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Why Are Shot Puts Thrown at 31°? Using Autograph for Applications of the Parabola

Autograph is a two- and three-dimensional dynamic statistics and graphing utility, developed in England, that has grown out of direct classroom experience. A simple select–and-right-click interface, together with tools such as Autograph's unique Slow Plot, Scribble Tool, and dynamic Constant Controller help make the classroom experience interactive.

Autograph can be used to introduce most students to their first nonlinear function. To their surprise, they will discover that parabolas are everywhere.

THE BEST SHOT-PUT ANGLE

Most students who have been introduced to the quadratic function quickly realize that a thrown object follows a parabolic path. Further studies reveal that the longest throw can be achieved by projecting the object at an angle of 45° to the horizontal. It is a bit surprising, therefore, to discover that in the sporting world, when the objective is to achieve a maximum distance, this rule is rarely followed.

The shot put, for example, is an Olympic field event that involves launching a heavy metal ball. The weight of the ball varies with the age and gender of the participants, with Olympic events featuring a 16-pound shot for men and an 8.82-pound shot for women. The event is so named because the shot (the ball) is pushed (put) away from the competitor's body. In the Beijing Olympics in 2008, the men's gold medal was won with a put of 21.51 meters, while the women's winning put was 20.56 m. Competitors in this event usually launch their shots at about a 31° angle to the horizontal. In the javelin toss, an Olympic event involving a long spearlike pole, the angle of elevation of the toss is even smaller. Similarly, contestants in the long jump, yet another Olympic field event, leap at an angle of around 30° or less. Why?

To model this problem in Autograph, first choose the Advanced mode on start-up (to ensure that the degrees-radians and calculus options are available), set the angle measurement to degrees (via the degrees button on the toolbar, located just below the Axes drop-down menu), and choose the right-click option: Enter Equation (see **fig. 1**).

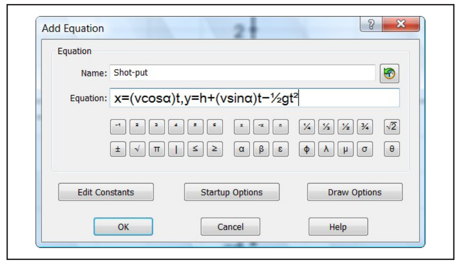

Fig. 1 A right click of the mouse allows users to choose the Enter Equation option, enter the parametric equation, and edit its name.

Now enter the two parametric equations, separated by a comma:

$$x=(v\cos\alpha)t, y=h+(v\sin\alpha)t-\tfrac{1}{2}gt^2$$

This department offers a forum where teachers can share innovative classroom activities and ideas related to teaching and learning mathematics through the use of technology. Ideas using all types of classroom technology are welcome. Send tips to the "Technology Tips" editors.

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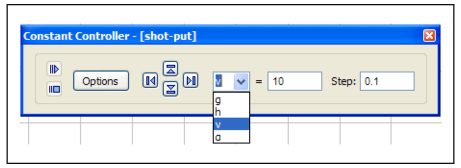


Fig. 2 The Constant Controller allows us to change the value of the constant and the step size by which it varies.

Here, g is the acceleration due to gravity (in m/s²), h is the height of the throw (put) off the ground (in meters), v is the velocity of projection (in meters per second), and α is the angle of projection (in degrees). Press Edit Constants and set g = 9.81, h = 2, v = 10, and α = 45. Press OK.

Then press Start-up Options. Autograph will think that this equation is trigonometric and will automatically choose inappropriate settings for this plot. So, to achieve a smooth plot, choose Manual plotting and set t-start = 0, t-finish = 10, and t-step = 0.1 (sec). Then OK this information and OK a second time to exit the Equation Entry dialog box and to plot the curve. Click on Axes and then Edit Axes; then set x:min to –1, x:max to 20, y:min to –1, and y:max to 10 and click on Equal Aspect. Doing so will adjust the scales and window appropriately. One final step will make everything appear larger: Access the Appearance tab and set Themes to Whiteboard. Click on OK to exit to the graph.

Before moving on, right-click to obtain the Enter Equation option and enter a new equation $y = h$, using Draw

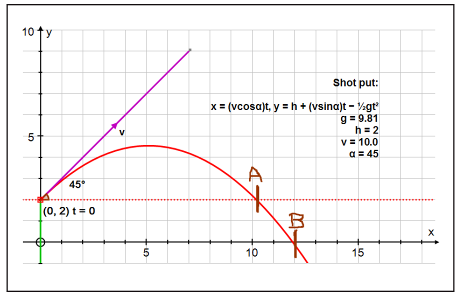


Fig. 3 The shot's maximum horizontal distance (marked by A) is obtained when the angle of elevation is 45°.

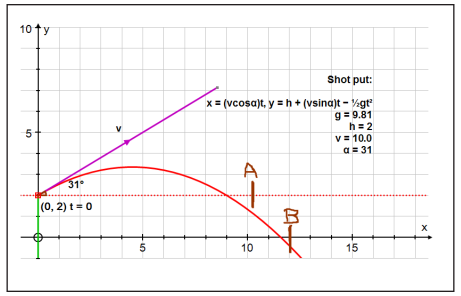


Fig. 5 Using an angle of elevation of 31° shortens the length of the put.

Options to set the dash style to dotted before plotting. The graph of this equation will be used to monitor the horizontal progress of the shot.

Autograph's Constant Controller icon (see **fig. 2**) looks like a box with a k in it and arrows below. It is turned on by a button on the top toolbar (just below and slightly right of the Object drop-down menu) and can be used to vary the parameters α and v. For the chosen parameter, the up and down arrows vary

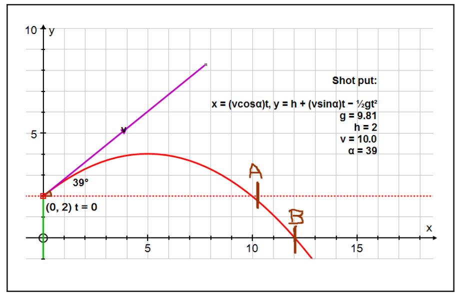


Fig. 4 The maximum put, however, is obtained when the angle of elevation is 39°.

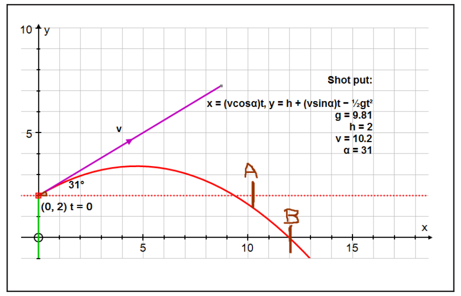


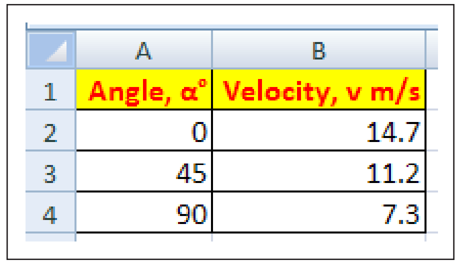
Fig. 6 A modest increase in the value of v will maximize the length of the put.

the value, and the left and right arrows vary the step, affording full dynamic control over the constant's values.

Classical physics suggests that a missile should be projected at 45° to achieve the maximum range. The assumption here is that the velocity of projection can be the same for all angles. Can the human body match this velocity? Think about throwing a ball for maximum distance, for example. At what angle of projection does the thrower feel the most power behind the throw. To investigate this issue, let's keep the velocity of projection constant at 10 m/s and vary the angle of projection.

Using the Constant Controller allows us to see that the shot achieves its maximum horizontal distance, as expected, at A, when α = 45°, and its maximum ground distance at B, when α = 39°(see **figs. 3** and **4**). But when α = 31°, the angle used by shot-putters, the ground distance is reduced (see **fig. 5**).

Now let's keep α fixed at 31° and increase v to try to improve on B. Surprisingly, increasing v to only 10.2 m/s will accomplish this (see **fig. 6**). Whereas classical physics assumes constant v, the maximum throw may be determined as much by the speed of the throw as by the angle of projection.



	A	B
1	Angle, α°	Velocity, v m/s
2	0	14.7
3	45	11.2
4	90	7.3

Fig. 7 Because the data are not quite linear, we use a quadratic model.

Can we predict, for a typical athlete, the possible projection speeds from different projection angles? A Web search reveals that work done by researchers at Brunel University in London suggests an almost linear relationship between the angle and the velocity (see **fig. 7**).

To explore this relationship, open a new 2D page in Autograph and use the right-click option Enter XY Data Set. If this information is in Excel, just select it and paste it into the space below the x. The column headers will accompany the data automatically. Otherwise, enter the data by typing them in. To set the column headers, right-click in each column. To use the two column headers as the axes labels, check the two boxes on the right. With Perform auto-scale checked, click OK to create a data set object with three data points, nicely scaled. Now, to illustrate the two angles we are studying, use Enter equation and plot $x = 31$ and $x = 45$ (see **fig. 8**).

The data are not quite linear, so let's try to fit a quadratic to them. Select the data (click on any one of the three points) and use the right-click option Best Fit with Order = 2. Click OK. To display the equation, select the quadratic and use the right-click option Text Box, obtaining Quadratic: {{y=–9.877E-005x^2–0.07333x+14.7}}. Note the very small coefficient of x^2. The section in

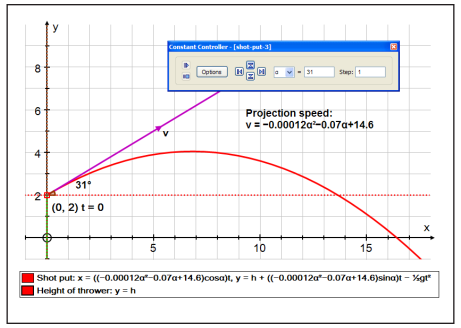


Fig. 10 The distance along the ground can be maximized when the angle of elevation is 31°.

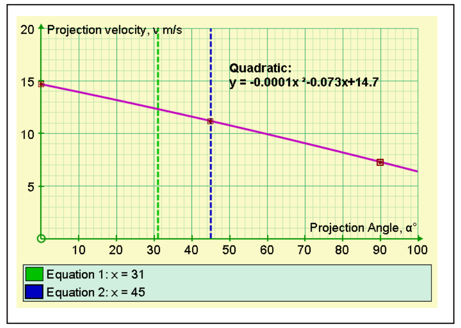


Fig. 8 The quadratic model has a lead coefficient very close to 0.

red is dynamic information, which will change if the data change or are dragged about. Use the option convert to static text and take the opportunity to tidy it up to $y = -0.0001x^2 - 0.073x + 14.7$.

We notice that as the angle drops from 45° to 31°, the possible projection velocity increases. Is this enough to improve performance at the lower angle, and is 31° the best angle? (Explore the Axes => Edit Axes dialogue box to manipulate the various style settings for the graph.)

To copy this relationship to the clipboard, double-click on the text box and select $-0.0001x^2 - 0.073x + 14.7$ and copy (Ctrl-C). Click Cancel and return to the first graph page with the shot put on it.

Double-click on the original equation—x=($v$cosα)t, y=h+($v$sinα) t–½gt^2—to open up its equation entry box. Two instances of v need converting to $-0.0001x^2 - 0.073x + 14.7$, but with x changed to α. Select v and replace with (); then paste (Ctrl-V) the formula from the clipboard in between the parentheses. Change the four x characters to α. Press OK to plot.

Now, with only one parameter, α, to vary, we can really explore what's going on. Use the Constant Controller to set α

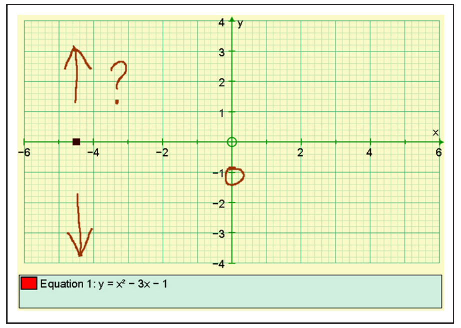


Fig. 11 Pausing the graphing of the quadratic allows for discussion of what the graph will look like.

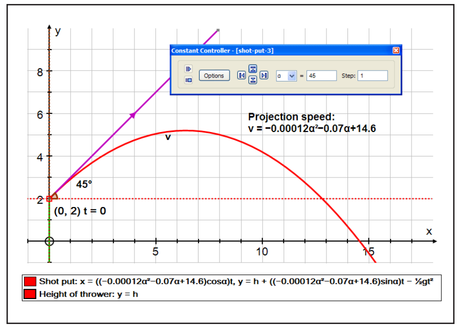


Fig. 9 Using an angle of elevation of 45° produces a put of about 14.5 m.

to 45°, noting that the range along the ground is 14.6m (see **fig. 9**). Varying α then reveals the greatest range is achieved at around 31°—the angle that most Olympic shot-putters use (see **fig. 10**).

COMPETING THE SQUARE

Visualizing possibilities can help students learn about the various factored forms for a quadratic. On a new 2D page, be sure to press Slow Plot (indicated by the turtle icon) to enable stop-start plotting and also press the Whiteboard Mode button (indicated by the white screen icon) to improve the visibility of the graphing.

Whiteboard Mode should be set up so that it is unnecessary to press Shift to build up multiple selections and the onscreen keyboard is present. To do so, go to View => Preferences => Whiteboard and make sure that all four options are checked. Press OK.

Use Enter Equation and type $y = xx - 3x - 1$ to enter $y = x^2 - 3x - 1$ (enter xx for x^2) and press OK. As soon as the plotting starts (because the graph is off screen, we see a black dot travelling along the x-axis), press the pause button (or the space bar) (see **fig. 11**). With the plotting paused, we can ask students what they

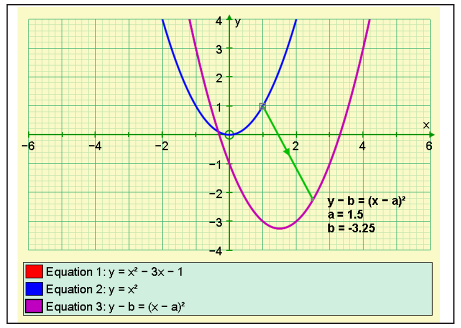


Fig. 12 Adding a vector to the graph clearly demonstrates the translation of the curve.

know about the graph of this function. The following answers are correct:

- x^2 means it will be a happy quadratic—one that opens upward
- $-3x$ will pull it up on the left and down on the right
- -1 means it intercepts the y axis at $(0, -1)$

Discuss these answers with the students.

Are we ready to take the brakes off?

Plot $y = x^2$, the father of all parabolas. We will now try to transform $y = x^2$ to $y = x^2 - 3x - 1$ by using the form $y = (a - x)^2 + b$ or, if preferred, the symmetrical form: $y - b = (x - a)^2$. Enter one of these forms in the Equation Entry dialog box, but before leaving the box, use Edit Constants to set both a and b to zero and press OK (twice). Doing so will ensure that the new graph initially draws right on top of $y = x^2$.

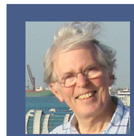
Then we can use the Constant Controller to vary a and b. To be sure that the students understand the process, be sure to discuss what will happen before a button is pressed. For example, what will happen to the graph if a is increased from 0 to 1. Will the graph move right, left, up or down, and why?

The dynamic nature of the Constant Controller comes into play here, allowing us to adjust the steps as we home in on the final values of a and b, when the new graph sits right on top of the original quadratic. Note that the values are $a = 1.5$ and $b = -3.25$.

Finally, using the Point mode, move the mouse over $y = x^2$ and click; doing so will place a selected point on the curve. Now use the right-click option Vector to add a vector to that point (see **fig. 12**).

Change the default vector components from [1, 1] to $[a, b]$. Notice that the vector stretches exactly to the transformed quadratic. If you drag the point around $y = x^2$, it is clear that all points on $y = x^2$ are translated by this vector $[a, b]$ to the new graph.

Editor's note: All the activities described in this article have been recorded as Jing videos and are available on the special YouTube channel: www.youtube.com/nctmbutler.



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