −1, z = 4   Vector: [−3, 7, −2]
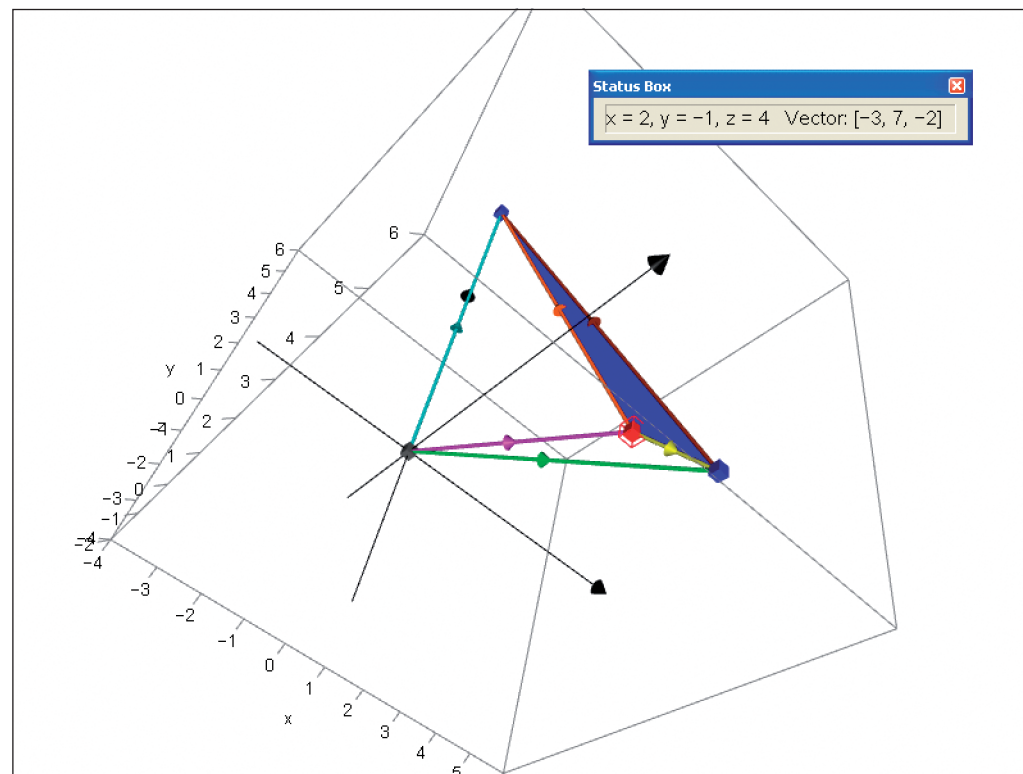
TASK 16: a.c = (i−j+k) . (−3i+7j−2k)

= −3 − 7 − 2
= −12

TASK 17: MATHS

TASK 18: MATHS

TASK 19: MATHS

MATHS
MATHS

Status Box
Vector: [1, −1, 1]

Results Box
Scalar Product: -12

Angle between Vectors: 151.6°

C (−1, 6, 2)

√62

A (2, −1, 4)   41.15°

√3

B (3, −2, 5)
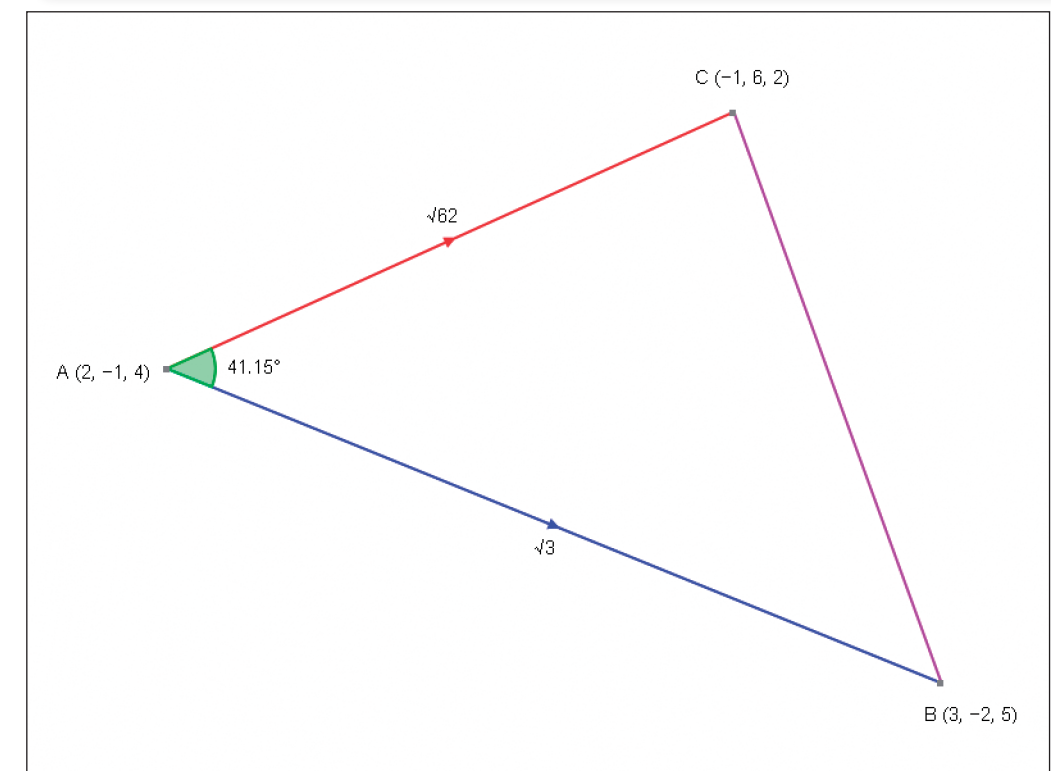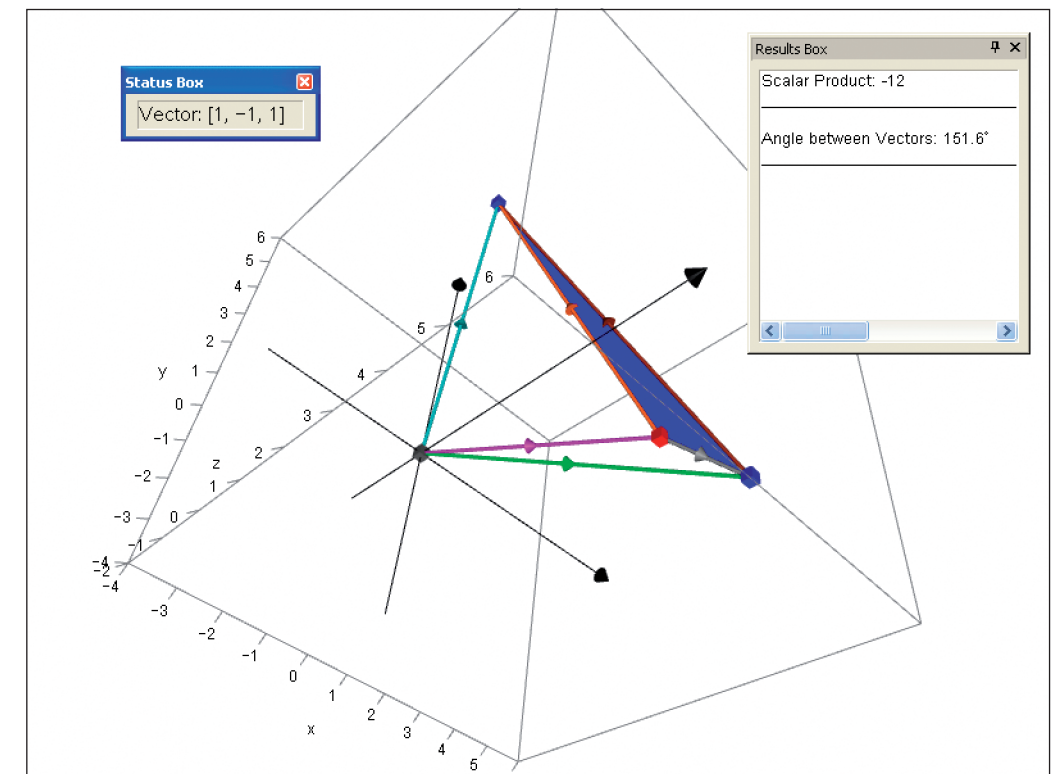
DRAWING OF A TRIANGLE (Nice to create in Autograph?)

TASK 20: Area of triangle = ½ ab sin(c)

= ½ √62 x √3 x sin(151.6)
= 3.24 units squared (three significant figures).

TASK 21: A (2, −1, 4)   C (−1, 6, 2)

Mid-point = (0.5, 2.5, 3)

TASK 22: i−j+k

Possible equation of line: MATHS
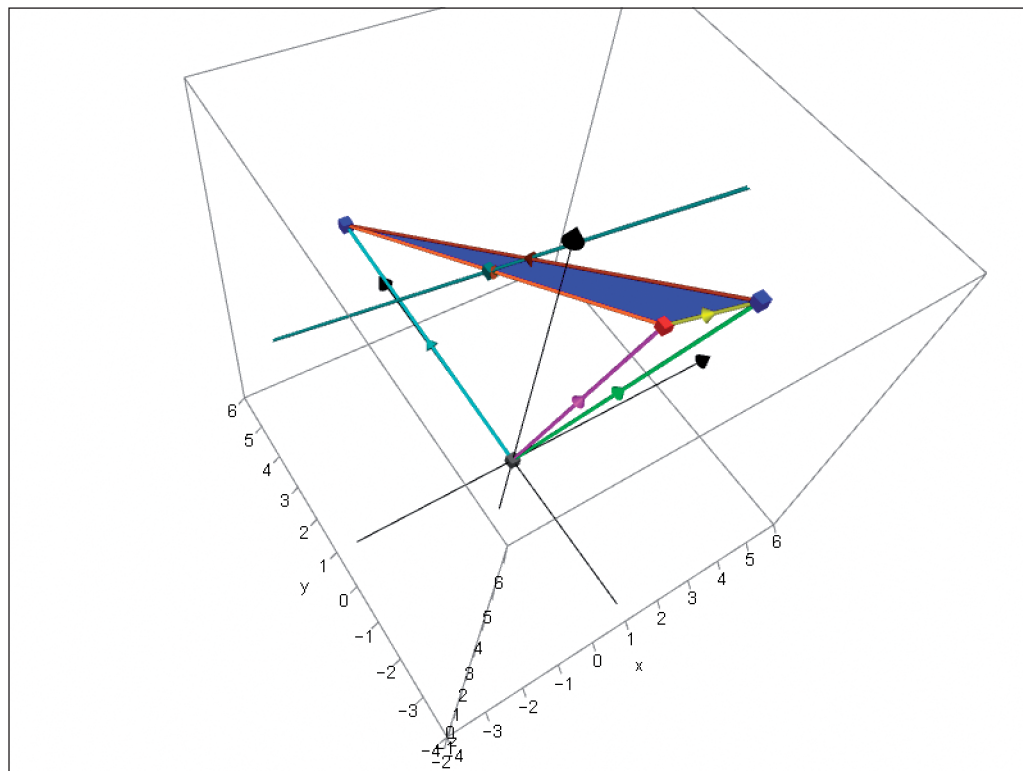
TASK 23: Possible equation of x-axis: MATHS

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

- To understand that some equations cannot be solved using conventional methods.

- To be able to construct iterative formulae in the form $x^n+1 = f(x^n)$.

- To be able to use a calculator efficiently to compute the sequence of numbers generated by an iterative formula.

- To understand and be able to visualise how the process of iteration works, and why it leads to the solution of otherwise unsolvable equations.

- To understand the concepts of convergence and divergence, as well as cob-web and staircase diagrams.

- To begin to appreciate what conditions lead to convergence and divergence, and the different forms this convergence or divergence will take.

- To reinforce the graphical features of different functions.

EXTENSION MATERIAL:

- To understand that one equation can produce different types of iterative formulae, and these formulae can lead to convergence on different solutions.

- To understand what conditions lead to convergence and divergence, and the different forms this convergence or divergence will take.

REQUIRED PRIOR KNOWLEDGE

- To be comfortable rearranging formula, including those involving powers and roots (also exponentials and logs for the Extension Material).

- To understand the concepts of solutions and roots to equations.

DESCRIPTION OF THE ACTIVITY

- This investigation is designed to provide a solid introduction to the concept of using numerical methods to solve difficult equations. Instead of just being told the iterative formulae, students are carefully introduced to the concept, and being able to relate the sequence of numbers generated on the calculator to the graphical representation of iteration should serve

to deepen understanding.

- In Activity 1, students are introduced to the need for numerical methods having been presented with an unsolvable equation. They are then guided through each of the steps necessary to form the iterative equation, and the sequence of numbers produced is related directly to the resulting cobweb graph. Students are then encouraged to use the iteration function on Autograph to try out different starting values and observe the effect on the type and rate of convergence.

- In Activity 2, students are presented with another unsolvable equation, and again encouraged to form the iterative equation that will help them find an approximation to the answer. This time they are faced with staircase convergence, and are encouraged to think why this might be.

- Activity 3 contains the extension material. Here students are presented with a more difficult equation, and given much less guidance of how to deal with it. They are also introduced to the concept of divergence, and to the fact that one equation may lead to different iterative formulae.

- In Activity 4, students are asked to use what they have learnt to produce a function which would lead to cob-web divergence, and then to find some general rules which would tell them if an iterative formulae will converge or diverge to a particular solution, and what might effect the rate of this process.

Pre–Activity Setup

None required.

Suggested Introduction

"You may think that with all the algebraic methods you have built up over the years, you are well equipped to deal with any equation that comes your way. However, there are some equations which, as nice as they might look, are actually incredibly difficult to solve. Today, we are going to look at one method of getting around this problem."

Suggested Prompts

The most important Autograph function employed during this investigation is, rather obviously, the iterative function. Be on hand to make sure the students get to grips with this during Activity 1, and then they should be fine.

- Another potential stumbling block is that students might not be able to program their calculators to perform iterative calculations quickly. I have written a brief guide about using the Ans function, but again be aware the some students might need a little extra guidance.

- Students can check their iterative sequence of numbers using the table in

Autograph, and it is often worthwhile reminding them of this fact.

- The whole investigation is designed to build up to the students themselves deriving the rules that govern whether or not an iterative formula will converge to a given solution, or diverge, and how fast this process will be. The students are challenged at various stages to offer their thoughts on this matter, but encourage them to move on with the activity if all the pieces to the puzzle are not in place just yet!

Suggested Plenary

- Can anybody summarise the key points they have learnt from this investigation?

- What conditions must be present in order for us to have convergence to a given solution?

- What affects the rate of this convergence?

- What determines whether we get a cobweb diagram, or a staircase diagram?

- Has anybody got any top tips, or way they have found successful, to help us avoid tripping up with work on numerical methods?

Ideas for Further Work

- Further questions and examples to consolidate understanding of the numerical solution to equations.

- The Newton Rhaphson method.

- Numerical integration using Trapezium Rule and Simpson's Rule.

S8 INVESTIGATING ITERATION

STUDENT WORKSHEET

NAME:...

ACTIVITY 1

TASK 1: Explain briefly the difficulty in solving the following equation: $1 - x^3 = \sqrt{x}$

Hint: Attempt to solve the equation for yourself, and you should quickly see!

TASK 2: Without using a calculator, find the two integer values which the solution to the equation $1 - x^3 = \sqrt{x}$ must lie between.

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Edit the axes as follows:
x: Minimum: −3   Maximum: 3   Numbers: 1   Pips: 0.2
y: Minimum: −5   Maximum: 5   Numbers: 1   Pips: 0.5
Remove all of the green ticks underneath **Auto**.

Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

On the top toolbar click on **Page** > **Edit Settings**, and change the Number significant figures to 10.

Click on **Slow Plot** mode (there should be a blue square around the button).

Enter the equation: $y = 1 - x^3$

Note: To enter x^3 use the little 3 button, or press "**alt 3**" together, or type "**xxx**".

Predict *What will this curve look like? Where will it cross the axes? Which direction will it slope?*

Click **OK**.

Enter the equation: $y = \sqrt{x}$

Predict *What will this line look like? Where will it cross the axes? Which direction will it slope? Will it ever be negative?*

Click **OK**.

Use the **Zoom In** and **Drag** tools to have a closer look at the point of intersection of these two curves. Notice how the scale automatically adjusts the closer you get.

TASK 3: Use Autograph to give the solution to the equation $1 - x^3 = \sqrt{x}$ to three decimal places.

Important Point: Arriving at an approximation to the solution by this method relies on the fact that we have powerful graphing software like Autograph available. We are now going to look at a way of getting an approximation to this solution which uses only our calculators.

TASK 4: Show that $1 - x^3 = \sqrt{x}$ can be re-arranged to give: $x = (1 - \sqrt{x})^{\frac{1}{3}}$

Open up a **New 2D Graph Page**.

Edit the axes as follows:
  x: Minimum: 0   Maximum: 1   Numbers: 0.1 Pips: 0.1
  y: Minimum: 0   Maximum: 1   Numbers: 0.1 Pips: 0.1
Remove all of the green ticks underneath **Auto**.

Enter the equation: $y = (1 - \sqrt{x})^{\frac{1}{3}}$
Note: Enter the equation as follows: y = (1 − √x)^(1/3).

Predict

What will this line look like over this section of the axes? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Enter the equation y = x.

Note: Because $y = (1 - \sqrt{x})^{\frac{1}{3}}$ is just a re-arrangement of $1 - x^3 = \sqrt{x}$, the point of intersection of the two graphs is exactly the same.

We are now going to use an **iterative formula** to help us find the solution to this equation.

The formula looks like this: $x_{n+1} = (1 - \sqrt{x_n})^{\frac{1}{3}}$

All this formula is saying is:

1. Pick a starting x value (x_0).

2. Substitute it into the right-hand side of the equation.

3. The answer gives us our next x value (x_1).

4. Substitute x_1 into the right-hand side of the equation to give us our next x value (x_2).

And so on…

We are going to use 0.5 as our starting value (x_0) as it is in the middle of the two integer bounds (0 and 1).

TASK 5: Ensure that you can get the first four values on your calculator, and then fill in the rest of the table:

Note: You can speed this process up considerably by using the answer function (Ans) of your calculator.

For example, on my calculator, I type 0.5 = , clear the screen, and then type $(1 - \sqrt{\text{Ans}})^{\frac{1}{3}}$.

Now, each time I press = the calculator generates the next term in the sequence.

x_i	Value	x_i	Value
x_0	0.5	x_6	
x_1	0.6641045243	x_7	
x_2	0.5698777976	x_8	
x_3	0.6258154615	x_9	
x_4		x_{10}	
x_5		x_{11}	

TASK 6: Continue finding values on your calculator until you can give the solution to $x = (1 - \sqrt{x})^{\frac{1}{3}}$ correct to four decimal places.

Important Point: We are now going to use Autograph to illustrate graphically exactly how this process of iteration arrives at an approximation to the solution of our equation.

Ensure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black)

Right-click and select **x=g(x) Iteration** from the menu.

Enter the value of x_0 as 0.5.

Click the right arrow.

Observe:

The end of the vertical line on the graph shows the value of y in $y = (1 − \sqrt{x})^{1/3}$ when $x = 0.5$.

The end of the horizontal line gives us the value of x which is equal to this y value.

This illustrates our value for x_1 (0.6641045243).

Click the right arrow again.

Observe:

Again, the end of the vertical line on the graph shows the value of y in $y = (1 − \sqrt{x})^{1/3}$ when $x = 0.6641045243$.

Again, the end of the horizontal line gives us the value of x which is equal to this y value.

This illustrates our value for x_2 (0.5698777976).

Clicking the right arrow continues this process.

Use the **Zoom In** and **Drag** tools to have a closer look at the point of intersection of these two curves. Notice how the scale automatically adjusts the closer you get. Notice also how the points on the graph correspond to the values in the table.

Just click **Undo** if you ever zoom too far in and loose the graph!

Note: You can extend the length of the **Setup x=g(x) Iteration** box to see more x values by dragging the bottom corner of the box.

Important Point:

Notice that with each iteration we get closer to the solution to the equation.

We say that our sequence of numbers converges on the solution.

The name for the diagram you have constructed on your graph is a converging cob-web.

When you are ready:

Press **Undo** enough times to return to the original view of the graph.

Ensure you are in **Select Mode**.

If the cob-web is still visible, **left-click** on the cob-web (it should turn black).

Right-click and select **Delete** from the menu, or simply press **delete** on the keyboard.

You should now be left with just the lines $y = x$ and $y = (1 − \sqrt{x})^{1/3}$.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black).

Right-click and select **x=g(x) Iteration** from the menu.

TASK 7: Try experimenting with different values of x_0 within the range $0 ≤ x ≤ 1$ and conducting the iteration process using the right arrow button. Are there any that values of x_0 that don't lead to convergence? Do some take longer to converge than others? Why do you think this is?

Note: You do not have to press **Undo** all the time. Just enter in a new initial value and Autograph will automatically reset the iteration.

ACTIVITY 2

TASK 8: Think about the shapes of the graphs of $y = 2^x$ and $y = x^3$. How do the shapes of these two graphs tell you that there is at most only one solution to the equation: $2^x = x^3$?

Hint: Draw a quick sketch in the space below if you find it helps.

[blank box]

TASK 9: Explain briefly the difficulty in solving the following equation: $2^x = x^3$ using normal algebraic methods.

[blank box]

TASK 10: Without using a calculator, find which two integer values the solution to the equation $2^x = x^3$ must lie between.

[blank box]

Open up a **New 2D Graph Page**.

Edit the axes as follows:
    x: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.2
    y: Minimum: −8   Maximum: 8   Numbers: 1   Pips: 0.5
Remove all of the green ticks underneath **Auto**.

Ensure **Slow Plot** mode is still turned on

Enter the equation: $y = 2^x$

Note: To enter x as a power use the little x button, or press "**alt x**" together.

What will this curve look like? Where will it cross the axes? Which direction will it slope? Will it ever be negative?

Click **OK**.

Enter the equation: $y = x^3$

Predict *What will this curve look like? Where will it cross the axes? Which direction will it slope? Will it ever be negative?*

Click **OK**.

Use the **Zoom In** and **Drag** tools to have a closer look at the point of intersection of these two curves. Notice how the scale automatically adjusts the closer you get.

TASK 11: Use Autograph to give the solution to the equation $2^x = x^3$ to three decimal places.

[blank box]

Important Point: Again, let's investigate how to get this solution armed only with our calculators.

TASK 12: Show that $2^x = x^3$ can be re-arranged to give the iterative formula:
$x_{n+1} = (2^{x_n})^{\frac{1}{3}}$

[blank box]

TASK 13: Using 1.5 as our starting value (x_0) ensure that you can get the first four values on your calculator, and then fill in the rest of the table:

x_i	Value	x_i	Value
x_0	1.5	x_6	
x_1	1.414213562	x_7	
x_2	1.386458584	x_8	
x_3	1.377596002	x_9	
x_4		x_{10}	
x_5		x_{11}	

TASK 14: Look at the sequence of values in your table compared to those in Activity 1. How are they different, and how do you think this will affect the diagram of the solution?

TASK 15: Use the table to give the solution to $x = (2^x)^{\frac{1}{3}}$ correct to four decimal places.

Important Point: Again, we are now going to use Autograph to illustrate graphically exactly how this process of iteration arrives at an approximation to the solution of our equation.

Open up a **New 2D Graph Page**.

Edit the axes as follows:
     x: Minimum: −8   Maximum: 10   Numbers: 1   Pips: 0.5

     y: Minimum: −6   Maximum: 6   Numbers: 1   Pips: 0.5
Remove all of the green ticks underneath **Auto**.

Enter the equation: $y = (2^x)^{\frac{1}{3}}$

Note: Enter the equation as follows: $y = (2^x)\wedge(1/3)$

Predict

What will this line look like over this section of the axes? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Enter the equation: $y = x$

Observe: Because $x = (2^x)^{\frac{1}{3}}$ is just a re-arrangement of $2^x = x^3$, the point of intersection of the two graphs is exactly the same.

Ensure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black).

Right-click and select **x=g(x) Iteration** from the menu.

Enter the value of x_0 as 1.5.

Click the right arrow.

As before, use the **Zoom In** and **Drag** tools to have a closer look at the point of intersection of these two curves. Notice how the scale automatically adjusts the closer you get. Pay particular attention to how the points on the graph correspond to the values in the table.

Important Point:

Again, we can see that our sequence of numbers *converges* on the solution, but this time they do not *oscillate* around the solution, rather head straight towards it.

The name for the diagram you have constructed on your graph is a **converging staircase**.

When you are ready:

Press **Undo** enough times to return to the original view of the graph.

Ensure you are in **Select Mode**.

If the cob-web is still visible, **Left-click** on the cob-web (it should turn black).

Right-click and select **Delete** from the menu, or simply press **delete** on the keyboard.

You should now be left with just the lines y = x and y = (2^x)$^{1/3}$.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black).

Right-click and select **x=g(x) Iteration** from the menu.

TASK 16: Try experimenting with different values of x$_0$ within the range −8 ≤ x ≤ 6 and conducting the iteration process using the right arrow button. Are there any that values of x$_0$ that don't lead to convergence? Do some take longer to converge than others? Why do you think this is?

Note: Again, you do not have to press **Undo** all the time. Just enter in a new initial value and Autograph will automatically reset the iteration.
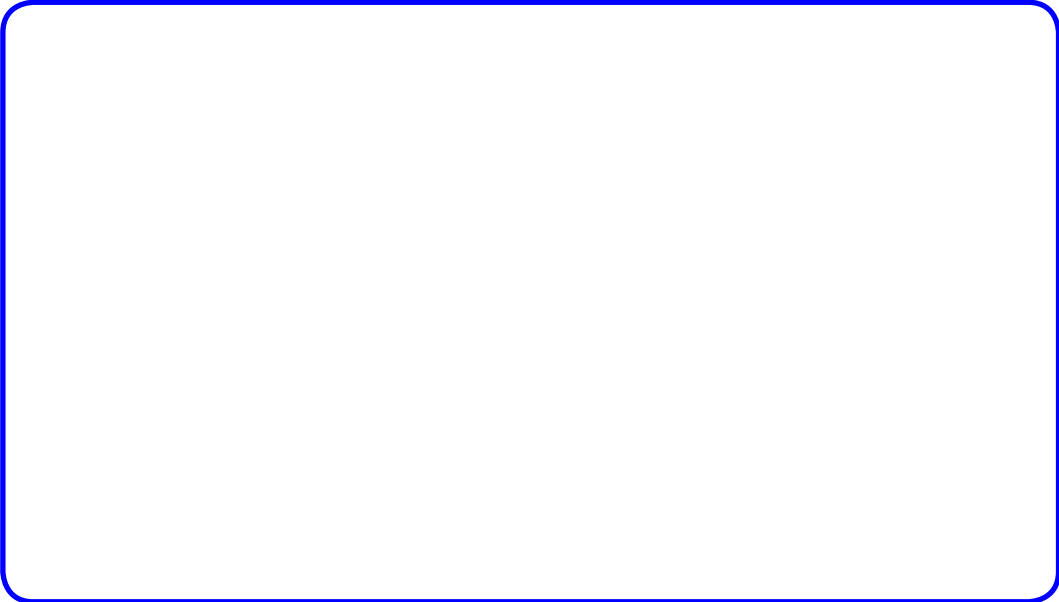
ACTIVITY 3

TASK 17: Using a calculator, show that one solution to the equation e^x − x − 2 = 0 lies between 1.1 and 1.2, and the other solution lies between −1.9 and −1.8.

TASK 18: Show that the equation: e^x −x − 2 = 0 can be rearranged to give the iterative formula: x$_{n+1}$ = e^{x_n} − 2

Open up a **New 2D Graph Page**.

Edit the axes as follows:
  x: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.25
  y: Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.5
Remove all of the green ticks underneath **Auto**.

Ensure **Slow Plot** mode is still turned on.

Enter the equation: y = e^x + 2

Predict

What will this line look like over this section of the axes? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Enter the equation y = x and click **OK**.

The line and the curve should now be on the screen with the two points of intersection representing the two solutions to our original equation.

Predict

Looking at the graph, if we use the iterative formula x$_{n+1}$ = e^{x_n} − 2 and a starting value of 1.2, what do you think will happen? Convergence? Divergence? Cob−web? Staircase? Which solution?

TASK 19: Using the iterative formula x$_{n+1}$ = e^{x_n} − 2 and using 1.2 as our starting value (x$_0$) ensure that you can get the first two values on your calculator, and then fill in the rest of the table:

X$_i$	Value
x$_0$	1.2
x$_1$	1.320116923
x$_2$	
x$_3$	

x_i	Value
x_4	
x_5	

TASK 20: Look at the sequence of values in your table. Briefly describe what is happening.

Ensure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to *deselect everything*.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black).

Right-click and select **x=g(x) Iteration** from the menu.

Enter the value of x_0 as 1.2.

Click the right arrow.

Pay particular attention to how the points on the graph correspond to the values in the table

Important Point:

Again, we can see that our sequence of numbers *diverges* from the solution

The name for the diagram you have constructed on your graph is a *diverging staircase.*

Predict

Looking at the graph, if we use the iterative formula $x_{n+1} = e^{x_n} - 2$ and a starting value of 1.1, *what do you think will happen? Convergence? Divergence? Cob–web? Staircase? Which solution?*

TASK 21: Using the iterative formula $x_{n+1} = e^{x_n} - 2$ and using 1.1 as our starting value (x_0) ensure that you can get the first two values on your calculator, and then fill in the rest of the table:

x_i	Value	x_i	Value
x_0	1.1	x_6	
x_1	1.004166024	x_7	
x_2		x_8	
x_3		x_9	
x_4		x_{10}	
x_5		x_{11}	

TASK 22: Look at the sequence of values in your table. Briefly describe what is happening.

Checking Your Answer

In the **Setup x=g(x) Iteration** box, change the value of x_0 to 1.1 and use the right arrow to observe the iteration process and check your answer.

TASK 23: Experiment with different values of x_0. Are there any values which lead to convergence towards the positive solution? Using what you have observed in the previous two activities, can you explain this?

TASK 24: Show that the equation: $e^x - x - 2 = 0$ can also be rearranged to give the iterative formula: $x_{n+1} = \ln(x_n + 2)$

Open up a **New 2D Graph Page**.

Edit the axes as follows:
> **x:** Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.25
> **y:** Minimum: −4   Maximum: 4   Numbers: 1   Pips: 0.5

Remove all of the green ticks underneath **Auto**.

Enter the equation: y = ln(x + 2)

Predict

What will this line look like over this section of the axes? Where will it cross the axes? Which direction will it slope?

Click **OK**.

Enter the equation y = x and click **OK**.

The line and the curve should now be on the screen with the two points of intersection representing the two solutions to our original equation.

Predict

Looking at the graph, if we use the iterative formula $x_{n+1} = \ln(x_n + 2)$ and a starting value of −1.5, what do you think will happen? Convergence? Divergence? Cob-web? Staircase? Which solution?

TASK 25: Using the iterative formula $x_{n+1} = \ln(x_n + 2)$ and using −1.5 as the starting value (x_0) ensure that you can get the first two values on your calculator, and then fill in the rest of the table:

x_i	Value	x_i	Value
x_0	−1.5	x_6	
x_1	−0.6931471806	x_7	
x_2		x_8	
x_3		x_9	
x_4		x_{10}	
x_5		x_{11}	

TASK 26: Look at the sequence of values in your table. Briefly describe what is happening.

Ensure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to deselect everything.

Hold down the **Shift** button to select more than one object:

Left-click on both the curve and the straight line (they should both turn black)

Right-click and select **x=g(x) Iteration** from the menu.

Enter the value of x_0 as −1.5.

Click the right arrow.

As before, use the **Zoom In** and **Drag** tools to have a closer look at the point of intersection of these two curves. Notice how the scale automatically adjusts the closer you get. Pay particular attention to how the points on the graph correspond to the values in the table.

TASK 27: Can you explain why we have convergence towards the positive solution here, whereas we did not with the previous iterative formula.

Hint: Have a look back at your previous graph and compare.

ACTIVITY 4

Open up a **New 2D Graph Page**.

TASK 28: Can you come up with a function that, along with the graph y = x,

will lead to a diverging cob-web when Autograph's function **x=g(x) Iteration** is employed?

Note: You may need to Edit the Axes.

TASK 29: Using the work you have done during this investigation, can you answer the following questions:

1. What determines whether we get convergence or divergence to a solution?
2. What determines whether the type of convergence or divergence is cob-web or staircase?
3. What determines the rate of convergence?

Hint: Have a look back at all previous graphs and compare.

ANSWERS

ACTIVITY 1

TASK 1: All of the standard methods to solve such equations fail. For example, squaring both sides:

$$1 - x^3 = \sqrt{x}$$
$$1 - 2x^3 + x^6 = x$$
$$x^6 - 2x^3 - x + 1 = 0$$

No obvious solution from here!

TASK 2:

When $x = 0$, we get: $1 - x^3 > \sqrt{x}$
When $x = 1$, we get: $1 - x^3 < \sqrt{x}$

As both functions are continuous in that range, that implies an intersection of the graphs of the two functions between 0 and 1, and hence a solution to the equation.
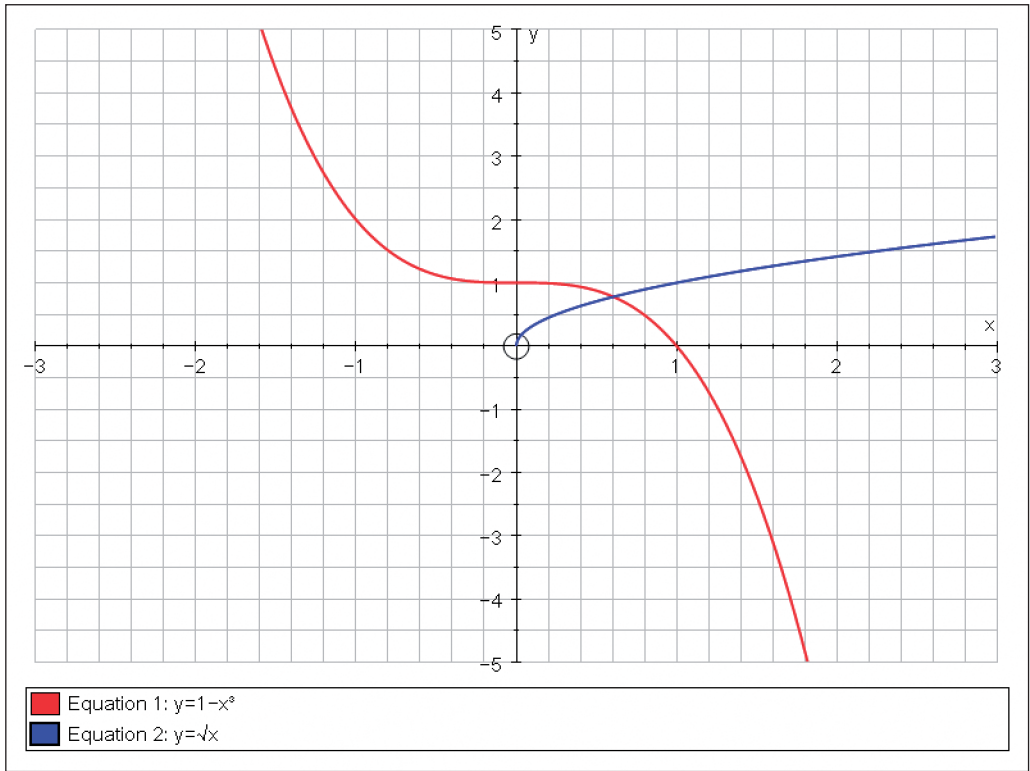
Equation 1: $y = 1 - x^3$
Equation 2: $y = \sqrt{x}$

TASK 3: To three decimal places, the solution is: $x = 0.605$

Equation 1: y=1−x³
Equation 2: y=√x

TASK 4:

$1 - x^3 = \sqrt{x}$
$1 - \sqrt{x} = x^3$
$x = (1 - \sqrt{x})^{1/3}$

Equation 1: y=(1−√x)^(⅓)
Equation 2: y=x
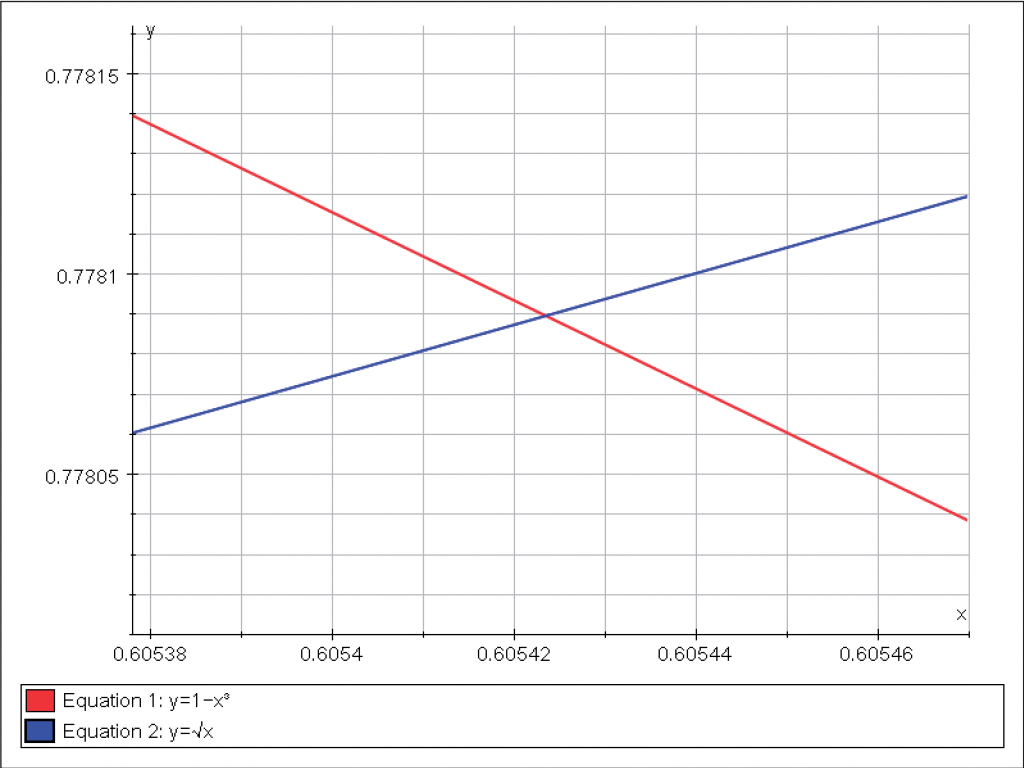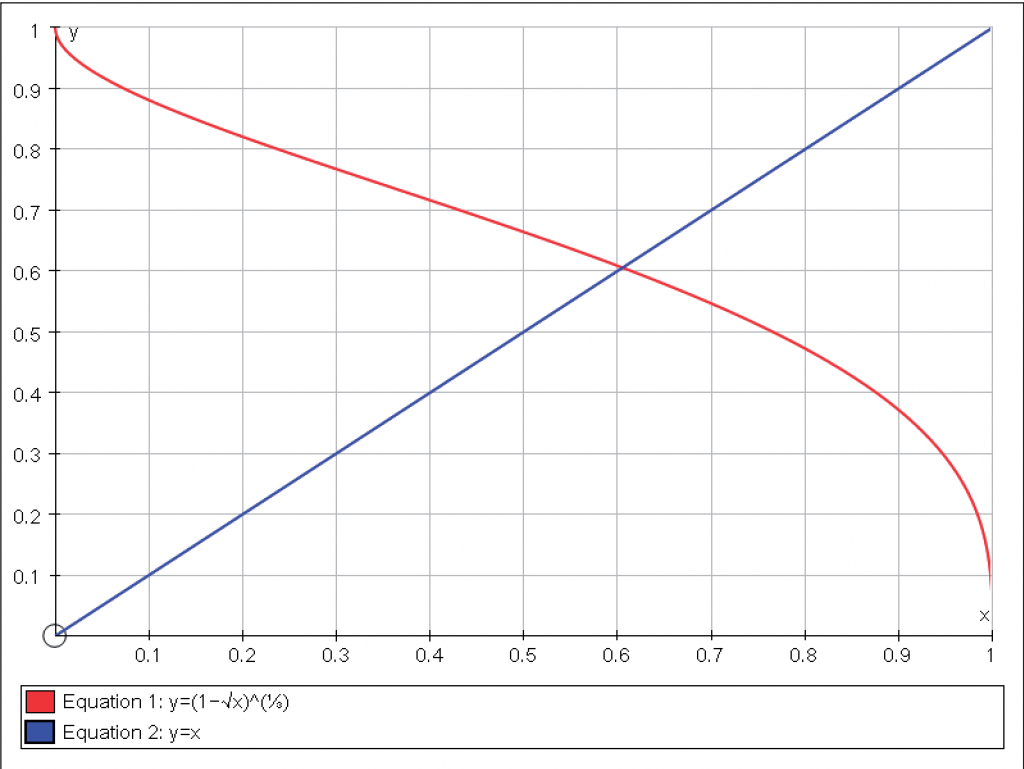
TASK 5: MATHS

x_i	Value	x_i	Value
x_0	0.5	x_6	0.6013167076
x_1	0.6641045243	x_7	0.6078179364
x_2	0.5698777976	x_8	0.6040222414
x_3	0.6258154615	x_9	0.6062416246
x_4	0.5933667611	x_{10}	0.6049450609
x_5	0.6124233689	x_{11}	0.6057028988

TASK 6: To four decimal places, the solution is: x = 0.6054

Setup x=g(x) Iteration

Parameters
x₀: 0.5

Values

x	|Δx|	n=2
0.6641045243	0.1641045243	
0.5698777976	0.09422672675	

Equation 1: y=(1−√x)^(⅓)
Equation 2: y=x

Equation 1: y=(1−√x)^(⅕)
Equation 2: y=x
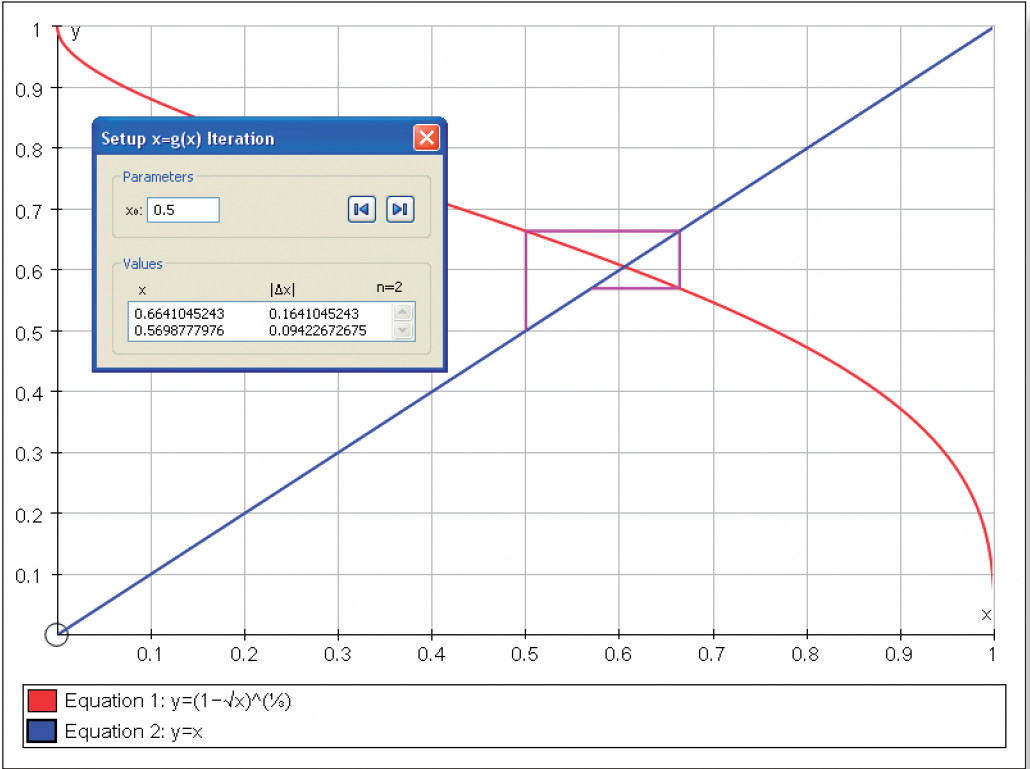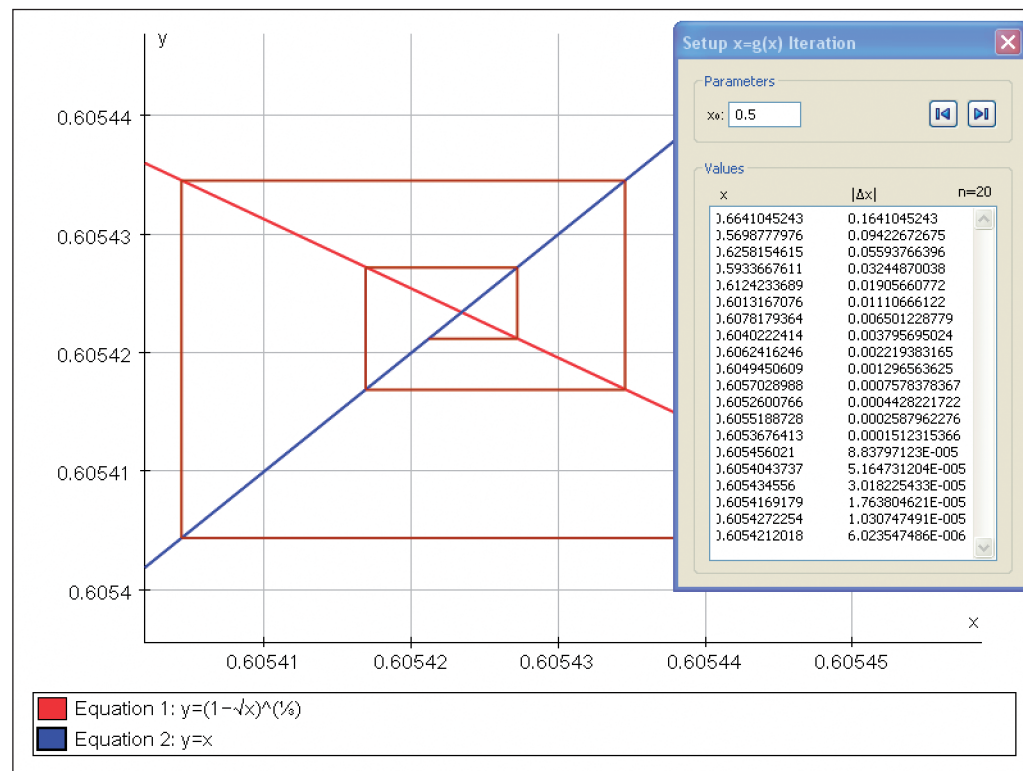
TASK 7: The only values of x_0 which do not lead to convergence are x = 0 and x = 1. Values further away from the solution seem to take longer to converge.

Equation 1: y=(1−√x)^(⅕)
Equation 2: y=x
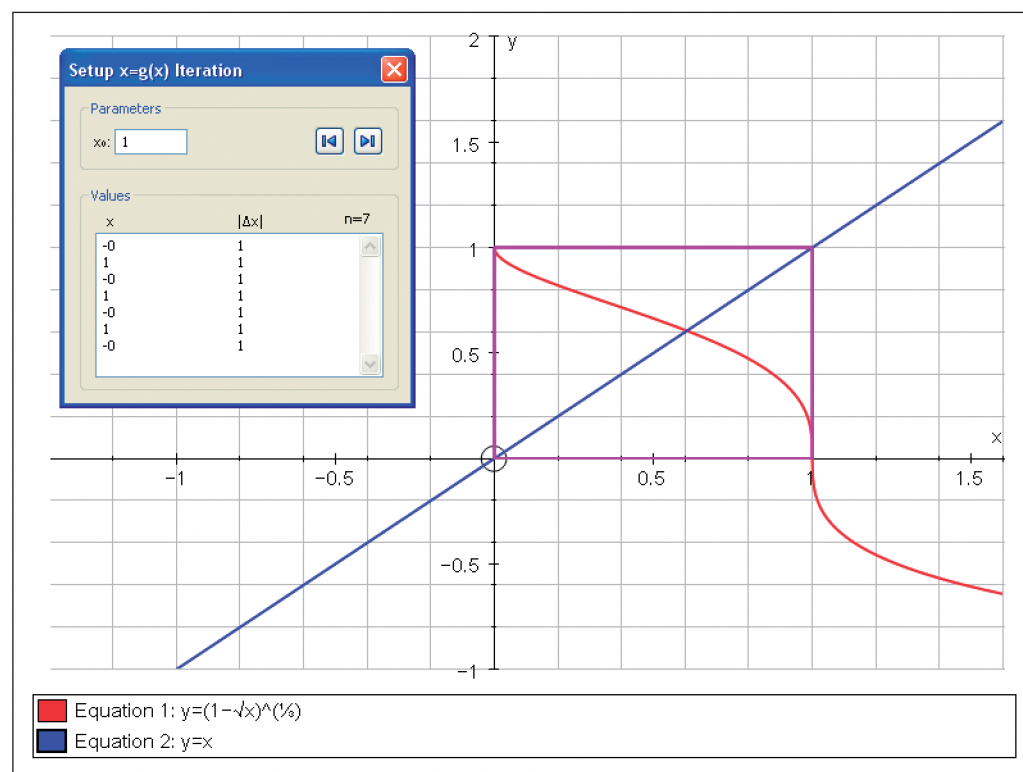
ACTIVITY 2

TASK 8: y = 2^x is always positive and y = x^3 is negative when x < 0, so the two curves cannot cross when x < 0. The gradients of both functions are always

increasing, and hence they can only cross each other at most once, which has to occur when x ≥ 0.

TASK 9: Again, all of the standard methods to solve such equations fail. For example, taking logs:

$2^x = x^3$
$\ln(2^x) = \ln(x^3)$
$x\ln(2) = 3\ln(x)$

No obvious solution from here!

TASK 10:

When x = 1, we get: $2^x > x^3$
When x = 2, we get: $2^x < x^3$

As both functions are continuous in that range, that implies an intersection of the graphs of the two functions between 1 and 2, and hence a solution to the equation.

Equation 1: y=(2^x)
Equation 2: y=x^3
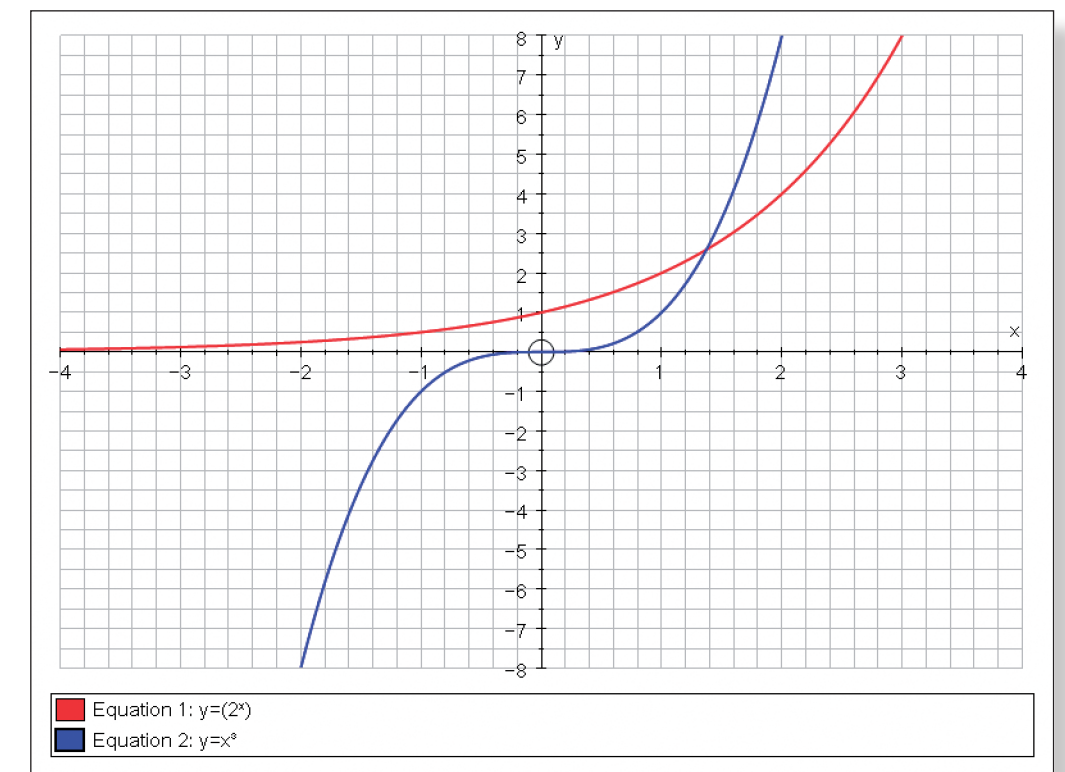
TASK 11: To three decimal places, the solution is: x = 1.373

Equation 1: y=(2ˣ) → $y = (2^x)$
Equation 2: y=x³ → $y = x^3$
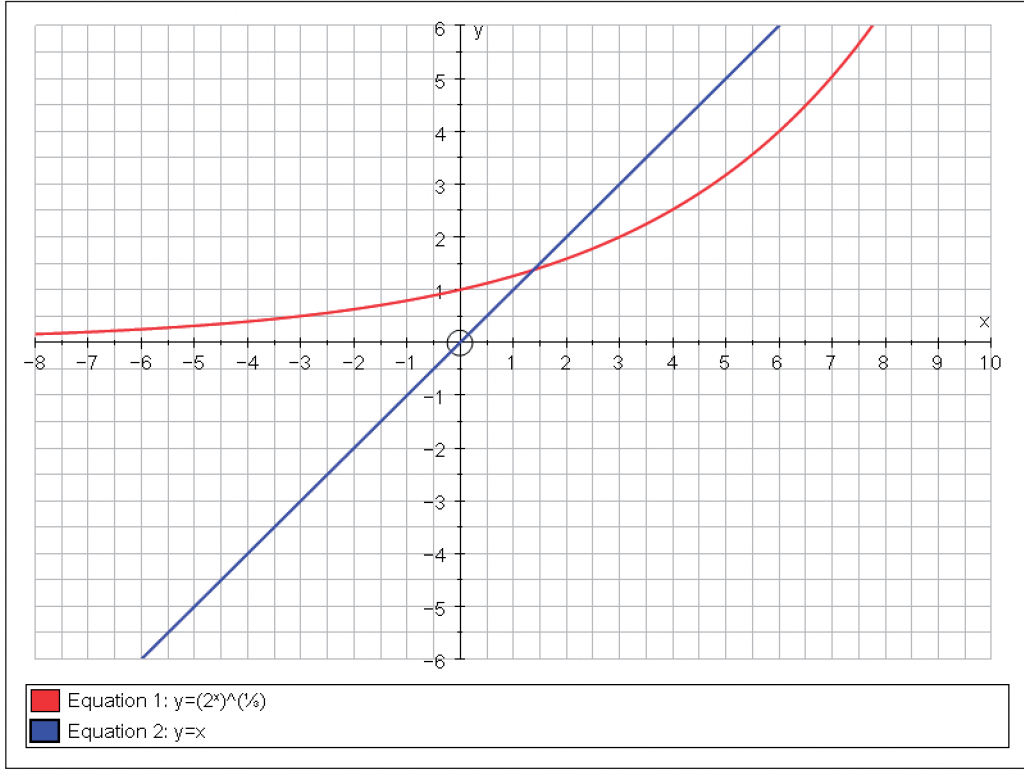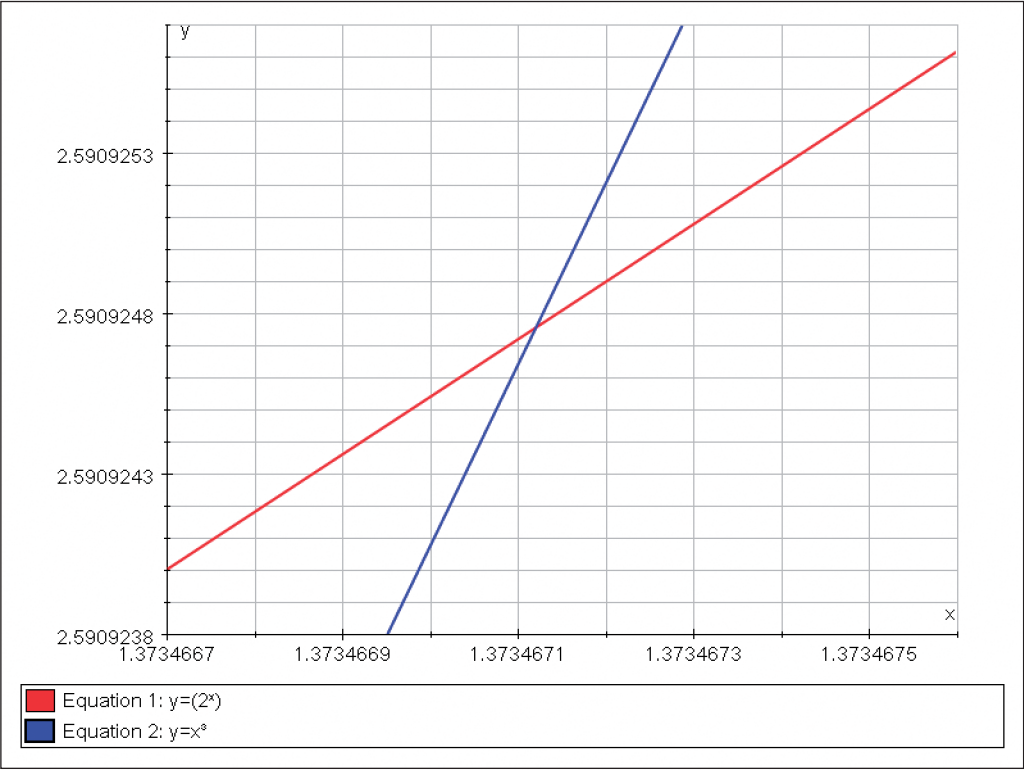
TASK 12:

$2^x = x^3$

$x = (2^x)^{\frac{1}{3}}$

$x_{n+1} = (2^{x_n})^{\frac{1}{3}}$

TASK 13: $x_{n+1} = (2^{x_n})^{\frac{1}{3}}$

x_i	Value	x_i	Value
x_0	1.5	x_6	1.373599156
x_1	1.414213562	x_7	1.373509021
x_2	1.386458584	x_8	1.373480417
x_3	1.377596002	x_9	1.373471339
x_4	1.374777997	x_{10}	1.373468459
x_5	1.373883174	x_{11}	1.373467545

TASK 14:
The values in the table are decreasing towards the solution, not oscillating around it.

TASK 15:
The solution to $x = (2^x)^{\frac{1}{3}}$ correct to four decimal places is: 1.3735

Equation 1: $y=(2^x)^{\wedge}(\frac{1}{3})$
Equation 2: $y=x$

Equation 1: $y=(2^x)^{\wedge}(\frac{1}{3})$
Equation 2: $y=x$
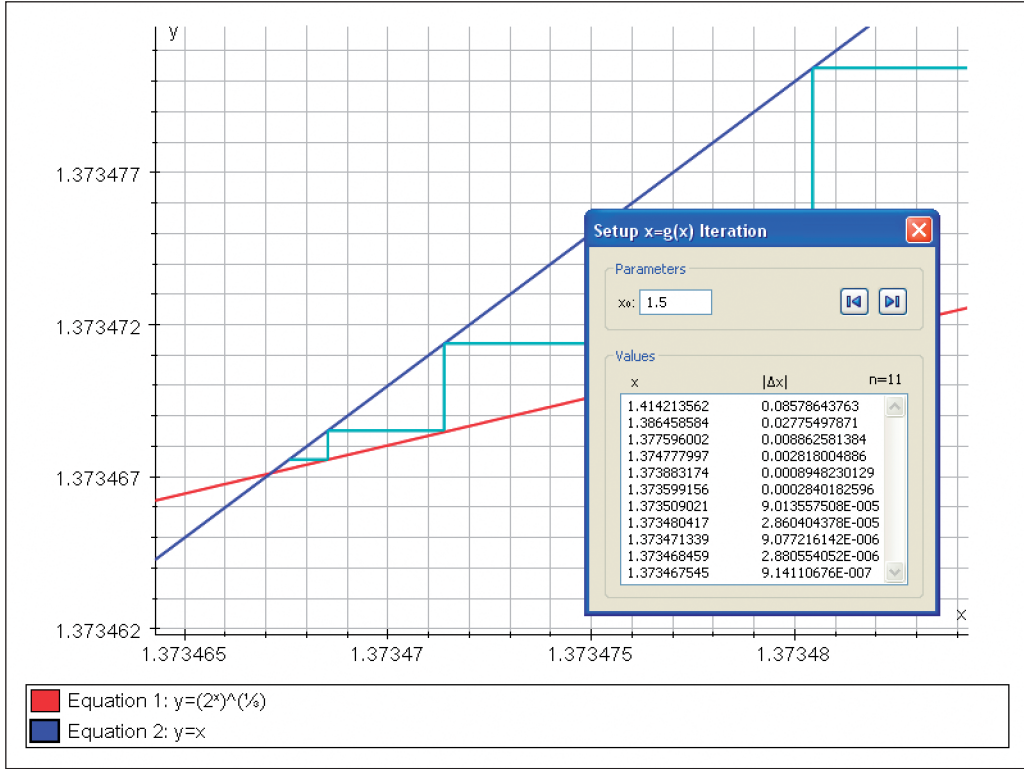
TASK 16:
We converge to the solution from all starting values in the given range. The starting values to the left hand side of the solution tend to converge more quickly. This is because the gradient of the curve is shallower at these points.

ACTIVITY 3

TASK 17: $e^x - x - 2 = 0$

When x = 1.1 $e^x - x - 2 = -0.9583…$
When x = 1.2 $e^x - x - 2 = 0.12011…$
Change of sign in a continuous function implies a solution between x = 1.1 and x = 1.2.

When x = −1.9 $e^x - x - 2 = 0.04956…$
When x = −1.8 $e^x - x - 2 = -0.03470…$
Change of sign in a continuous function implies a solution between x = −1.9 and x = −1.8.

TASK 18:

$e^x - x - 2 = 0$
$x = e^x - 2$
$x_{n+1} = e^{x_n} - 2$

Equation 1: y=e^x−2
Equation 2: y=x
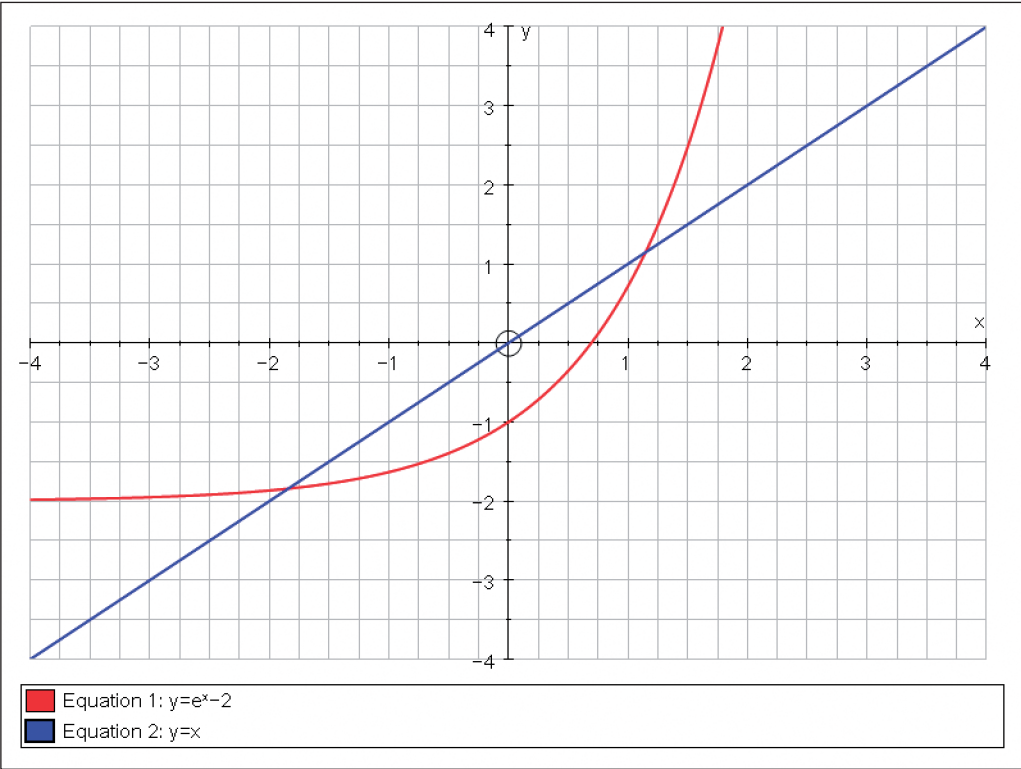
TASK 19:

$x_{n+1} = e^{x_n} - 2$

x_i	Value
x_0	1.2
x_1	1.320116923
x_2	1.743859094
x_3	3.719372485

x_i	Value
x_4	39.2385082
x_5	1.09917×10^{17}

TASK 20: The numbers appear to be diverging from our solution.

Equation 1: y=e^x−2
Equation 2: y=x
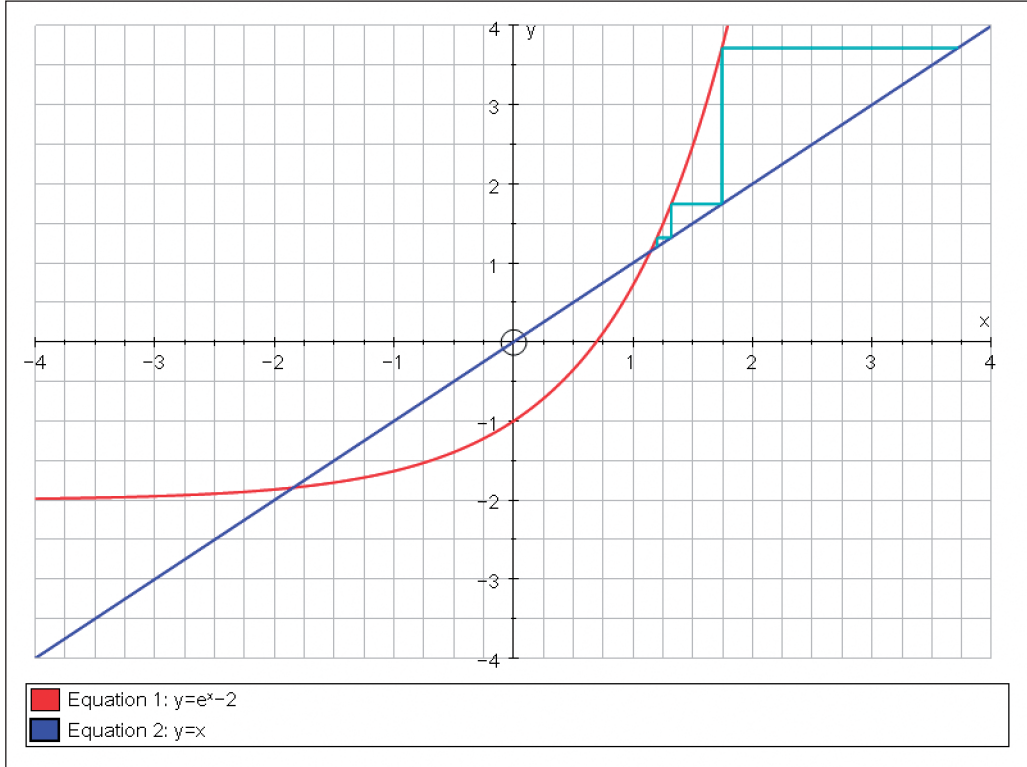
TASK 21:

x_i	Value	x_i	Value
x_0	1.1	x_6	−1.798631289
x_1	1.004166024	x_7	−1.83447471
x_2	0.7296298774	x_8	−1.840302633
x_3	0.0743127154	x_9	−1.84123063
x_4	−0.9228564077	x_{10}	−1.841377899
x5	−1.602617668	x_{11}	−1.841401258

TASK 22: The numbers appear to be diverging from one solution but converging towards the other.

Setup x=g(x) Iteration

Parameters

xe: 1.1

Values

n=10

x	|Δx|
1.004166024	0.09583397605
0.7296298774	0.2745361466
0.07431271547	0.6553171619
-0.9228564077	0.9971691232
-1.602617668	0.6797612599
-1.798631289	0.1960136213
-1.83447471	0.03584342159
-1.840302633	0.005827922457
-1.84123063	0.0009279970979
-1.841377899	0.0001472691714

Equation 1: y=eˣ−2
Equation 2: y=x

Equation 1: y=ln(x + 2)
Equation 2: y=x

TASK 23: There appears to be no starting value which will lead to convergence towards the positive solution. This again appears to be due to the gradient around the positive solution. It is greater than 1 (as can be seen by comparing it to the line y = x), which appears to lead to divergence, whereas the gradient around the negative solution is less than 1.

TASK 24:

$e^x - x - 2 = 0$
$e^x = x + 2$
$x = \ln(x + 2)$
$x_{n+1} = \ln(x_n + 2)$
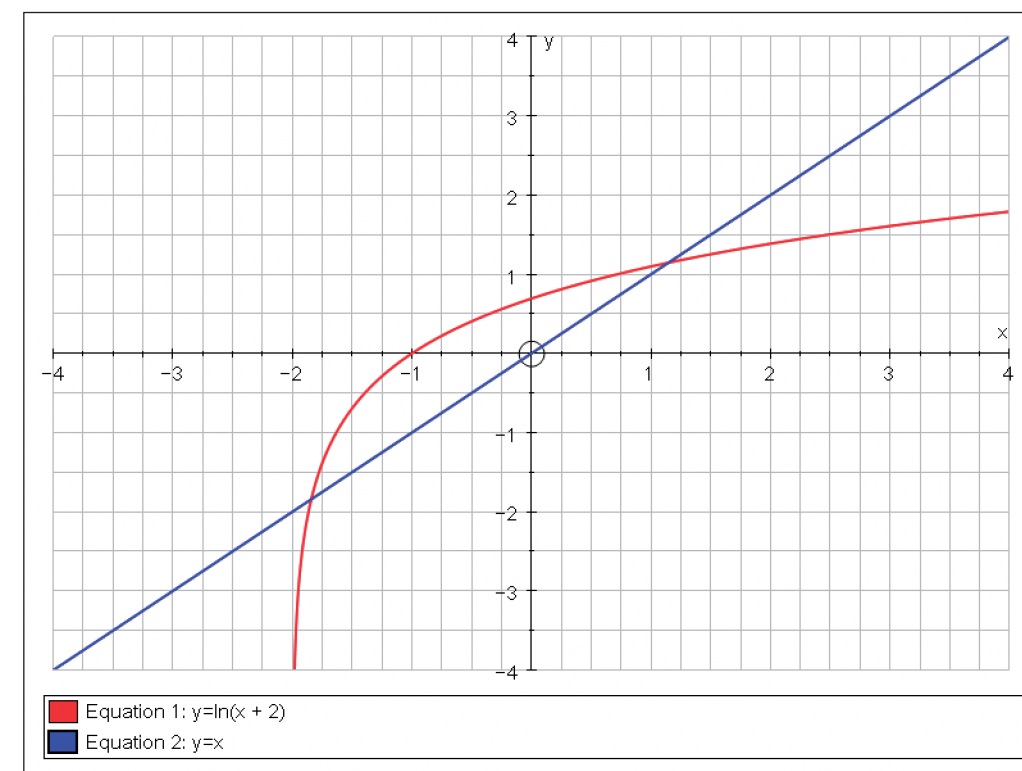
TASK 25: $x_{n+1} = \ln(x_n + 2)$

x_i	Value	x_i	Value
x_0	−1.5	x_6	1.134827011
x_1	−0.6931471806	x_7	1.142573993
x_2	0.2676218188	x_8	1.145042207
x_3	0.8187316254	x_9	1.14582731
x_4	1.036287006	x_{10}	1.146076911
x_5	1.110635389	x_{11}	1.146156252

TASK 26: The numbers appear to be diverging from the negative solution but converging towards the positive one.

Equation 1: y=ln(x + 2)
Equation 2: y=x

Setup x=g(x) Iteration

Parameters
x₀: -1.5

Values

x	|Δx|	n=10
-0.6931471806	0.8068528194	
0.2676218188	0.9607689994	
0.8187316254	0.5511098066	
1.036287006	0.2175553801	
1.110635389	0.07434838362	
1.134827011	0.02419162135	
1.142573993	0.007746981982	
1.145042207	0.002468214445	
1.14582731	0.0007851034456	
1.146076911	0.0002496009328	

Equation 1: y=2−x³
Equation 2: y=x

Setup x=g(x) Iteration

Parameters
x₀: 1.01

Values

x	|Δx|	n=5
0.969699	0.040301	
1.088176369	0.1184773691	
0.7114600989	0.3767162701	
1.639876348	0.9284162494	
-2.409946355	4.049822703	

TASK 27: Again, this appears to be due to the gradient of the curve around the solution. The gradient is less than −1 around the negative solution, but more than 1 at the positive solution, and so the sequence of numbers converges towards the positive solution.

ACTIVITY 4

TASK 28: Example of function leading to cob-web divergence: $y = 2 − x^3$

(Need negative gradient, smaller than −1 around solution.)
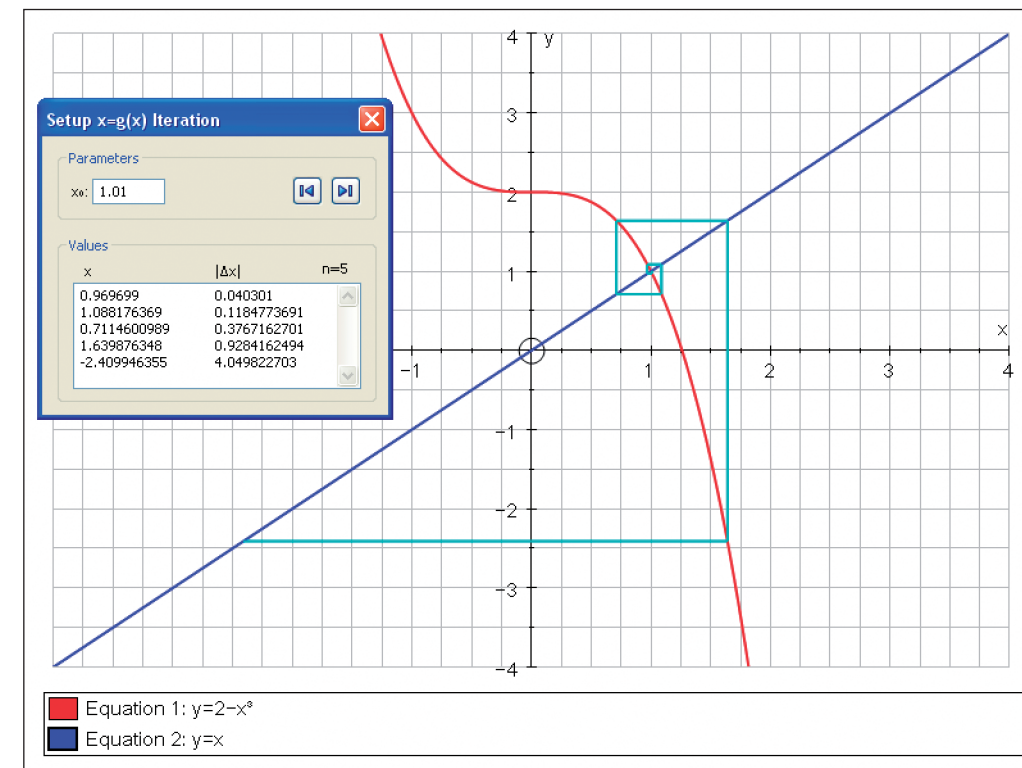
TASK 29:

1. What determines whether we get convergence or divergence to a solution?

   This is due to the gradient of the curve near the solution. If the gradient is between −1 and 1, then we have convergence. However, if it lies outside of this range, then we get divergence.

2. What determines whether the type of convergence or divergence is cob-web or staircase?

   If the gradient of the curve is positive, then we have the staircase, and if it is negative, the iterative process will oscillate, and we have cob-web.

3. What determines the rate of convergence?

   The rate of convergence is determined by how close the gradient of the curve is to 1 or −1. If it is close, then the rate of convergence is relatively slow, however if the gradient is closer to 0, then the rate of convergence is relatively quick.

TEACHER NOTES

LEARNING OBJECTIVES

CORE CONTENT:

- To be able to visualise how the methods of linear programming lead to optimal solutions by constructing constraints and feasible regions, and manipulating constants to dynamically **Shift** the objective function.

- To be able to understand the relationship between the graphical and algebraic solution to a linear programming problem.

- To consolidate how to form and solve linear programming problems, including problems involving blending, infinite optimal solutions, and three variables.

EXTENSION MATERIAL:

- To be able to understand and visualise how changing the initial conditions of the problem can affect the optimal solution.

- To understand the relationship between the simplex algorithm and linear programming solutions.

REQUIRED PRIOR KNOWLEDGE

- To be able to draw linear functions.

- To be able to construct linear inequalities and use them to find a feasible region.

- To be able to find the point of intersection of two lines using simultaneous equations.

- To have some prior experience of linear programming problems, in the sense that the students understand the concepts of constraints, objective functions, feasible regions, and optimal solutions.

DESCRIPTION OF THE ACTIVITY

- This activity is intended to consolidate, deepen and extend students' knowledge of solving linear programming problems. It is best tackled after a lesson's introduction to the topic of linear programming, covering how to formulate objective functions and constraints based on information given.

- In Activity 1, students are briefly reminded about how to draw linear

inequalities and use them to construct a feasible region.

- In Activity 2, students are presented with a standard linear programming problem concerning a company wishing to minimise the cost of production. After the solution is reached, the problem is extended by altering the initial conditions.

- In Activity 3, students are presented with a more complex three variable linear programming problem, which requires them to substitute for one variable, and also encompasses aspects of blending. Once again, the activity is extended, and the relationship between the graphical and algebraic solutions is emphasised.

- In Activity 4, students are asked to reflect on the work they have done during this investigation to see if they can spot anything about the graphical features of all the optimal solutions. This could be used to provide an introduction into the Simplex Algorithm.

Pre–Activity Setup

None required.

Suggested Introduction

"Linear programming is a topic which has many real life applications. It was used for scheduling, transportation and resource allocation applications during World War Two, in the 1980s, American Airlines cut their costs by 10% using linear programming methods, and it is the basis for between 10% and 25% of all scientific computation today. During this investigation, you will use Autograph to help you get a better feel for just how linear programming works, and why it leads to an optimal solution."

Suggested Prompts

- The Autograph functions employed during this investigation are relatively straight forward, and the majority of them are introduced in a simple context in Activity 1 for that very reason. The only exception is the **Constant Controller**, which is introduced in Activity 2, but is also relatively straight forward once the students have experienced it.

- Often, the final, simplified expression for the objective functions and constraints are given, and the students must justify them, or show how to arrive at them. This is to ensure that the important concepts covered later in the activity are not compromised by an algebraic slip–up earlier on.

- When it comes to finding the optimal solution, the students can use Autograph to confirm the answer derived by simultaneous equations. Not only is this a nice checking device, it also highlights the connections between graphical and algebraic solutions to linear programming problems.

Suggested Plenary

- Can anybody summarise the key points they have learnt from this investigation?

- What are some of the ways you used to help you determine the objective functions and constraints?

- Can anybody suggest a non–graphical way to solve linear programming problems?

- Has anybody got any top tips, or way they have found successful, to help us avoid tripping up when tackling linear programming questions?

Ideas for Further Work

- Further questions and examples to consolidate understanding of linear programming

- Activity 4 hints at a non–graphical solution to linear programming problems. This could be followed up by work on the Simplex Algorithm.

STUDENT WORKSHEET

NAME:...

ACTIVITY 1: FINDING REGIONS

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

On the top toolbar, go to **View** > **Preferences**.

On the **Plotting** screen, ensure the line thickness is set to 1½ pt, and place a tick in the box next to **All Thick Lines**.

Click the **General** tab, and ensure that **Inequalities** are set to **Shade Reject Region**.

Edit the axes as follows:
- x: Minimum: −8 Maximum: 8 Numbers: 1 Pips: 0.5
- y: Minimum: −10 Maximum: 10 Numbers: 1 Pips: 1

Remove all of the green ticks underneath **Auto**.
Note: You must ensure all the ticks under **Auto** are removed or Autograph will attempt to re-scale your axes for you.

Enter the equation: y = ½ x − 1

Predict | *What will this line look like? Where will it cross the axes?*

Click **OK**.

TASK 1: If you were to draw y ≥ ½ x − 1, and shade the region not satisfied by the inequality, which side of the line would you shade?

Press **Undo** so the line disappears and you are left with the blank set of axes again.

Enter the equation: y ≥ ½ x − 1, and click **OK**.

Check that this matches your answer to Task 1 before moving on.

Enter the equation: 2x + y = 8

Predict | *What will this line look like? Does it have a positive or a negative gradient? Where will it cross the axes?*

Click **OK**.

TASK 2: If you were to draw 2x + y < 8, and shade the region not satisfied by the inequality, which side of the line would you shade?

Press **Undo** so the previous line disappears.

Enter the equation: 2x + y < 8, and click **OK**.

Check that this matches your answer to Task 2 before moving on.

Note: Autograph represents inequalities in the form **>** and **<** with dashed line, and **≥** and **≤** with solid lines.

TASK 3: Use Autograph to find the region satisfied by the following inequalities (the feasible region):

y ≥ ½x − 1   2x + y < 8   5x + 2y > −10   y ≤ 2x + 5

Give the co-ordinates of a point that lies within the feasible region.

TASK 4: What is the largest integer of x that falls within the feasible region? Give the co-ordinates.

Hint: Be careful of the dashed lines!

![Σ✏ icon] **TASK 5:** What is the largest integer of y that falls within the feasible region? Give the co-ordinates.

![Σ✏ icon] **TASK 6:** Using only integer co-ordinates, what is the largest possible value of x + y that falls within the feasible region? Give the co-ordinates.

ACTIVITY 2: THE CHOCOLATE PROBLEM

The Problem

A struggling chocolate company owns two coco production plants, each of which is able to produce various quantities of their three best selling chocolate products: Swedish Delight, Venus Bar and Dig Dog, for the daily costs shown in the table below.

Coco Plant	Cost of Daily Operation (£'000)	Production (Crates of Chocolate Bars per Day)		
		Swedish Delight	Venus Bar	Dig Dog
X	18	6	3	4
Y	16	1	1	6

The future of the company was looking rather bleak, but fortunately, a local chain of supermarkets has recently placed a weekly order for 12 crates of Swedish Delights, 8 crates of Venus Bars, and 24 crates Dig Dogs.

Question: How many days a week should the chocolate company open each coco production plant to fulfil the order, and what will be the weekly cost to the company?

The Variables

Let: x = the number of days Plant X is open.
      y = the number of days Plant Y is open.

The Objective Function

![Σ✏ icon] **TASK 7:** Explain briefly why the Objective Function of the chocolate company can be expressed as follows: "Minimise: 18x + 16y"

The Constraints

![Σ✏ icon] **TASK 8:** In terms of x and y, write three constraints facing the chocolate company based on the numbers of crates of Swedish Delights, Venus Bars, and Dig Dogs produced.

Hint: Think about the supermarket's order.

![Σ✏ icon] **TASK 9:** Assuming each coco production plant can only open on weekdays (Monday to Friday), write an additional two constraints facing the company.

![Σ✏ icon] **TASK 10:** What is the logic behind imposing the additional constraint: x,y ≥ 0.

Graphing the Problem

Open up a **New 2D Graph Page**.

Edit the axes as follows:
- x: Minimum: 0   Maximum: 8   Numbers: 1   Pips: 0.25
- y: Minimum: 0   Maximum: 8   Numbers: 1   Pips: 0.5

Remove all of the green ticks underneath **Auto**.

Predict — *What will each of the constraints in Task 8 look like when plotted? Will they have a positive or a negative gradient? Where will they cross the axes? Which sides of the lines will be shaded?*

Enter the three constraints you gave as answers to Task 8, being careful of the inequality signs.

Predict — *What will each of the constraints in Task 9 look like when plotted? How will they limit the size of the Feasible Region?*

Enter the two constraints you gave as answers to Task 9, again being careful of the inequality signs.

You should be left with an un-shaded region in the middle.

This is the Feasible Region.

TASK 11: Pick a point which lies inside the Feasible Region, and write down its co-ordinates below. Test that the co-ordinates of the point satisfy each of the constraints you have listed.

Enter the equation: 18x + 16y = k

This is our Objective Function.

Still on the **Enter Equation** screen, click on **Edit Constants**

Change the value of k to 30.

Predict — *What will this line look like? Which way will it slope? Where will it cross the axes?*

Click **OK** twice.

Click on the **Constant Controller** on the top toolbar.
The **up-down** buttons alter the value of the constant, k.
The **left-right** buttons alter the value of the step.

Predict — *If we increase the value of k, how will this affect the line 18x + 16y = k?*

Adjust the value of k, and the size of the step, so the line moves towards the Feasible Region.

TASK 12: To two decimal places, what are the co-ordinates of the first part of the Feasible Region to be touched by the objective function?

Reduce the size of the step, and use the hand and zoom functions to zoom in on the area. Notice how the scale on both axes automatically adjusts the closer you get.

Press the **Undo** button several times until you return to the original view of the graph.

TASK 13: Explain briefly why your answer to Task 12 gives you the optimal solution to the chocolate company's problem?

TASK 14: Use simultaneous equations (or another method) to find the exact co-ordinates of the optimal solution. Leave your answer as a fraction.

Hint: Think about the equations of the intersecting lines.

Enter your answer as a co-ordinate to check that your answer to Task 14 is correct.

TASK 15: Use your answer to Task 14 to answer the original problem: how many days each week should the chocolate company open each of its plants, and what are its weekly costs (in £'000)?

Extending the Problem

TASK 16: If each plant could only open for a whole number of days (i.e. no fractions of days), what would the optimal solution be in terms of the numbers of days open and the weekly costs?

Hint: Use Autograph and the **Constant Controller** to help you.

TASK 17: Have a look at the original question, and in particular the table of figures. Imagine that Coco Production Plant Y could suddenly produce twice as many crates of each type of chocolate. How do you think this will affect the optimal solution in terms of the number of days each plant is open, and the weekly costs to the company?

Hint: Think about the effect on the gradient of the constraint lines, and how this would affect the location of the optimal solution.

TASK 18: Use Autograph to alter the constraints to match the new production capabilities of Plant Y, and find the new optimal solution to two decimal places. Use this to state the number of days each plant must stay open, and the new weekly costs. Does this match your prediction in Task 17?

Note: To edit the equation of a line/inequality, just double **left-click** on that line whilst in select mode.

TASK 19: Have another look at the original question. Can you think of any unrealistic assumptions which have been made to simplify this question?

Hint: Think about other possible costs, and the amount of crates actually produced.

Activity 3: Pocket Money

The Problem

Eager to give them a taste of the real world, a mathematics teacher decides that instead of simply giving his three children pocket money each week, he will lend them a lump sum at the start of the year, and charge them an annual rate of interest on that lump sum. After much deliberation, he decides that the interest rates charged to each of his three children will be as follows:

Child	Interest Rate (%)
Fermat	14
Lagrange	20
Euler	25

The mathematics teacher has a total of £1000 to lend to his three children, and he must lend it all! However, the mathematics teacher's wife imposes three constraints on her husband's lending:

1. Fermat and Lagrange have a rather violent history of sibling rivalry, and Lagrange always seems to come out on top, and so to restore the balance she states that Fermat must receive at least 50% of the sum of money dished out between himself and Lagrange.

2. Euler is a bit of a wild child, and she is worried that if he receives too much money all at once he will once again go out and blow it all on mathematical textbooks and books of SuDoku puzzles, and so states that Euler cannot receive more than 30% of the total sum given to all children.

3. To ensure her husband raises enough money to buy her something other than a new scientific calculator for her birthday, she stipulates that the average interest rate on all loans must be greater than 18%.

Question: How can the mathematics teacher seek to maximise the income extracted from his children, whilst also staying out of trouble by sticking to his wife's conditions?

The Variables

Let x be the amount of money loaned to Fermat.
Let y be the amount of money loaned to Lagrange.
Let z be the amount of money loaned to Euler.

The Objective Function

TASK 20: Write the objective function of the maths teacher in terms of x, y and z.

Hint: Remember, he is ruthless, and wants as much money from his children as possible.

TASK 21: Looking at the objective function, what problem immediately becomes apparent if we want to solve this problem using a graphical method?

The Constraints

TASK 22: The maths teacher is limited to a lending budget of £1000. Write this constraint in terms of x, y and z.

Hint: Remember, he must lend all his budget, so this constraint will be an equation (=), and not an inequality (≥ or ≤).

TASK 23: Re-arrange the equation from Task 22 to make variable z the subject.

Crucial Point: Because we can always re-write variable z in terms of the other two variables, we have changed a three variable problem into a two variable problem, and so we can solve it graphically!

TASK 24: Use your answer to Task 23 to show that the objective function can be expressed as follows: "Maximise: 250 − 0.11x − 0.05y"

TASK 25: Express Constraint 1 in terms of x and y.

Hint: Try imagining some numerical values of x and y if it helps!

TASK 26: Express Constraint 2 in terms of x, y and z.

TASK 27: Use your answer to Task 23 to show that Constraint 2 can be expressed as follows:

x + y ≥ 700

TASK 28: Express Constraint 3 in terms of x, y and z (be careful, this is quite tricky!)

TASK 29: Use your answer to Task 23 to show that Constraint 3 can be expressed as follows:

0.11x + 0.05y ≤ 70

Graphing the Problem

Open up a **New 2D Graph Page**.

Edit the axes as follows:
- **x:** Minimum: 0   Maximum: 1000   Numbers: 100 Pips: 50
- **y:** Minimum: 0   Maximum: 1000   Numbers: 100 Pips: 100

Remove all of the green ticks underneath **Auto**.

Predict

> *What will each of the Constraints 1, 2 and 3 look like when plotted? Will they have a positive or a negative gradient? Where will they cross the axes? Which sides of the lines will be shaded?*

Enter Constraints 1, 2 and 3 being careful of the inequality signs.

You should be left with an un-shaded region in the middle.

This is the Feasible Region.

Enter the equation: 250 − 0.11x − 0.05y = c

This is our Objective Function.

Still on the **Enter Equation** screen, click on **Edit Constants**.

Change the value of **c** to 220.

Predict

> *What will this line look like? Which way will it slope? Where will it cross the axes?*

Click **OK** twice.

Click on the **Constant Controller** on the top toolbar.

Predict

> *If we increase the value of c, how will this affect the line 250 − 0.11x − 0.05y = c?*

Adjust the value of **c**, and the size of the step, so the line moves towards the Feasible Region.

TASK 30: Moving from the origin outwards, is the optimal solution to this

problem at the first point of the feasible region touched by the objective function, or the last point? Explain your answer.

Hint: If stuck, try a few solutions and see which one gives the maths teacher most money.

TASK 31: Looking at the equations of two of the lines on your graph, how do you know that there is no unique optimal solution to this problem?

TASK 32: Give the co-ordinates of one of the optimal solutions, and use this to work out how much pocket money each of the three children gets, and the total interest the maths teacher receives off his children at the end of the year.

TASK 33: Using Autograph (or another method), find the optimal solution which results in Euler receiving the lowest amount of money. Exactly how much money do all three children receive with this solution?

Hint: Think about how the values of x and y affect what Euler receives, and maybe draw another line (with a constant) on your graph.

ACTIVITY 4: ANOTHER WAY?

TASK 34: Look back at the graphical solutions to Activities 2 and 3, and in particular, the location on the feasible region where these optimal solutions lie. What do you notice?

TASK 35: Using your answer to Task 34, can you suggest a different way of finding the optimal solution to problem like these without drawing graphs?

SUMMARY

Use the space below to summarise what you have learnt during this investigation:
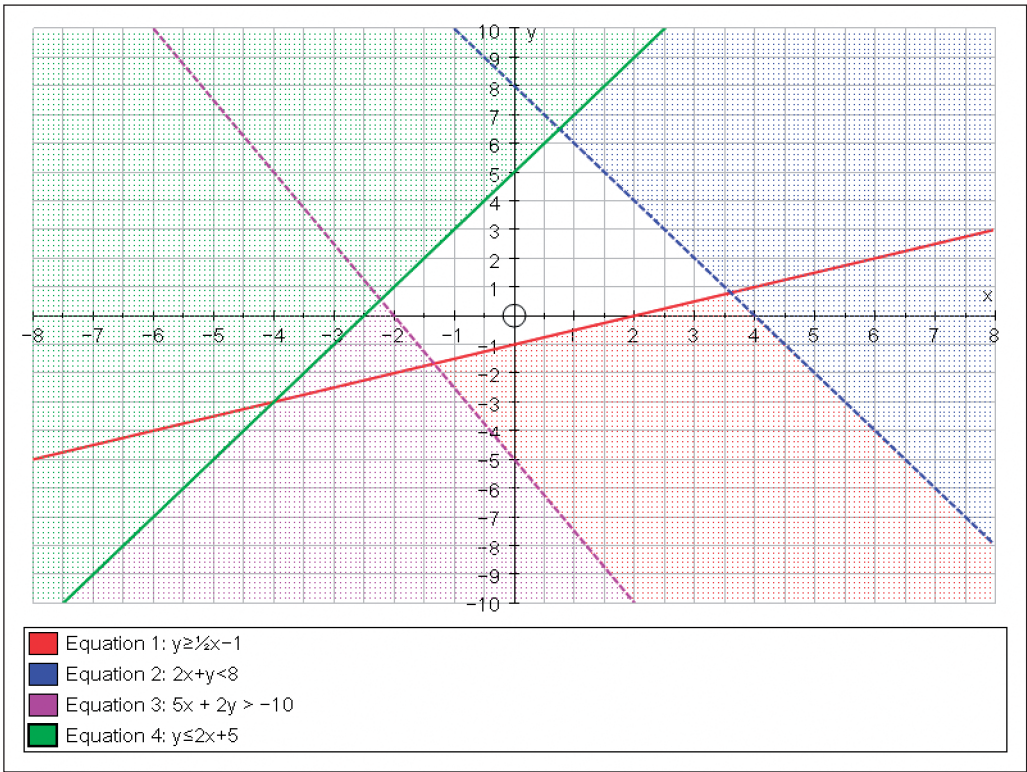
ANSWERS

ACTIVITY 1: FINDING REGIONS

TASK 1: $y \geq \frac{1}{2}x - 1$

You would shade the side below the line to represent the region not satisfied.

TASK 2: $2x + y < 8$

You would shade the side to the right of the line to represent the region not satisfied.

TASK 3: Co-ordinates of a point within the feasible region: (1, 2)

Equation 1: y≥½x−1
Equation 2: 2x+y<8
Equation 3: 5x + 2y > −10
Equation 4: y≤2x+5

TASK 4: Largest integer value of x = 3 at (3, 1).

TASK 5: Largest integer value of y = 6 at (0.5, 6).

TASK 6: Largest integer value of x + y = 6 at (1, 5).

ACTIVITY 2: THE CHOCOLATE PROBLEM

TASK 7: 18x gives an expression for the cost of running Plant X (£18,000 a day for x days), and likewise 16y gives an expression for the cost of running Plant Y (£16,000 a day for y days). The objective of the firm is simply to keep costs to a

minimum, and hence the objective function is to Minimise: 18x + 16y.

TASK 8:

Swedish Delight: $6x + y \geq 12$
Venus Bar: $3x + y \geq 8$
Dig Dog: $4x + 6y \geq 24$

TASK 9: $x \leq 5$ and $y \leq 5$.

TASK 10: x, y ≥ 0 as is makes no sense to talk about a negative number of days.

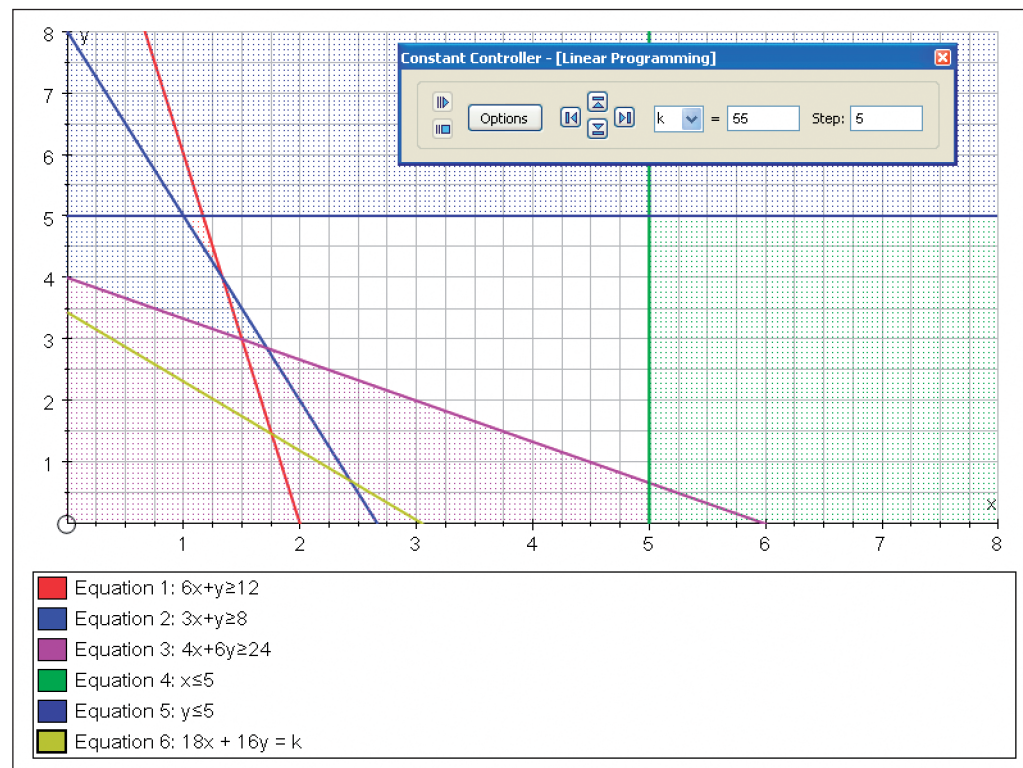**TASK 11:** Co-ordinates of a point within the feasible region: (3, 4)

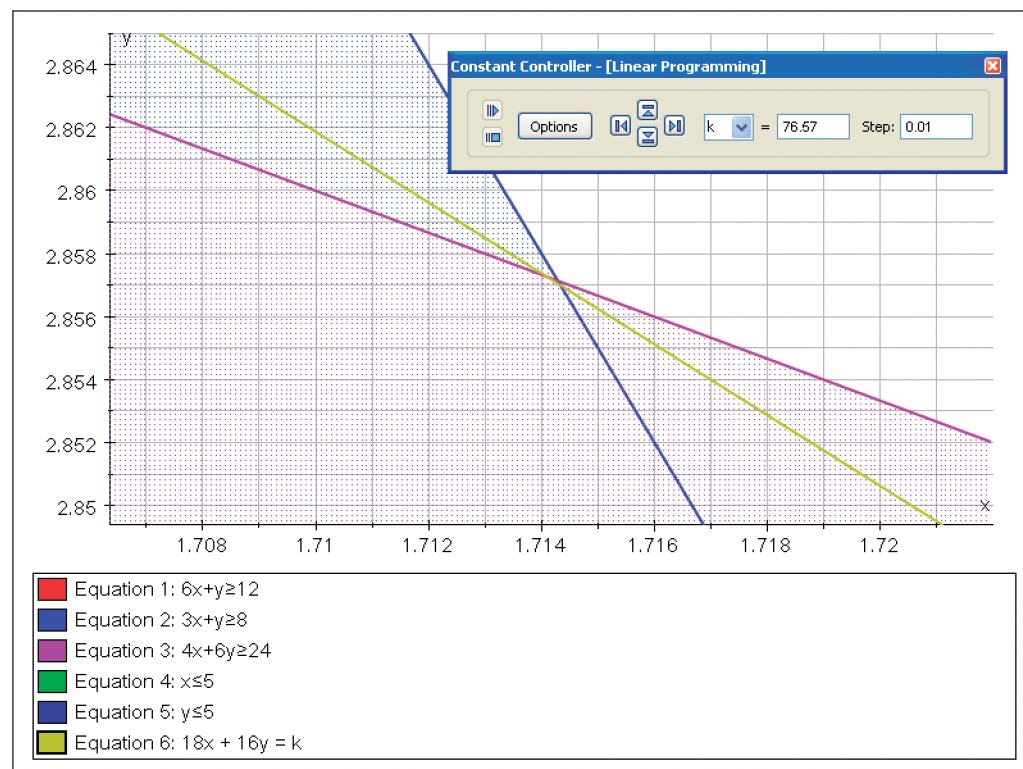Equation 1: 6x+y≥12
Equation 2: 3x+y≥8
Equation 3: 4x+6y≥24
Equation 4: x≤5
Equation 5: y≤5
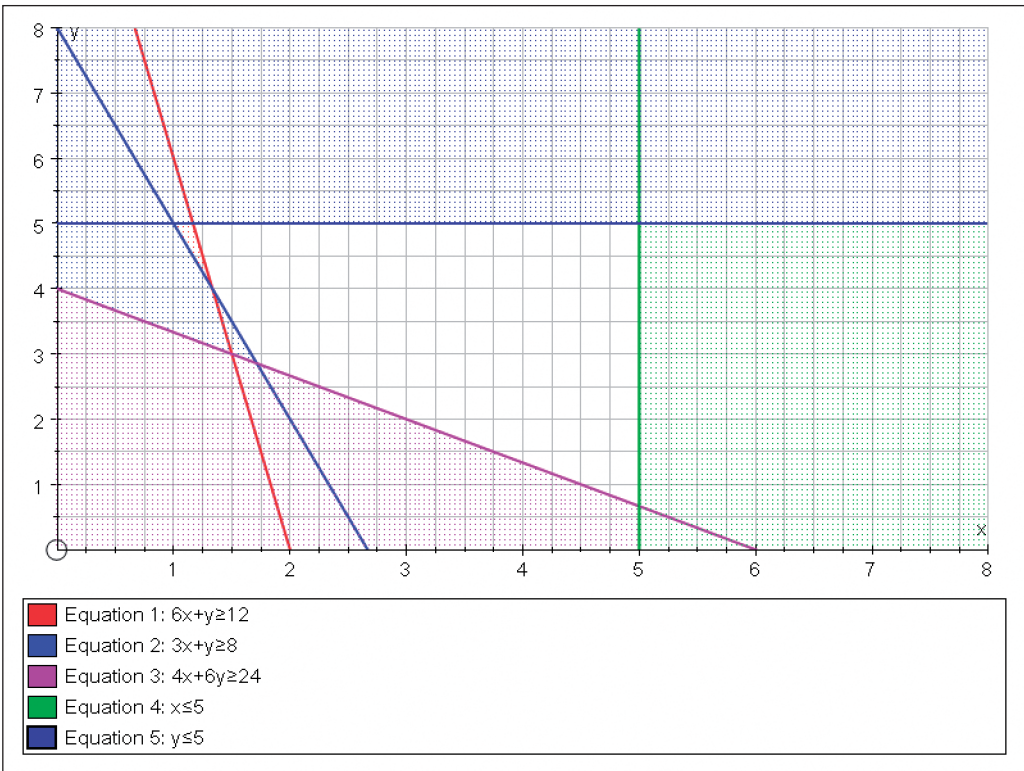
Test:
Swedish Delight: (6 × 3) + 4 ≥ 12
Venus Bar: (3 × 3) + 4 ≥ 8
Dig Dog: (4 × 3) + (6 x 4) ≥ 24

TASK 12: Co-ordinates of the point of intersection = (1.71, 2.86) (2dp).

TASK 13: This must be the optimal solution, because it is a feasible combination of x and y (in that it satisfies all the constraints) that can be obtained for the lowest cost (the smallest value of k).

TASK 14: The solution is the intersection of the lines: 3x + y = 8 and 4x + 6y =

24.

3x + y = 8
4x + 6y = 24

4x + 6(8 − 3x) = 24
4x + 48 − 18x = 24
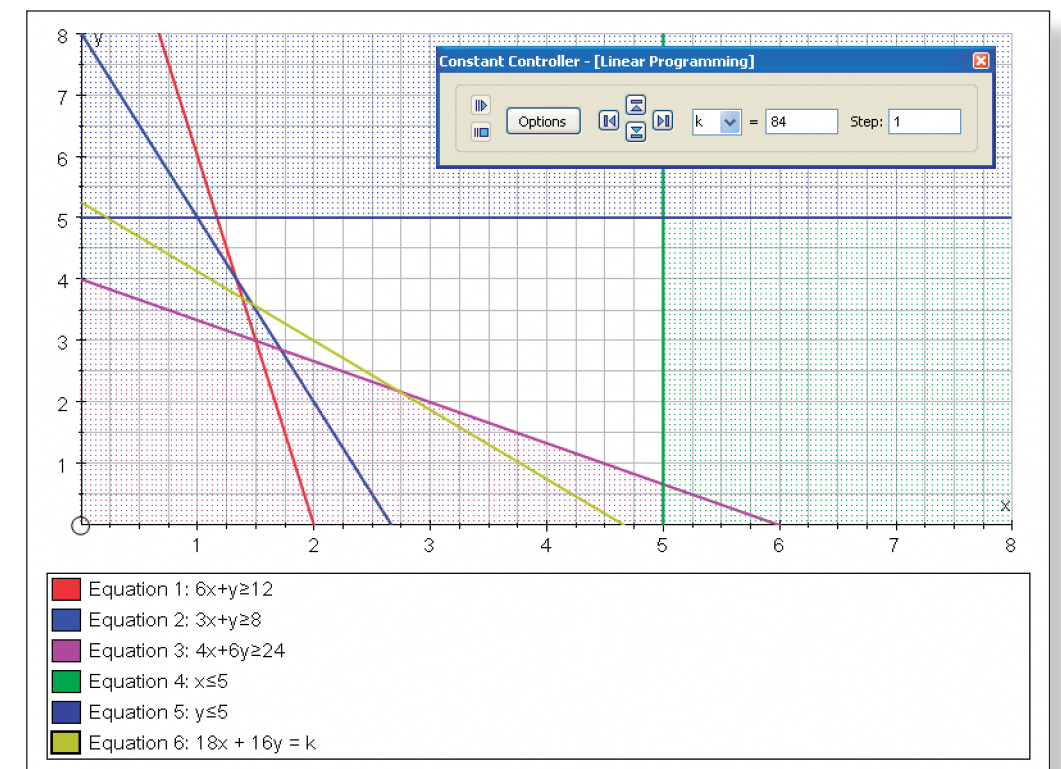14x = 24

$x = \frac{12}{7}$

$y = \frac{20}{7}$

TASK 15: Solution to problem:

The company should open Plant X for 1.71 days, and Plant Y for 2.86 days, which would have a weekly cost of (18 × 1.71) + (16 × 2.86) = £76,571 (nearest pound).

TASK 16: Co-ordinates of the best integer solution: (2, 3).

Total Weekly Costs: (18 × 2) + (16 × 3) = £84,000

TASK 17: If Plant Y was able to produce twice as many crates of each type of chocolate, then you would expect the company to be able to produce the required amount of chocolate in a shorter time frame, and hence for lower costs. You might expect production to **Shift** more towards Plant Y, but then if you look at the data in the table, Plant X is still much more efficient at producing the first two types of chocolate bar. Therefore, you might expect the number of days of production in Plant X to remain roughly the same, but production

in Plant Y to decrease. This will be reflected in a changing of the gradient of the constraint lines (becoming less steep), and a **Shifting** of the location of the optimal solution to the left.
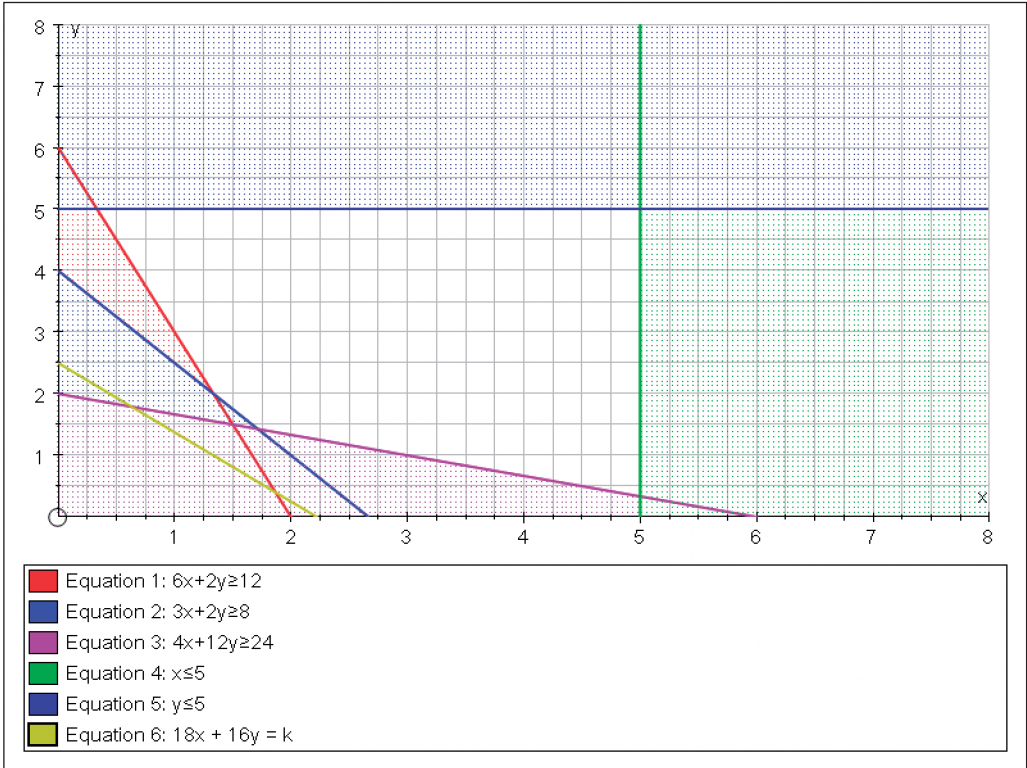
TASK 18: Co-ordinates of the point of intersection = (1.71, 1.43) (2dp).

This gives total weekly costs of (18 × 1.71) + (16 × 1.43) = £53,700 (nearest £'00)
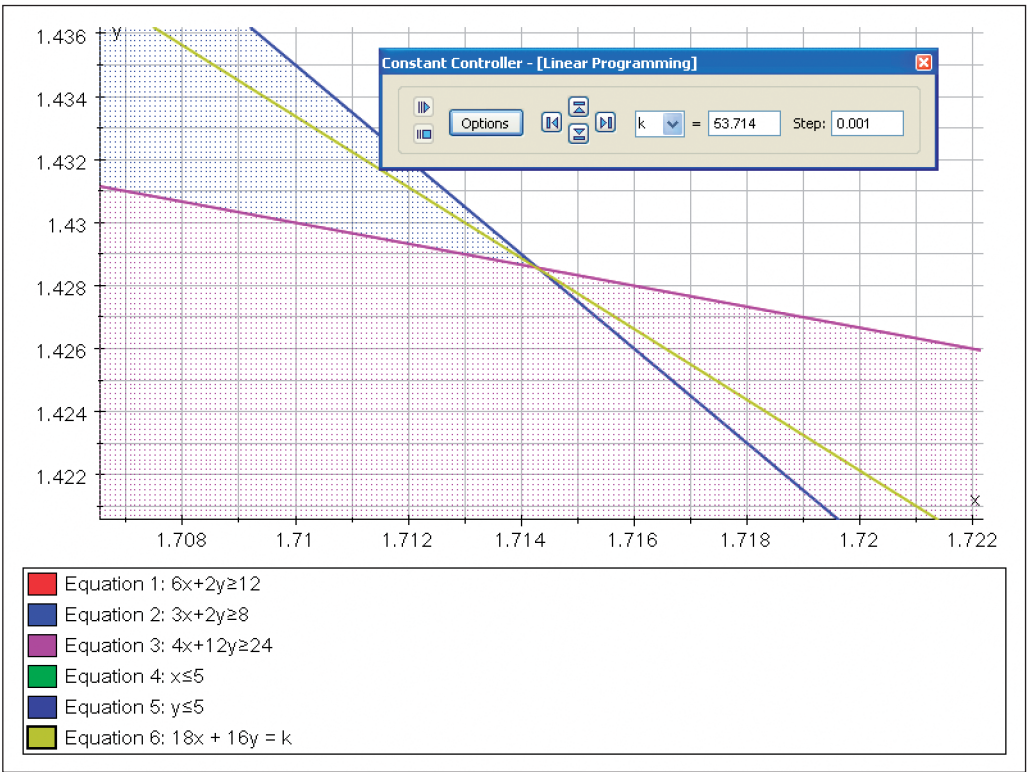
This supports the belief that operation time would simply be reduced in Plant Y, but kept at a similar level in Plant X. In fact, operation time in Plant Y has halved!

TASK 19: Examples of unrealistic assumptions include:

- There are no transportation costs.
- There is no mention of what happens to the stock which is surplus to the requirements of the supermarket
- There is no mention of the price the supermarket is paying for the chocolate, so we can only assume that the chocolate company is making a profit at the optimal level of weekly costs.

Activity 3: Pocket Money

TASK 20: Objective Function: Maximise: 0.14x + 0.2y + 0.25z

TASK 21: There are three variables, but we can only graph any two of them on the usual set of axes.

TASK 22: Budget Constraint: x + y + z = 1000

TASK 23: z = 1000 − x − y

TASK 24:

0.14x + 0.2y + 0.25z
0.14x + 0.2y + 0.25(1000 − x − y)
0.14x + 0.2y + 250 − 0.25x − 0.25y
250 − 0.11x − 0.05y

So, Objective Function is Minimise: 250 − 0.11x − 0.05y

TASK 25: Constraint 1: x ≥ 0.5(x + y)
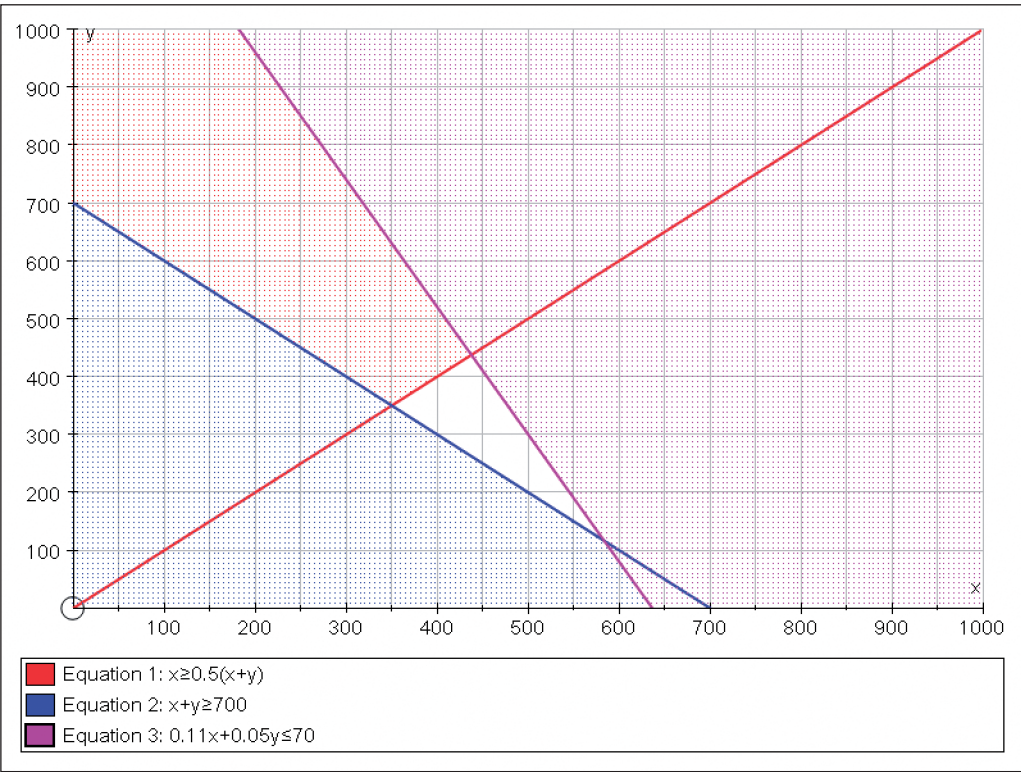
TASK 26: Constraint 2: $z \leq 0.3(x + y + z)$

TASK 27:

$$z \leq 0.3(x + y + z)$$
$$1000 - x - y \leq 0.3(x + y + 1000 - x - y)$$
$$1000 - x - y \leq 300$$
$$700 \leq x + y$$
$$x + y \geq 700$$

**TASK 28:** Constraint 3: $0.14x + 0.2y + 0.25z \geq 0.18(x + y + z)$

TASK 29:

$$0.14x + 0.2y + 0.25(1000 - x - y) \geq 0.18(x + y + 1000 - x - y)$$
$$0.14x + 0.2y + 250 - 0.25x - 0.25y \geq 180$$
$$-0.11x - 0.05y + 250 \geq 180$$
$$70 \geq 0.11x + 0.05y$$
$$0.11x + 0.05y \leq 70$$

Equation 1: x≥0.5(x+y)
Equation 2: x+y≥700
Equation 3: 0.11x+0.05y≤70

TASK 30: The optimal solution must be the last point the objective function touches in the feasible region, moving from the origin outwards. The maths teacher is trying to maximise the value of the objective function, and this last point gives the greatest value of the objective function, whilst also meeting all the criteria imposed by the constraints.

TASK 31: Looking at the equations of the lines, the objective function is parallel to the constraint $0.11x + 0.05y \leq 70$, and so there will not be one unique "last" point in the feasible region.

Equation 1: x≥0.5(x+y)
Equation 2: x+y≥700
Equation 3: 0.11x+0.05y≤70
Equation 4: 250−0.11x−0.05y=k
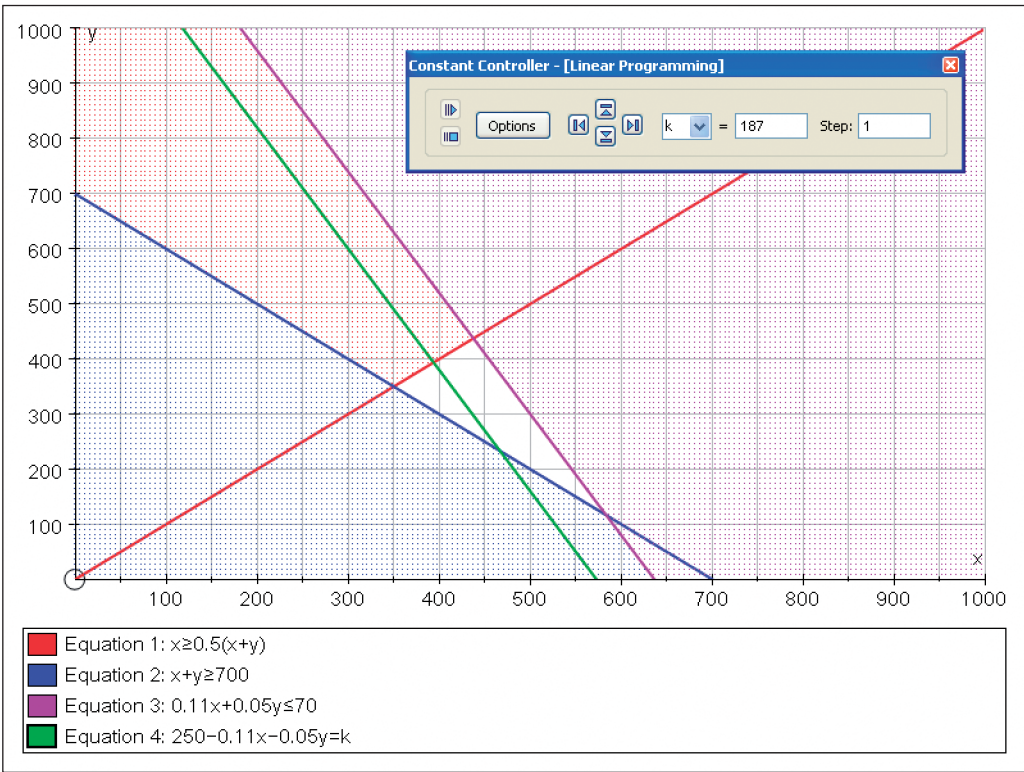
TASK 32: Co-ordinates of one optimal solution = (500, 300).

This gives the children the following amounts of money:

Child	£
Fermat	500
Lagrange	300
Euler	200

This gives the maths teacher the following amount of interest:

$(0.14 \times 500) + (0.2 \times 300) + (0.25 \times 200) = £180$

TASK 33: For Euler to receive the lowest amount of money, the value of x + y must be maximised. So, draw the line, x + y = m, adjust the value of the constant, and see the greatest value of m that lies on the line of optimal solutions, and still within the feasible region!

Equation 1: x≥0.5(x+y)
Equation 2: x+y≥700
Equation 3: 0.11x+0.05y≤70
Equation 4: x+y=m

Equation 1: x≥0.5(x+y)
Equation 2: x+y≥700
Equation 3: 0.11x+0.05y≤70
Equation 4: 250−0.11x−0.05y=k
Equation 5: x+y=m
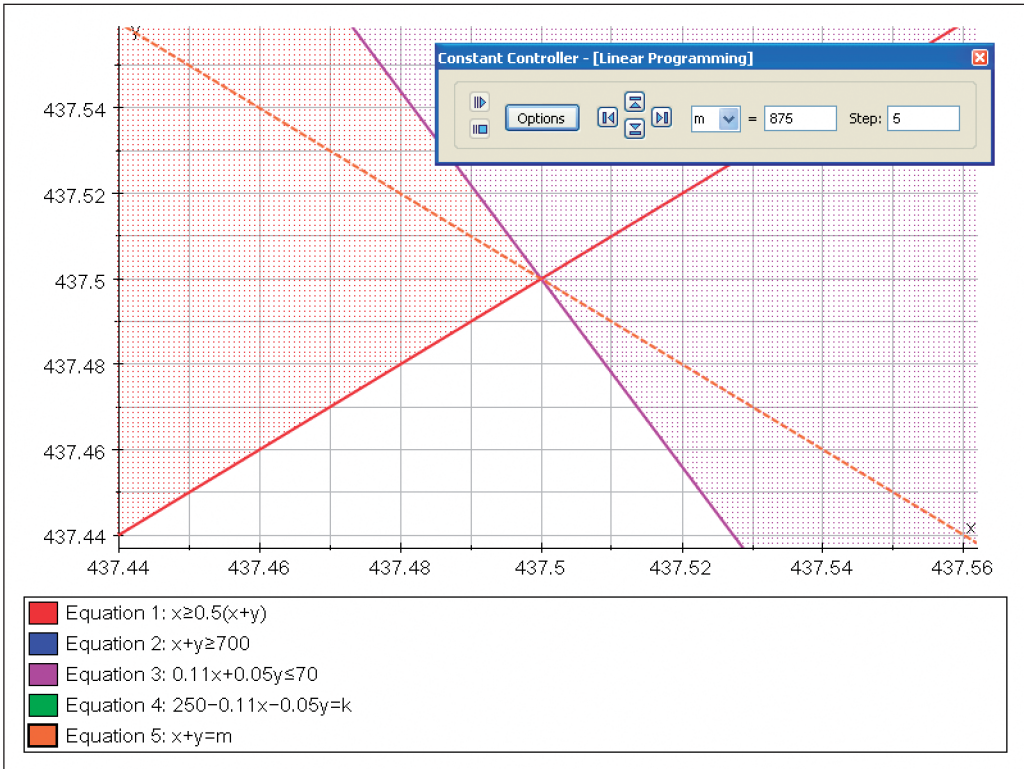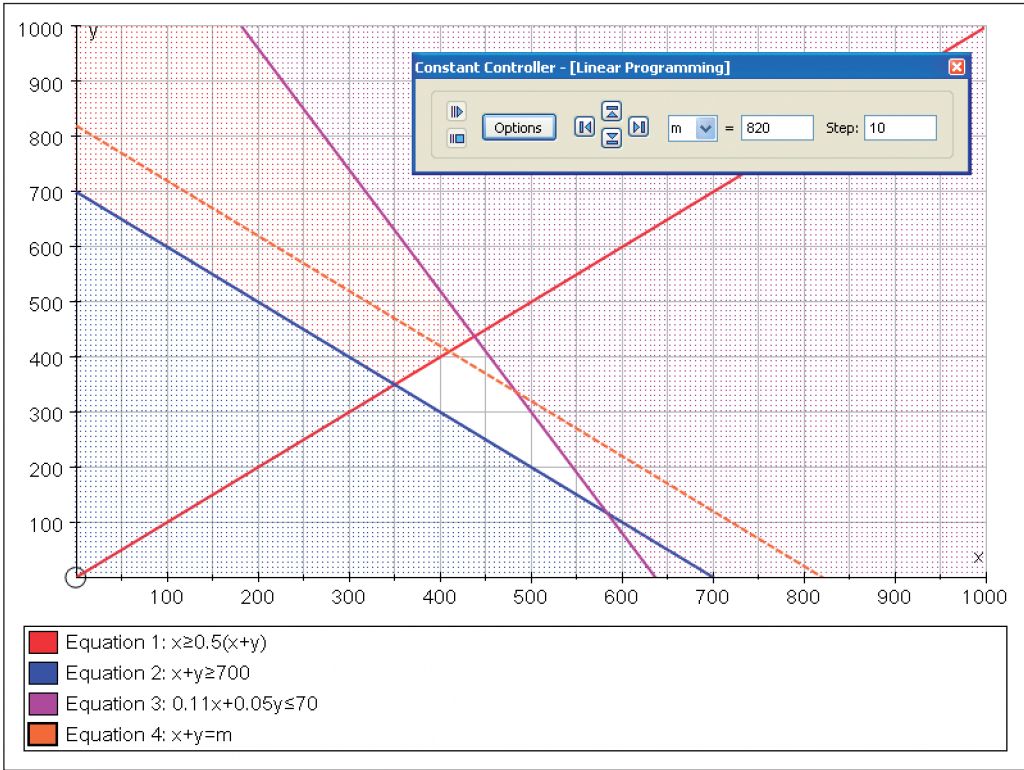
The co-ordinates of the optimal solution are: (437.5, 437.5)

At this solution, the three children receive the following sums of money:

Child	£
Fermat	437.50

Child	£
Lagrange	437.50
Euler	125

Activity 4: Another Way?

Task 34: All the optimal solutions lie on one (or more) of the vertices of the feasible region. This may not appear to be the case in Activity 3 where there are an infinite number of optimal solutions, but two of these still lie on the vertices.

Task 35: This suggests that another way to solve linear programming problems is simply to work out the equations of the constraints, and find the co-ordinates of the intersections of each of these constraints, (using simultaneous equations), and then test each one of these solutions in using the objective function to find which is optimal. This is the basis for the Simplex Algorithm.

LEARNING OBJECTIVES

CORE CONTENT:

- To understand and carry out an investigation involving a practical application of mathematics.

- To be able to appreciate how we can collect and model data using functions to enable us to make predictions.

- To be able to make important decisions to make the process of data collection as clear and easy as possible.

- To be able to conduct a controlled experiment to gather useful data.

- To be able to fit a quadratic function to a set of data.

- To be able to analyse the model using tools such as differentiation.

- To be able to interpret the main features of the model.

- To be able to evaluate and critique the model.

- To be able to use the model to make predictions.

- To suggest improvements for the model.

- To be able to appreciate the real-world uses for such a modelling technique.

EXTENSION MATERIAL:

- To be able to fit a trigonometric function to a set of data.

REQUIRED PRIOR KNOWLEDGE

- Be able to recognise the shapes of quadratic functions.

- Be able to differentiate quadratic functions and use this to find the co-ordinates of stationary points.

- To have some knowledge of how the constants in a quadratic function effect its shape.

DESCRIPTION OF THE ACTIVITY

- Students will be introduced to the Flying Gonzo game, and they will be given an opportunity to familiarise themselves with the controls at the

start of the activity. This should also highlight to them that without some kind of model/system, the game is pretty difficult to succeed in.

- Students are then encouraged to conduct a controlled experiment to gather data which will then be plotted on Autograph.

- The students will then attempt to fit a quadratic function to the data using knowledge of transformations, before allowing Autograph to find the "curve of best fit".

- Students will then use differentiation to find the maximum distance which Gonzo can travel.

- They will then be asked to discuss the important features of the model, before testing it out.

- Students will then be asked to evaluate the model in the light of their findings and suggest improvements.

- The Extension Activity involves fitting a sine function to the data to see if this gives us a closer fit.

Pre–Activity Setup

Ensure the Flying Gonzo game is installed and available on all workstations.

It can be copied from the accompanying CD, or played on–line at the following website:

http://www.funny-games.biz/flying-gonzo.html

Note: This is a Flash game, and so all computers must also have the latest Flash software installed on them. If you have Autograph installed, then you should already have flash installed because it is needed to access the Autograph Help files. However, it is also available to download for free at the following address:

http://www.adobe.com/products/flashplayer/

Suggested Introduction

"Today we are going to try something a little different. The technique we are going to learn today has many extremely important uses in the real world. Your challenge is to help Gonzo survive repeated trips in a cannon and guide him safely into a bucket. And a little bit of mathematics might just help you along the way…"

Suggested Prompts

- One possible hindrance to the success of this investigation is the amount of time the students spend "familiarising themselves" with how the Flying Gonzo game works. I would suggest that five minutes is more than enough time.

- The Autograph functions employed during this investigation are relatively straight forward, so the students shouldn't have too much difficulty with them.

- It is also worth having a quick check to make sure your students' data, when plotted, is the correct shape. It should be an upside-down U. If not, suggest they try collecting the data again.

- One possible area where they might get stuck is attempting to fit a quadratic function (and later a trigonometric function) to the shape of the data. Once they realise that the constant in front of the $x2$ term has to be negative, and that depending on their chosen distances some of the constants might be very small decimals, they should be up and running. If any students do get bogged down by this part of the investigation, encourage them to simply move on.

- In terms of the extended material, suggesting to students that they think what effect each constant has on the shape of the graph should be enough to get them on the right track.

Suggested Plenary

- How successful were your models at helping guide Gonzo into the bucket compared to just estimates alone?

- What problems did you encounter when using the model?

- How would you improve your model if you could do the activity again?

- What are some real life applications of this modeling technique?

Ideas for Further Work

- Take a look at some real life applications of this modeling technique.

- Have a look at some other real life applications of the mathematics the students are studying.

STUDENT WORKSHEET

NAME:..

ACTIVITY 1: GETTING USED TO THE GAME

Open up the file called: *The Great Gonzo's Ridiculously Dangerous and Downright Foolish Stunt Game.*

The aim of the game is simple: poor Gonzo is repeatedly fired out of a cannon, and you must try your hardest to help him have a soft landing in the bucket of water.

To help you do this, you have control over two variables:
1.   The **Voltage** at which the cannon propels Gonzo out at.
2.   The **Angle** at which the cannon is pointing.

To **View** the current location of the bucket, click on the right arrow by the word **View**. Every time you successfully land him in the bucket, the location changes.

TASK 1: Spend a few minutes familiarising yourself with the game and its controls, and try to get Gonzo to land safely in the bucket.

We are now going to devise a mathematical method which will (hopefully) significantly improve Gonzo's chances of surviving his little trip in the cannon.

ACTIVITY 2: MAKING SOME DECISIONS

Distance

You may have noticed that whilst the location of the bucket changes after every successful landing, there are no distances marked on the screen. However, you may also have noticed that as Gonzo flies through the air, the *background screen continually repeats itself*. Specifically, the pipes running down the wall containing the panel of switches recur at regular intervals. We can use this feature to enable us to *estimate distances*.

TASK 2: Decide what length you wish the distance between two pipes to represent. Write your answer in the space below, and briefly explain your choice.

Voltage

Continually altering both the angle and the voltage would not make for a very controlled experiment. So, we are going select one value for the voltage and hold it constant whilst varying the angle of the cannon in our *High Voltage Setting*. We are then going to do the same, but with another constant value of the voltage in our *Low Voltage Setting*. Hopefully this will enable us to successfully land Gonzo in buckets at a whole range of possible distances.

TASK 3: Decide upon what value the voltage will take for your High Voltage Setting and what value it will take for your Low Voltage Setting, and briefly explain your choices.

Hint: You must ensure Gonzo can reach all possible ranges of bucket locations!

ACTIVITY 3: GATHERING THE DATA

TASK 4: Set the voltage at the level you decided for the *High Voltage Setting*.

Select *10 different angle settings*, fire the cannon, and each time use your choice in Task 2 to estimate the distance from the cannon where Gonzo lands.

Record your answers in the table below.

Remember: The aim of this part of the investigation is not to land Gonzo in the bucket, but to collect data to help us develop a mathematical model. Any pain suffered by Gonzo now is for his own good.

TASK 5: Repeat Task 4, but this time for the *Low Voltage Setting*.

High Voltage Setting		Low Voltage Setting	
Angle of Cannon	Estimate of Distance	Angle of Cannon	Estimate of Distance

High Voltage Setting	Low Voltage Setting

TASK 6: How did you decide upon which angles to use?

TASK 7: Briefly describe any problems you encountered when collecting the data.

TASK 8: Looking at your data, if you were to plot distance (y-axis) against the angle (x-axis), roughly what shape function would both sets of data form?

ACTIVITY 4: PROCESSING THE DATA

Open up Autograph in **Advanced Mode** and ensure you have a blank **2D Graph Page**.

Make sure you are not in **Whiteboard Mode** (there should not be a blue square around the button).

Adjusting the Axes

Look at your data. Notice the maximum and minimum values for both the distance travelled and the angle of the cannon. We want to edit the axes accord-

ingly so that all the data will fit on.

Note: We are going to plot the Angle of the Cannon on the x-axis, and the Distance Travelled on the y-axis.

Click on **Edit Axes**.
Adjust the maximum and minimum values of x and y.
Click on the Labels tag and re-label your axes: Angle of the Cannon (x) and Distance Travelled (y).
Click **OK.**
Note: You can probably improve the appearance of your graph by revisiting **Edit Axes** and altering the values of the pips. Remove the green ticks and try entering some numbers of your own until you are happy with both the numbers of the axes and the lines of the graph itself.

Entering the Data

Click on **Enter XY Data Set**.

Change the name of the Data Set to *High Voltage Setting.*

Carefully enter your 10 pairs of data values, remembering that x is the angle, and y is the distance.

Click **OK**.

Your ten pieces of data should now be displayed on your graph.

TASK 9: Now that you can see the data in front of you, are you still happy with your answer to Task 8 about the rough shape of function that the data forms?

ACTIVITY 5: FINDING A FUNCTION TO MATCH THE DATA

Now, if your data appeared to form part of a linear function, then we could simply draw a line of best fit, or calculate the equation of a regression line.

However, as the data appears to be in the shape of a curve, a straight line would not be appropriate, so we can try to find a curve which makes a good representation of the shape of the data – "a curve of best fit"

We will try with the simplest functions that involve curves – quadratic functions.

Enter the equation: y = ax² + bx + c

Note: To write x², just type x then press "**Alt 2**" together, or writing "**xx**" does the job as well!

The values of the constants are all set to 1. *What will this curve look like? Where will it cross the axes? Which direction will it slope?*

Click **OK**.

Because the values of the three constants are automatically set to 1, this gives you the graph of y = x² + x + 1.

Left-click on the newly drawn curve (it should turn black).

Select **Text Box** from the toolbar.
Tick the box next to **Show Detailed Object Text** and click **OK**.

The **Text Box** should now display the equation of the curve, together with the current values of the constants a, b and c.

Click on the **Constant Controller** on the top toolbar.
The **drop-down** menu allows you to select different constants.
The **up-down** arrows alter the value of the selected constant.
The **left-right** arrows adjust the size of the step.

Predict

How will altering the value of each of the constants affect the shape of your graph?

TASK 10: Use the **Constant Controller** to adjust the values of the three constants in an attempt to find a curve whose shape roughly matches that of the pattern of your data. Try for a few minutes and using the information contained in the **Text Box**, write the equation of your best effort below.

Hint: At least one of the constants will have to be *negative*!

If you ever loose sight of your curve, just hit the **Undo** button.

Activity 6: Finding a Model

We can actually use Autograph to work out the "curve of best fit".

Finding the Curve of Best Fit

Left-click on one of your data points (they should all have squares around them).

Right-click and select **Best Fit** from the menu.

Adjust the Order to 2 (this will give us a quadratic), and click **OK**.

The "curve of best fit" should now appear on the screen.

Left-click on the curve of best fit (it should turn black).

Select **Text Box** from the toolbar.

Change the word "Quadratic" to "High Voltage Setting" and click **OK**.

The equation of your curve of best fit should now appear.

TASK 11: Write down the equation of your High Voltage curve of best fit.

TASK 12: Looking at the curve of best fit, approximately what appears to be the maximum distance you can fire Gonzo, and at what angle must the cannon be placed the achieve this?

TASK 13: Using the equation of your curve of best fit, use differentiation to find a more accurate estimate for the angle of the cannon (x) which fires Gonzo the greatest distance.

Hint: Think about the gradient of the curve of best fit at the point of maximum distance.

[blank box]

TASK 14: Use your answer the Task 13, together with the equation of the line of best fit, to find more accurately what the maximum distance Gonzo can reach actually is.

[blank box]

Checking your Answer

Left-click on the High Voltage curve y (it should turn black).

Right-click and select **Solve f'(x) = 0** from the menu.

The co-ordinates of this maximum point should now be displayed on the bottom of the screen.

TASK 15: What are some of the limitations of this particular curve as a model to the distances Gonzo travels on the High Voltage Setting?

Hint: Look at the range of your data, the extremes of the curve, and think about the y–intercept.

[blank box]

TASK 16: Repeat the set of instructions to find the curve of best fit for the Low Voltage Setting. Label it with a **Text Box** and write the equation below

Note: You do not need to use differentiation to find the maximum distance Gonzo can travel at this setting… unless you really want to!

[blank box]

Left-click on the curve you used to estimate the curve of best fit (it should turn black).

Right-click and select **Delete** from the menu, or simply press **Delete** on the keyboard.

You should now be left with your two labelled curves of best fit on the screen, together with the set of data points.

TASK 17: Looking at your graph, what range of distances can be obtained on both the High Voltage and Low Voltage settings?

[blank box]

TASK 18: If you had to obtain a distance in that particular range, which setting (high voltage, or low voltage) would you use? Explain your choice.

You now have two models which should enable you to more accurately guide Gonzo successfully into his bucket. But how good are your models?

Activity 7: Testing the Model

Re-open the Flying Gonzo game.

Use the right arrow next to the **View** button, together with your choice of the distance between the pipes all the way back in Task 2, to estimate how far away the bucket is.

Re-open your Autograph page.

Enter the equation: y = ? replacing the question mark with your estimate for the distance of the bucket.

A horizontal line should now appear on your graph.

Decide whether you are going to use the High Voltage or Low Voltage setting.

Ensure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Hold down the **Shift** button to select more than one object:

Left-click on the curve of your chosen setting, and **left-click** on the horizontal line (they should both be black).

Right-click and choose **Solve f(x)=g(x)** from the menu.

The points of intersection between the curve and the line should now be marked on the graph.

On the top toolbar select **View** > **Status Box**.

The co-ordinates of the points of intersection should now be displayed next to Equation Solver.

TASK 19: How do these co-ordinates help you know what angle to fire the cannon at?

Re-open the Flying Gonzo game.

Use your answer to Task 19, together with your chosen Voltage setting, to set up the cannon.

When you are ready… FIRE!

Repeat this activity a few times for different distances of bucket.

Note: The **Status Box** will only ever display results that are currently selected. If the co-ordinates of the points of intersection are not displayed, simply **left-click** on them and they should appear!

Activity 8: Evaluating the Model

TASK 20: How successful does your model appear to be compared with the estimates you were making when you first played the game?

TASK 21: Does your model seem to work better for certain distances/angles than others, and can you explain this?

TASK 22: If you were to start this investigation again, how would you change the procedure to improve the accuracy of your model?

TASK 23: Can you think of any real–life applications where a modelling technique like this might be useful?

ACTIVITY 9: A DIFFERENT MODEL?

A quadratic function is not the only type of function we could have chosen for our curve of best fit.

Another type of function that could give us a curve to match our data is the *sin function*.

Re-open your Autograph page.

Enter the equation: y = asin(bx + c) + d

Predict *The values of the constants are all set to 1. What will this curve look like? Where will it cross the axes? How many times will it repeat itself in the range of your axes?*

Click **OK**.

Left-click on the newly drawn curve (it should turn black).

Select **Text Box** from the menu.
Tick the box next to **Show Detailed Object Text**.

The **Text Box** should now display the equation of the curve, together with the current values of the constants a, b, c and d.

Click on the **Constant Controller** on the top toolbar.
The **drop-down** menu allows you to select different constants.
The **up-down** arrows alter the value of the selected constant.
The **left-right** arrows adjust the size of the step.

Predict *How will altering the value of each of the constants affect the shape of your graph?*

TASK 24: Use the **Constant Controller** to adjust the values of the four constants in an attempt to find a curve whose shape roughly matches the pattern of data for the High Velocity Setting. Try for a few minutes and using the information contained in the **Text Box**, write the equation of your best effort below.

Hint: Think about what effect each of the constants has on the sin function.

We will again use Autograph to find the curve of best fit, but it works a little differently this time:

Ensure you are in **Select Mode**.

Left-click on an unoccupied part of the graph area to *de-select everything*.

Hold down the **Shift** button to select more than one object:

Left-click on the sin curve (it should turn black), and **left-click** on one of the pieces of data from the High Voltage Setting (they should all now have squares around them).

Right-click and choose **Best Fit to Data** from the menu.

The curve of best fit should now appear.

TASK 25: Use the information contained in the **Text Box** to write the equation of the curve of best fit.

TASK 26: How does this curve of best fit compare to the quadratic one? Is one more useful for some distances/angles than others? Again, pay attention to the extremes of the data.

Summary

Use the space below to summarise what you have learnt during this investigation:

ANSWERS

Please Note: The majority of these answers are specific to the voltage, distance and angle settings that I chose. They are included for illustrative and guidance purposes. Each student will have a different set of answers.

ACTIVITY 1: GETTING USED TO THE GAME

TASK 1: Students play the game "Flying Gonzo".

ACTIVITY 2: MAKING SOME DECISIONS

TASK 2: I chose 10 metres to represent the distance between the two pipes because I felt this would make it as easy as possible to estimate distances in-between the pipes.

TASK 3: I chose a High Voltage setting of 80 and a Low Voltage setting of 50 to ensure I could cover a wide range of distances and have relatively high levels of accuracy on the closer distances.

ACTIVITY 3: GATHERING THE DATA

TASKS 4 AND 5: The data I generated was as follows:

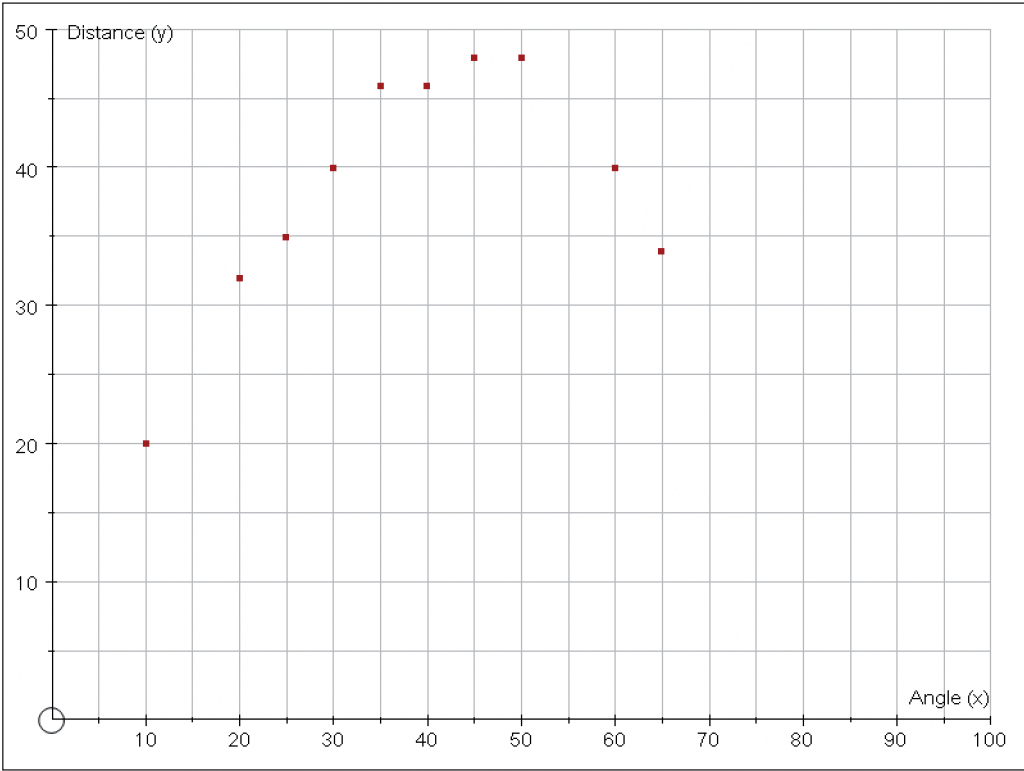| High Voltage Setting: 80		Low Voltage Setting: 50	
Angle of Cannon	Estimate of Distance	Angle of Cannon	Estimate of Distance
10	20	10	9
20	32	20	13
25	35	25	15
30	40	30	17
35	46	35	18
40	46	40	18
45	48	45	18
50	48	50	17
60	40	60	15
65	34	65	12

TASK 6: I chose my selection of angles to make sure I had covered the two extremes, and to make sure I had a greater concentration of angles in the middle of the range where the difference between the distances was less.

TASK 7: The problems I encountered when collecting the data included: the fact that it was hard to know precisely where Gonzo landed each time as he tends to slide along the ground; the background whizzed by so fast for some of the further distances that it was hard to count the number of pipes; and it was hard to distinguish between the shorter distances.

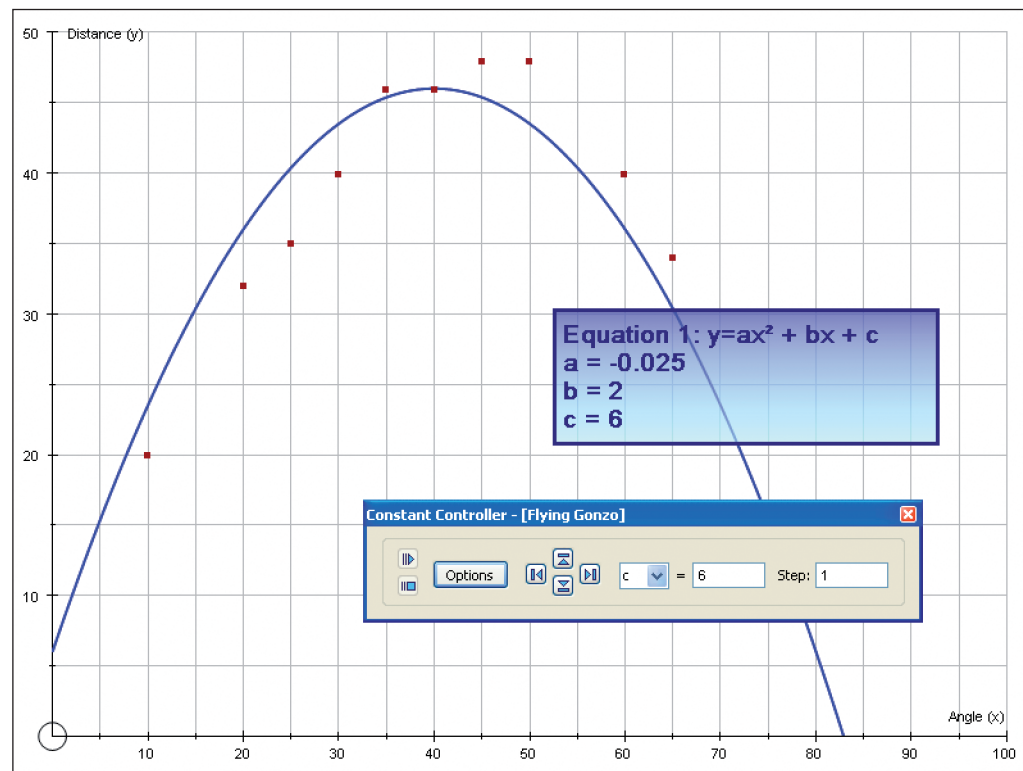TASK 8: A curve / an upside down U-shape / a negative quadratic function.

ACTIVITY 4: PROCESSING THE DATA

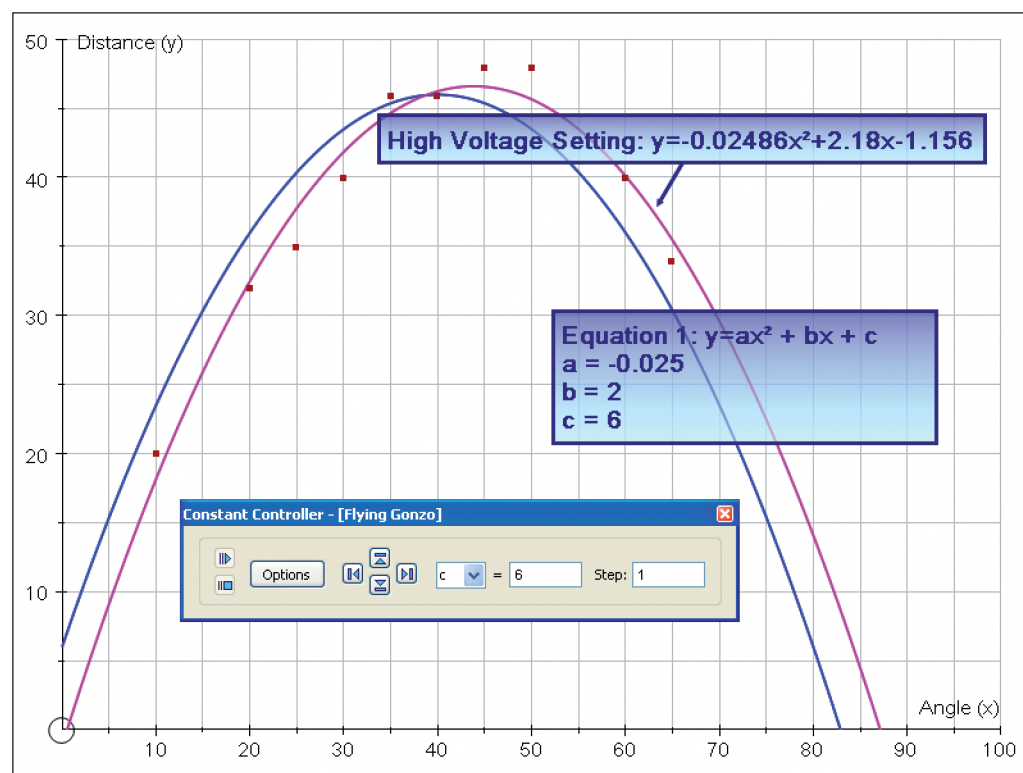TASK 9: My High Voltage data looked like this:

ACTIVITY 5: FINDING A FUNCTION TO MATCH THE DATA

TASK 10: After a few minutes of trying, I got the equation: $y = -0.025x^2 + 2x + 6$ and my screen looked like this:

Activity 6: Finding a Model

TASK 11: The equation of the curve of best fit for my High Voltage setting is: $y = -0.02486x^2 + 2.18x - 1.156$ and my screen looked like this:

TASK 12: Looking at the curve of best fit, it appears that the maximum distance is approximately 47 metres, and this is achieved at an angle of approxi-

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mately 45 degrees.

TASK 13: $y = -0.02486x^2 + 2.18x - 1.156$

Using differentiation:

$\frac{dy}{dx} = -0.04972x + 2.18$

$-0.04972x + 2.18 = 0$

$x = 43.85°$ (2 dp)

TASK 14: $y = -0.02486x2 + 2.18x - 1.156$

Using substitution:

$y = -0.02486(43.84…)^2 + 2.18(43.84…) - 1.156$

$y = 46.64$ m (1 dp)

TASK 15: One possible limitation of the model is that it assumes the cannon can fire at angles such as 80° and 5°, both of which are outside of its range. Also, the equation for my curve of best fit seems to imply that for an angle of 0 **Degrees**, Gonzo would travel a negative distance!

TASK 16: The equation of the curve of best fit for my Low Voltage setting is: $y = -0.009755x^2 + 0.802x + 1.501$ and my screen looked like this:
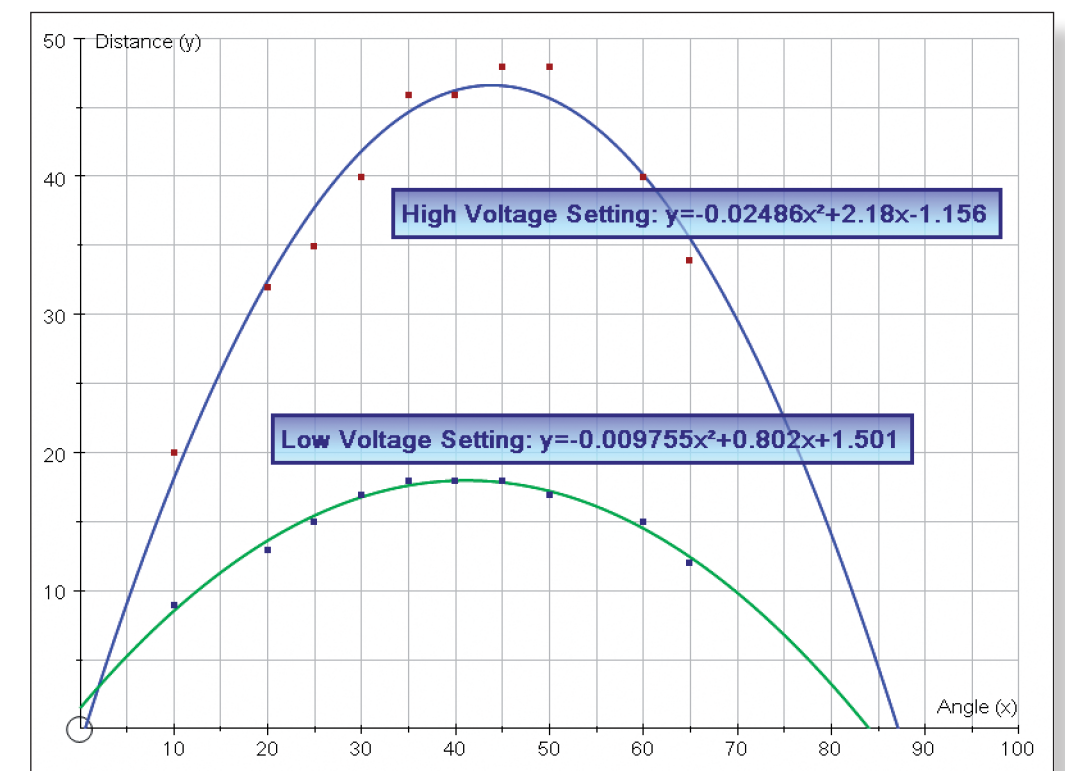
TASK 17: According to my graph, distances between 1.5m and 18m could be achieved on both settings.

TASK 18: Given a choice for distances in this range, I would choose the Low Voltage setting, as there is more data for those particular distances, and hence the model should be more accurate.

ACTIVITY 7: TESTING THE MODEL

TASK 19: Firstly, you must make sure that you choose a co-ordinate that is in the range of the cannon, and once you have done that, the x co-ordinate tells you what angle to aim at.

TASK 20: My model improved my performance at hitting the bucket significantly, but it was by no means perfect.

ACTIVITY 8: EVALUATING THE MODEL

TASK 21: I found that my model was particularly poor at landing Gonzo safely in the bucket for the closest distances. This seemed to be because the angle of elevation of the cannon was too low to generate the height and loop needed to get Gonzo into the bucket. The model was also fairly poor at the long range distances. This might have been because there was a greater variation in these distances compared to those in the middle.

TASK 22: To improve my model I might have selected a slightly higher number for my Low Voltage Setting, thus enabling me to generate the necessary loop to land Gonzo in the bucket. I would also have carried out more than ten experiments for each setting in order to improve the accuracy of my model. Alternatively, I might have tried holding the angle of the cannon constant and varying the voltage to see if this improved the performance of the model.

TASK 23: Real life applications might include: determining the maximum level a tide reaches in order to build adequate defences; determining the optimal price to charge for a product in order to secure the maximum profit; and determining what week to take a holiday given data about average hours of sunshine.

ACTIVITY 9: A DIFFERENT MODEL?

TASK 24: After a few minutes of trying, the closest I got was a function with equation: y = −13.8sin(0.08x + 0.14) + 34.96 and my screen looked like this:

Equation 1: y=asin(bx + c) + d
a = -13.7653
b = 0.0802132
c = 0.139601
d = 34.9624
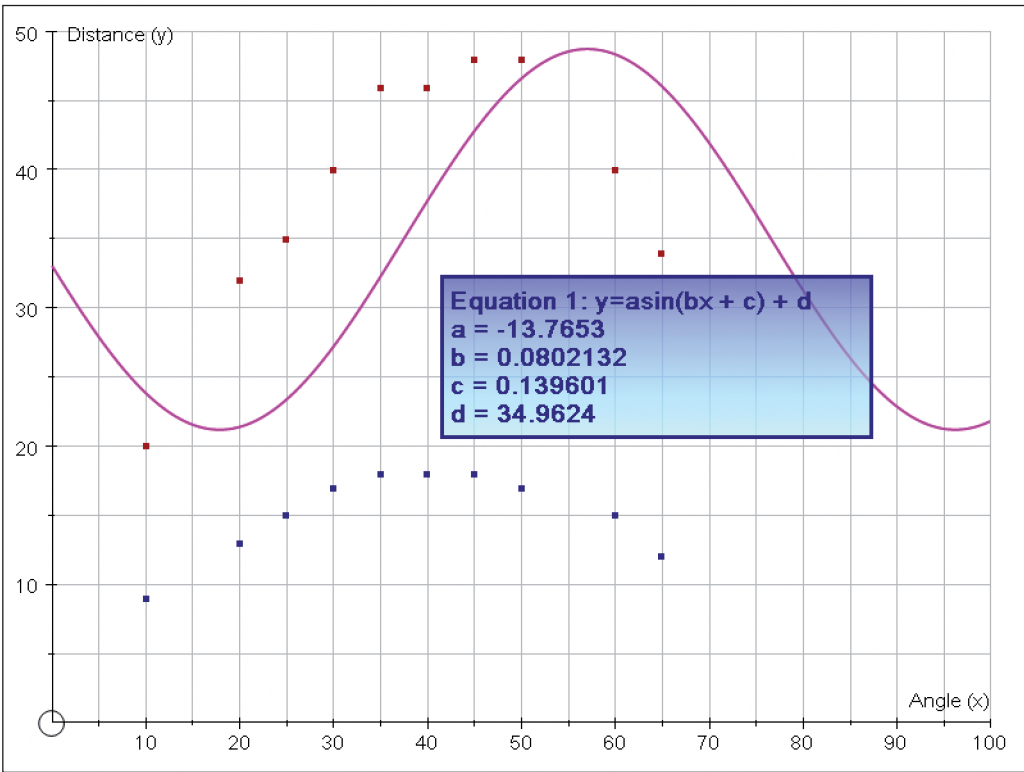
TASK 25: For the equation of the curve of best fit, I got: y = −11.9 sin(0.10x + 12.9) + 37.63 and my screen looked like this:

Equation 1: y=asin(bx + c) + d
a = -11.9419
b = 0.0969005
c = 12.9346
d = 37.6265
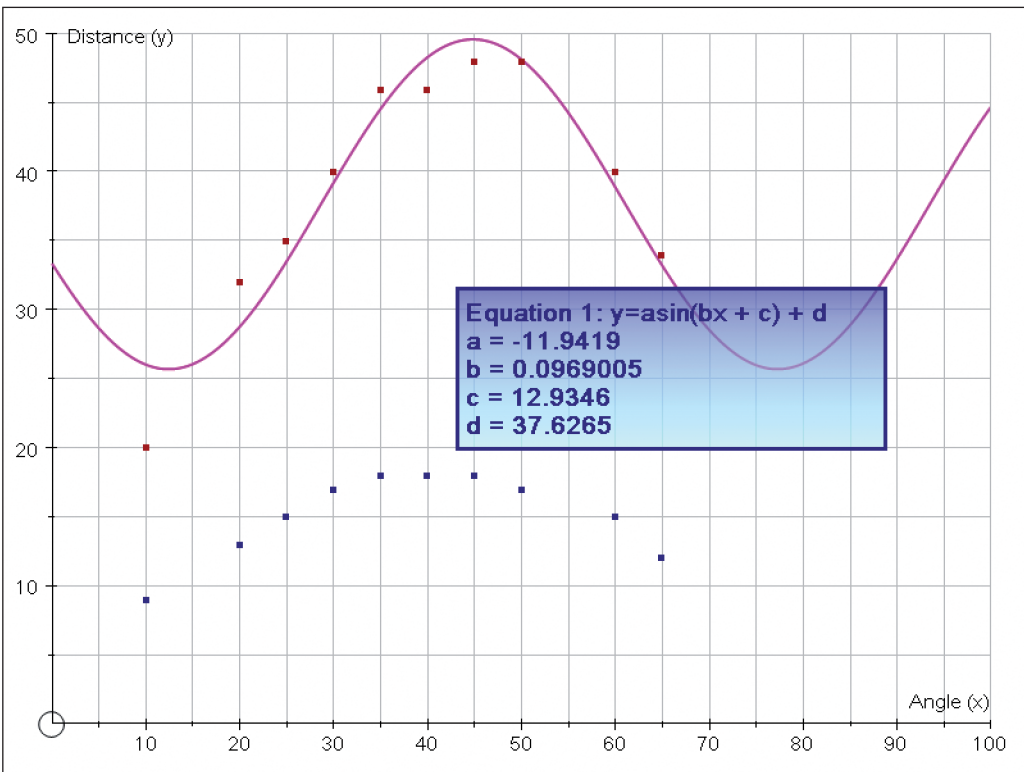
TASK 26: For my data, the sin curve was not as good a fit as the quadratic function. Furthermore, it gives an unrealistic representation of the pattern of the data at the extremes, suggesting that the distance travelled will increase at

very large or very small angles.