From FLATLAND to the FOURTH DIMENSION.

(This article does not involve any advanced mathematics. It only uses the Real Numbers and it was written so that a senior secondary student could follow it with ease.)

See video: _https://www.screencast.com/t/cQOsIqbk

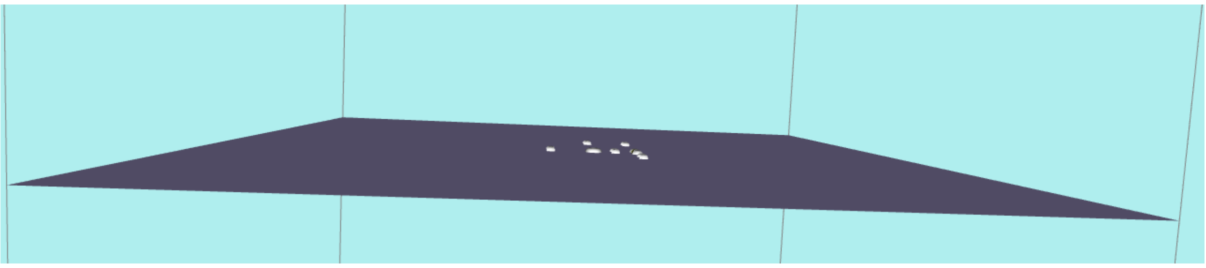
Edward Abbott Abbott published a book in 1884 called FLATLAND which was a story about beings which exist in two dimensions.

These creatures are free to move **forwards** and **backward**s and from **left** to **right** but they have no concept of "**up**" or "**down**".

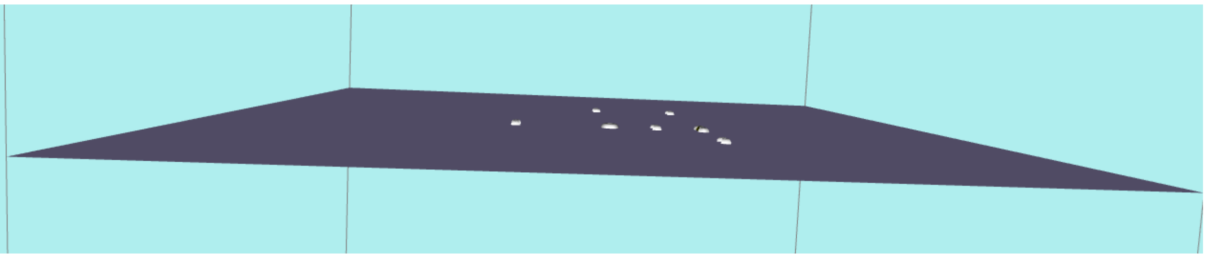
In this world, there is no such thing as "height" so strictly speaking we should say they lived "in" a plane surface rather than "on" the surface.

These two diagrams show the creatures have moved from position A to B.

Position A



Position B



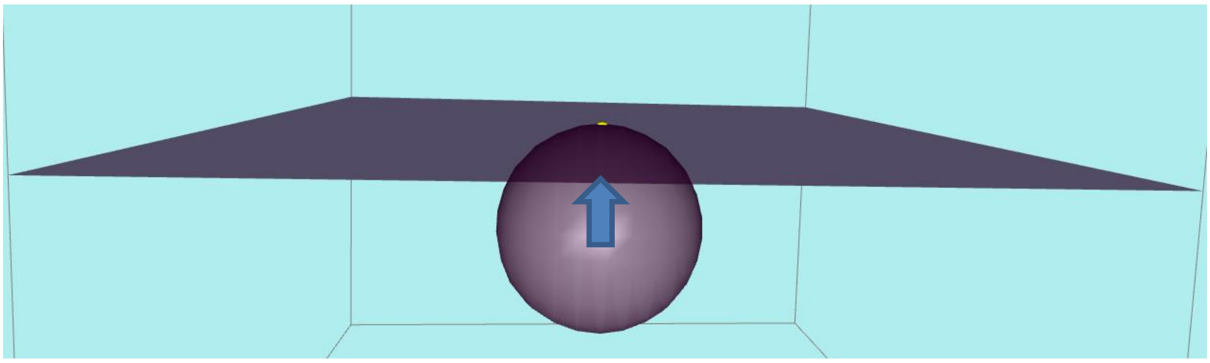
Suppose we humans could observe these 2D creatures.

They would not be aware of being observed because they have no concept of up or down.

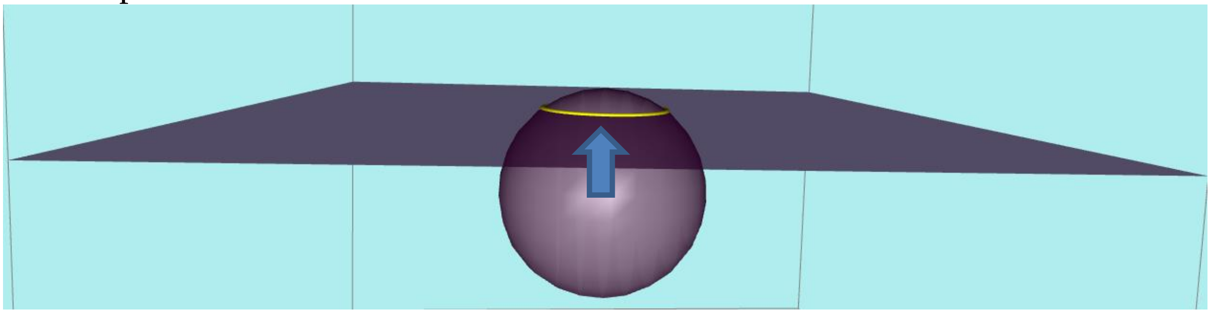
They could only become aware if we did something **in** their plane.

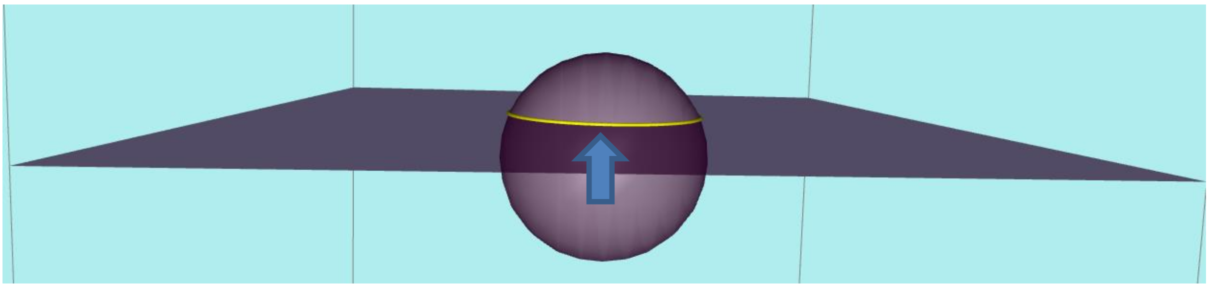
Imagine that we could pass a sphere through their plane.

As soon as the sphere touched the plane, they would be aware of ONE point as shown below:

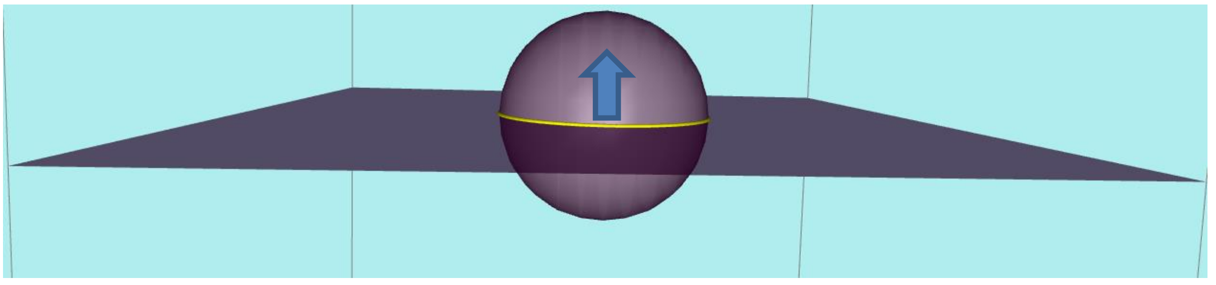


As the sphere rises, they would only detect that a circle was increasing in radius in their plane as below:

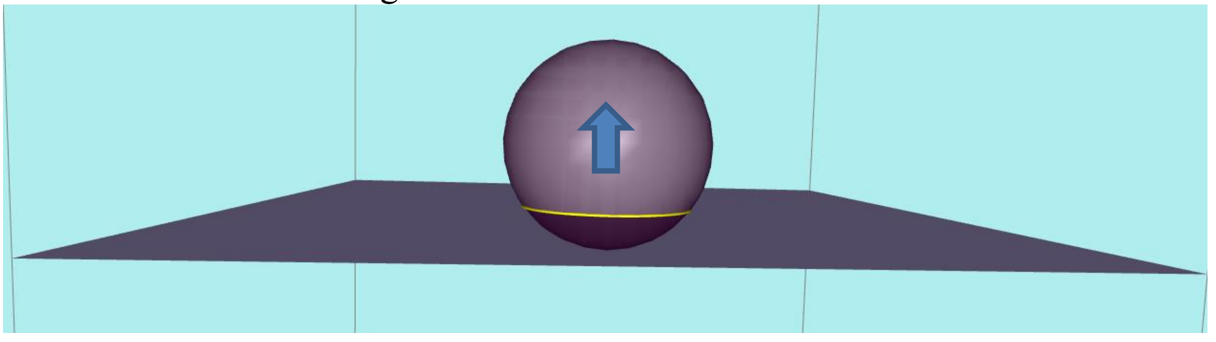




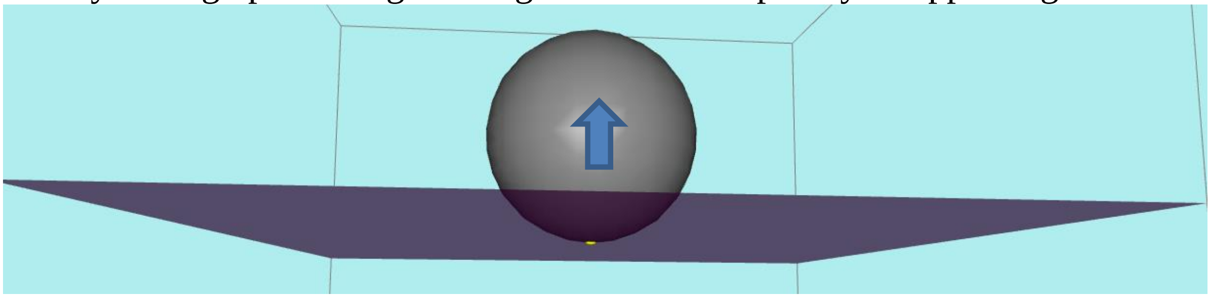
The circle would reach a maximum radius…



…then start to decrease again:



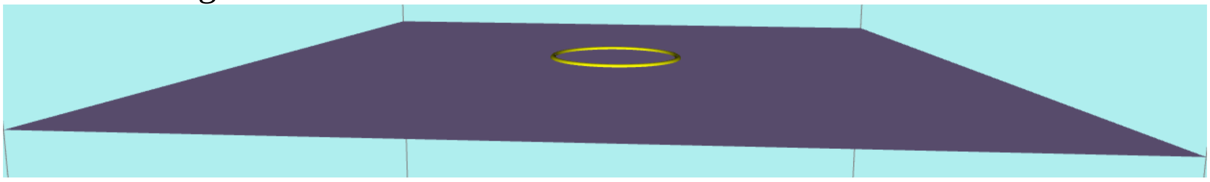
Finally ending up as a single dot again before completely disappearing:



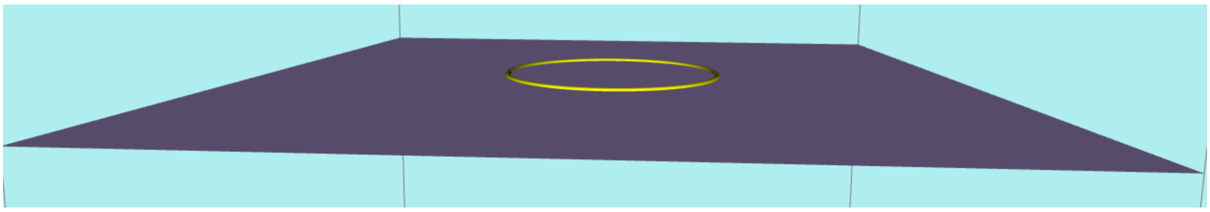
The creatures would not know anything about a SPHERE passing through their plane!
They would only be aware of a mysterious dot appearing; changing into a circle (whose radius increases then decreases again); becoming a dot again and finally disappearing.

First a dot appears:

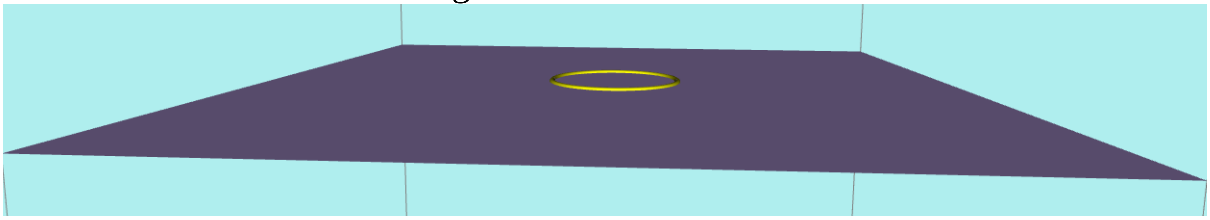
The dot changes into a circle and its radius increases:



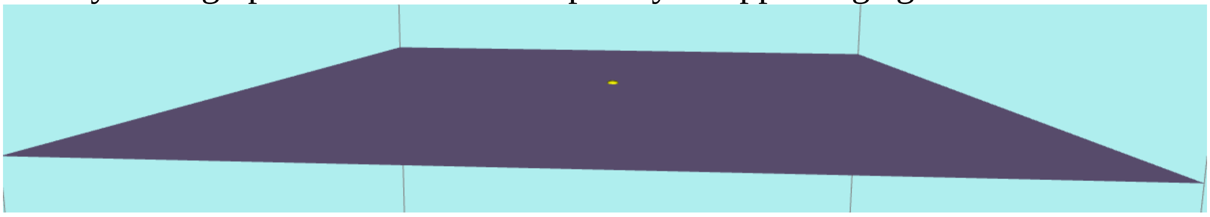
The radius reaches a maximum value:



The radius starts to decrease again:



Finally ending up as a dot before completely disappearing again:



The creatures could have no concept of what caused this phenomenon to occur. It would be a case of some “**superior 3D beings**” being able to observe these “**2D creatures**” and being able to cause **mysterious things** to happen in their “**2D world**”!

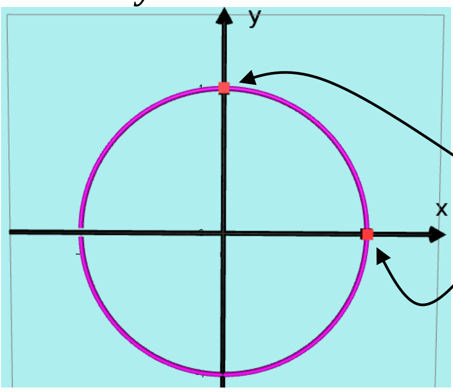
Now consider these 4 equations, each requiring an extra dimension.
I am only using very simple numbers so no calculators are needed.

1. $\boldsymbol{x^2 = 1}$

   This needs just a number line (ie 1 dimension)

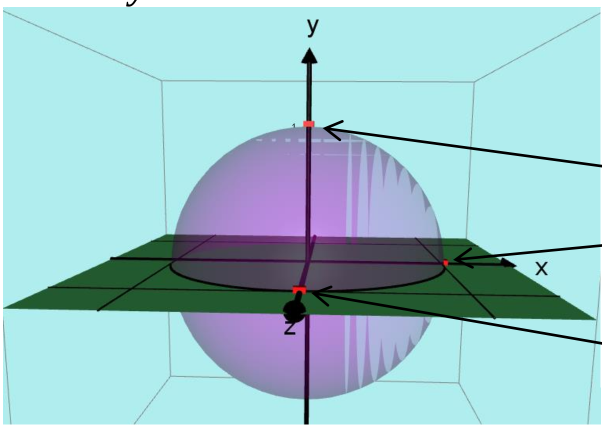
The only solutions are of course $\boldsymbol{x} = 1$ and $\boldsymbol{x} = –1$

2. $\boldsymbol{x^2 + y^2 = 1}$ This needs 2 dimensions because it is a circle graph:



All points on the circle are solutions of
$\boldsymbol{x^2 + y^2 = 1}$
I have just marked two particular points
(0, 1)
and
(1, 0) on this diagram.

3. $\boldsymbol{x^2 + y^2 + z^2 = 1}$ This needs 3 dimensions because it is a sphere.



All points on the sphere are solutions of
$\boldsymbol{x^2 + y^2 + z^2 = 1}$
I have just marked three particular points
(0, 1, 0)

(1, 0, 0)

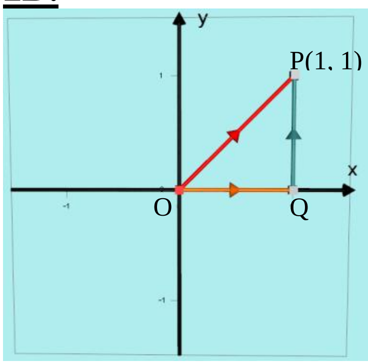
(0, 0, 1)

4. $\boldsymbol{x^2 + y^2 + z^2 + w^2 = 1}$ This needs 4 dimensions so I cannot draw it!
Points in 4D would be of the form ***(a, b, c, d)*** so whatever 4D shape this represents, the following “points” satisfy this equation.
(1, 0, 0, 0)
(0, 1, 0, 0)
(0, 0, 1, 0)
(0, 0, 0, 1)
Notice (**½, ½, ½, ½**) also satisfies the equation because
$(½)^2 + (½)^2 + (½)^2 + (½)^2 = ¼ + ¼ + ¼ + ¼ = \mathbf{1}$

Also notice that we can have **4D points** even though we cannot actually plot them in 4D space.

<u>Now consider DISTANCES in 2, 3 and 4 dimensions:</u>

<u>2D.</u>



By Pythagoras' Theorem:

$OP^2 = OQ^2 + PQ^2$

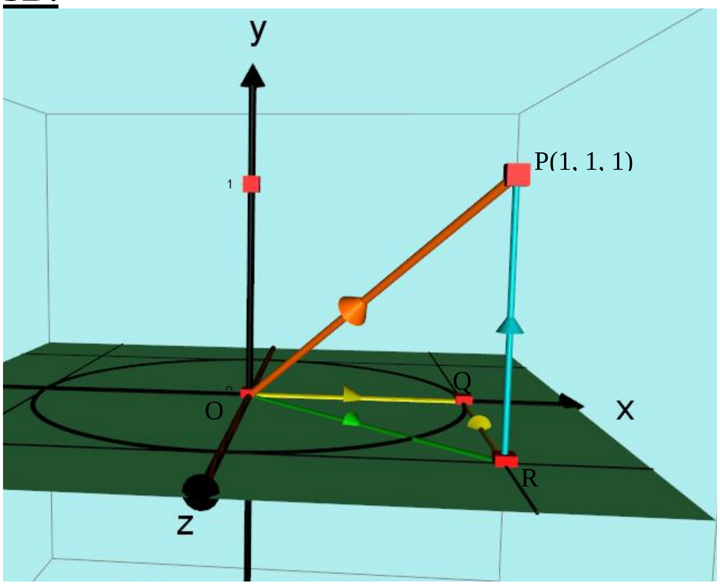
$= 1^2 + 1^2$

$= 2$

So OP = √2 cm

NB If coordinates of P were (***a, b***)
then OP = √($\boldsymbol{a^2 + b^2}$)

<u>3D.</u>



$OR^2 = OQ^2 + QR^2$

$= 1^2 + 1^2$

OR = √2

$OP^2 = OR^2 + QR^2$

$= 2 + 1$

OP = √3

Notice that:
So $OP^2 = 1^2 + 1^2 + 1^2$

OP = √3

So if coordinates of P were (***a, b, c***) then
OP = √($\boldsymbol{a^2 + b^2 + c^2}$)

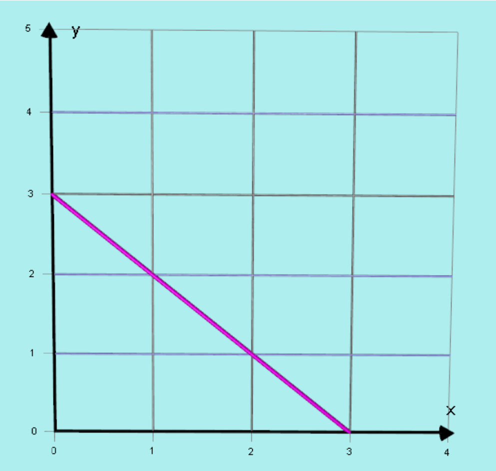
<u>4D.</u>

Obviously we can't draw this but following the logic above we could say that if P had coordinates (***a, b, c, d***) then OP = √($\boldsymbol{a^2 + b^2 + c^2 + d^2}$)

So if the coordinates of P were (1, 1, 1, 1)
then OP = √($\boldsymbol{1^2 + 1^2 + 1^2 + 1^2}$) = √4 = 2 cm

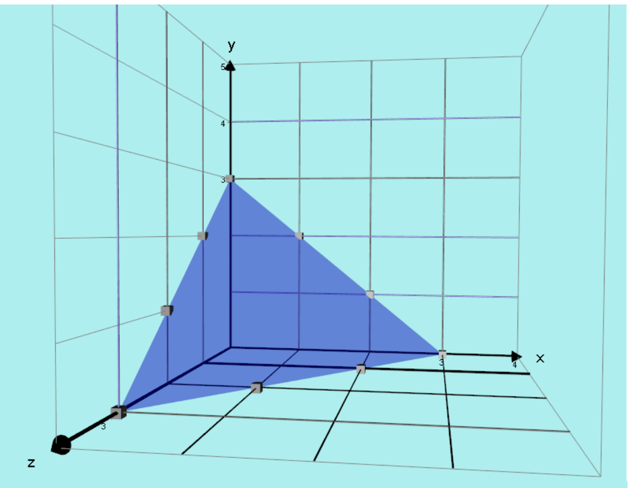
I must say that this seems very strange finding distances in a 4 dimensional space!

Consider the equation ***x + y = 3 in 2D***



Clearly ***x + y = 3*** is a **one** Dimensional LINE in 2 Dimensional space.

Now consider the equation ***x + y + z = 3 in 3D***



Clearly ***x + y + z = 3*** is a **TWO** dimensional PLANE in 3 Dimensional space.

Finally consider the equation *x + y + z + w = 3 in 4D*

We can't draw this but *x + y + z + w = 3* would be a THREE Dimensional shape in 4 Dimensional space but what shape could it be?

I want to go back to the idea of the sphere passing through Flatland.

I will let the 3D sphere have the equation $x^2 + y^2 + (z - c)^2 = 25$
The creatures in Flatland only know about coordinates x and y and when the sphere passes through Flatland the creatures can only see circles which have equations in x and y only. This is when $z = 0$
If c = 5 we have $x^2 + y^2 + 25 = 25$ or $x^2 + y^2 = 0$ which is a circle of radius 0 or just the point (0, 0).
If c = ±4 we have $x^2 + y^2 + 16 = 25$ or $x^2 + y^2 = 9$ which is a circle of radius 3
If c = ±3 we have $x^2 + y^2 + 9 = 25$ or $x^2 + y^2 = 16$ which is a circle of radius 4
If c = 0 we have $x^2 + y^2 = 25$ which has the largest radius of 5.

Using this same idea, I want to consider the 4 Dimensional shape whose equation is $x^2 + y^2 + z^2 + w^2 = 25$ moving through our 3D space.
(I will not call it a "sphere" or "super sphere" or "hyper sphere" because $x^2 + y^2 = r^2$ *was a* ***circle*** *and we don't call* $x^2 + y^2 + z^2 = r^2$ *a "****super circle****".)*
Firstly consider points on this 4D shape:
The obvious ones are:
(5, 0, 0, 0)
(0, 5, 0, 0)
(0, 0, 5, 0)
(0, 0, 0, 5)
So each one of these points is at a "distance" of 5 cm from the origin.

25 is a very good number to use because we can find several whole numbers to fit the equation. For example:
(3, 4, 0, 0)
(0, 3, 4, 0)
(0, 0, 3, 4)
(3, 0, 4, 0)
(3, 0, 0, 4)
etc and not forgetting ±3 and ±4 in all cases.

Other ideas consist of combinations of ±4, ±2, ±2, ±1 because
$4^2 + 2^2 + 2^2 + 1^2 = 25$ in all the combinations.

Recall that to move the sphere through **Flatland** I changed the equation to
$x^2 + y^2 + (z - c)^2 = 25$ and varied the value of c to move the sphere up and down the z axis through the x, y plane of Flatland.
I said that the sphere will contact Flatland when $z = 0$

So I will move the 4D shape $x^2 + y^2 + z^2 + w^2 = 25$ along the ***w* axis** by changing the equation to $x^2 + y^2 + z^2 + (w - c)^2 = 25$ ***and vary c between –5 and +5.***
This 4D shape will intersect with our 3D space when $w = 0$
That is when $x^2 + y^2 + z^2 + (0 - c)^2 = 25$

If $c = \pm 5$ ***then*** $x^2 + y^2 + z^2 + 25 = 25$ ***so*** $x^2 + y^2 + z^2 = 0$ ***which is just the point*** $(0, 0, 0)$
If $c = \pm 4$ ***then*** $x^2 + y^2 + z^2 + 16 = 25$ ***so*** $x^2 + y^2 + z^2 = 9$ ***which is a sphere of radius 3***
If $c = \pm 3$ ***then*** $x^2 + y^2 + z^2 + 9 = 25$ ***so*** $x^2 + y^2 + z^2 = 16$ ***which is a sphere of radius 4***
This means that as the 4D shape (*whatever it is!*) moves along the 4th dimensional ***w*** axis we will see spheres all centred at (0, 0, 0) starting with a zero radius which increases to a maximum value of 5 cm then decreases.

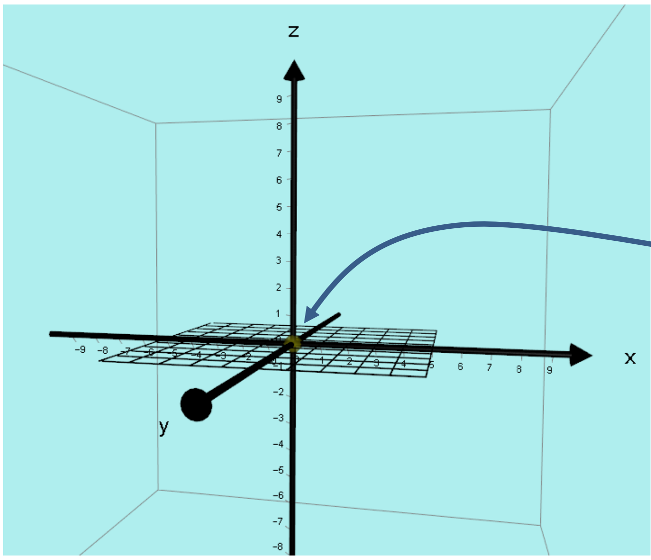


FIG 1
c = – 5 and the 4D shape has just become visible in our 3D space as the point (0, 0, 0)

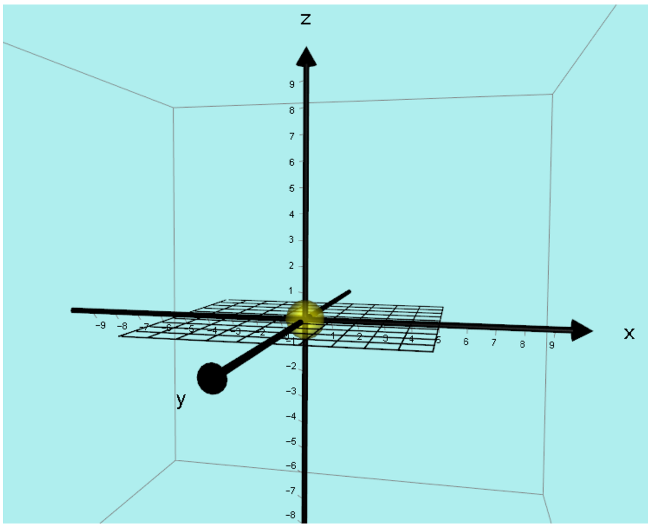


FIG 2
$c \approx -4.9$ and the 4D shape has changed into a sphere in 3D space.

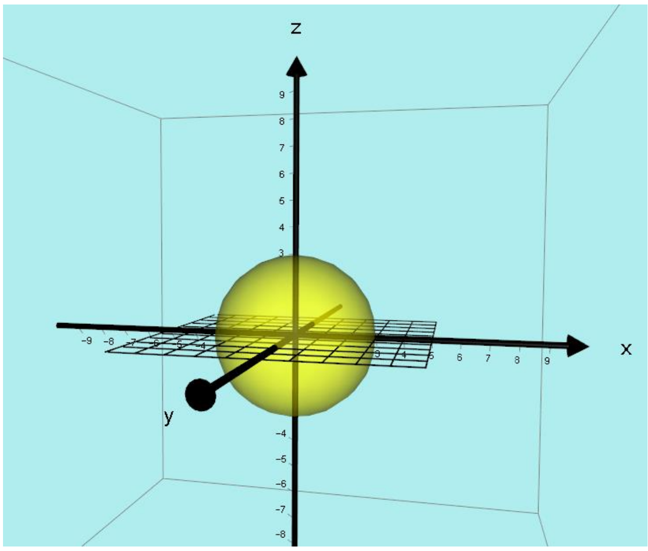


FIG 3
c = –4 and the 3D projection of the 4D shape has increased its radius to 3cm.

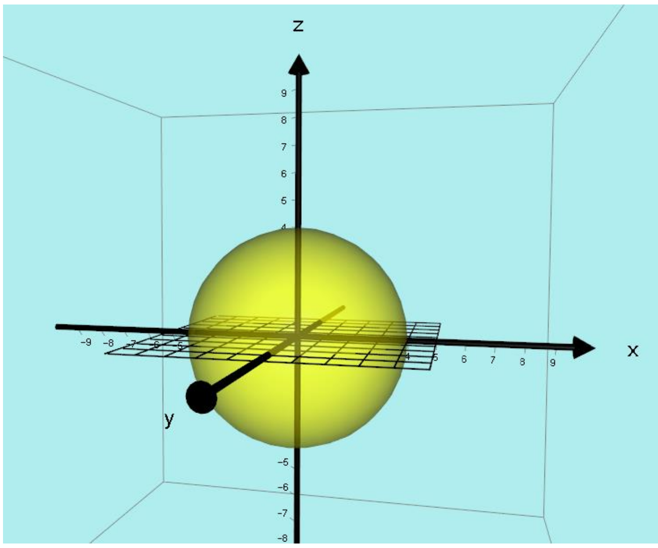


FIG 4
c = –3 and the 3D projection of the 4D shape has increased its radius to 4cm.

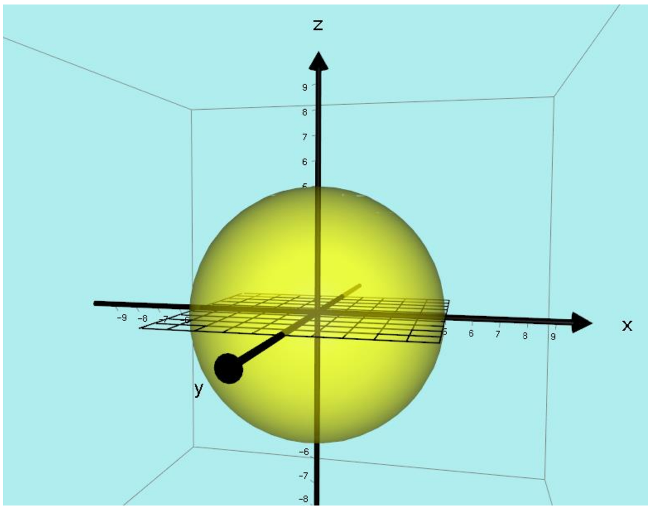


FIG 5
c = 0 and the 3D projection of the 4D shape has reached its maximum radius of 5cm.

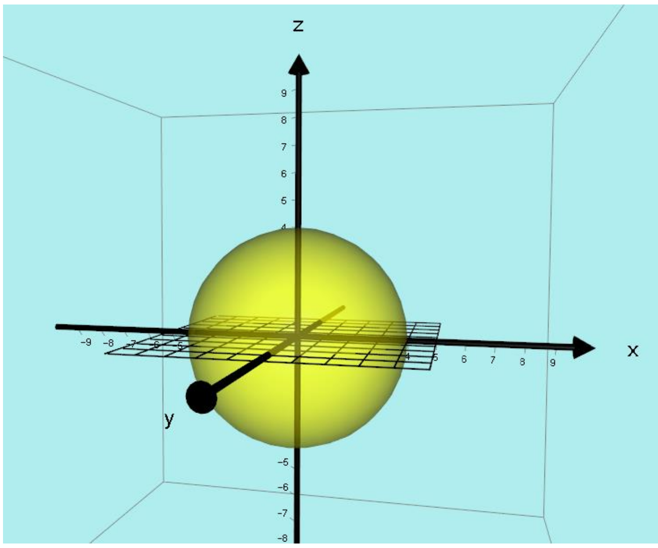


FIG 6
c = +3 and the 3D projection of the 4D shape has decreased its radius to 4cm.

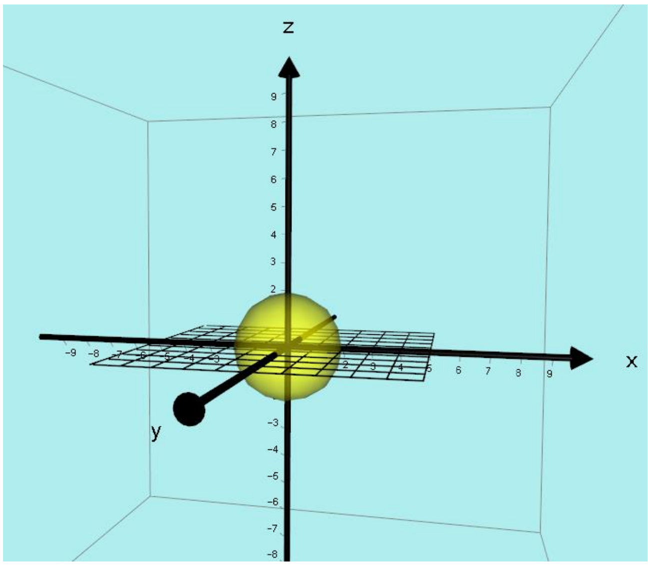


FIG 7
c ≈ +4.6 and the 3D projection of the 4D shape has decreased its radius to approximately 2cm.

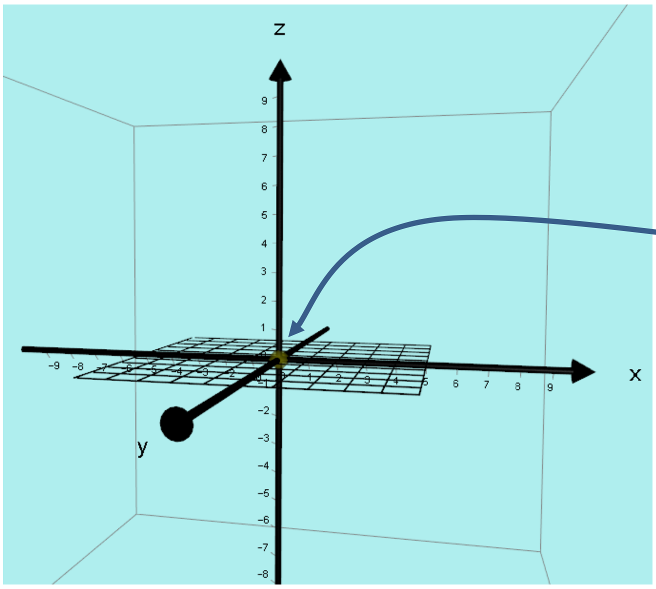


FIG 8
c = +5 and the 3D projection of the 4D shape has decreased its radius to 0cm which is the point (0, 0, 0)