

Geared Up: Exploring Families of Curves with Autograph

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One of the many benefits of using an ICT package such as Autograph to explore graphs in mathematics is the fact that learners get instant, visual feedback. Graphical outputs have a different quality to discrete 'right' or 'wrong' comments.

They can suggest to the learner how close they are to achieving their intended goal, and allow them to work in a step-by-step way towards a solution. In this way the software package can enable pattern spotting, generalisation, and even the occasional serendipitous discovery. It also opens the door to the discovery and manipulation of some unusual families of curves.

Recently we conducted an activity with the mathematics PGCE students at the University of Warwick which focused on two such sets of curves. The exercise was particularly successful, with the students producing some insightful pieces of work. At each stage Autograph enabled the students to get to grips with the underlying mathematics in a way that would have otherwise been impossible.

The first set of curves was the **gear curves**. This is a particularly unusual set of curves which closely resemble gears. One form of gear curve can be drawn using the parametric equations:

$$\begin{aligned}x &= (1 + 0.1 \tanh(10 \sin nt)) \cos t \\y &= (1 + 0.1 \tanh(10 \sin nt)) \sin t\end{aligned}$$

where n is a constant and t is a parameter ranging from 0 to 2π radians. (Although the expressions involved can be a little intimidating for learners who have not yet met the hyperbolic tangent, no knowledge of this function is required for what follows, and it can be validly considered as a 'black box'.)

Fig 1 shows this curve when $n=5$.

The Unit circle is entered as
 $x = \cos t, y = \sin t$

The basic gear is entered as
 $x = (1 + 0.1 \tanh(10 \sin nt)) \cos t, y = (1 + 0.1 \tanh(10 \sin nt)) \sin t$

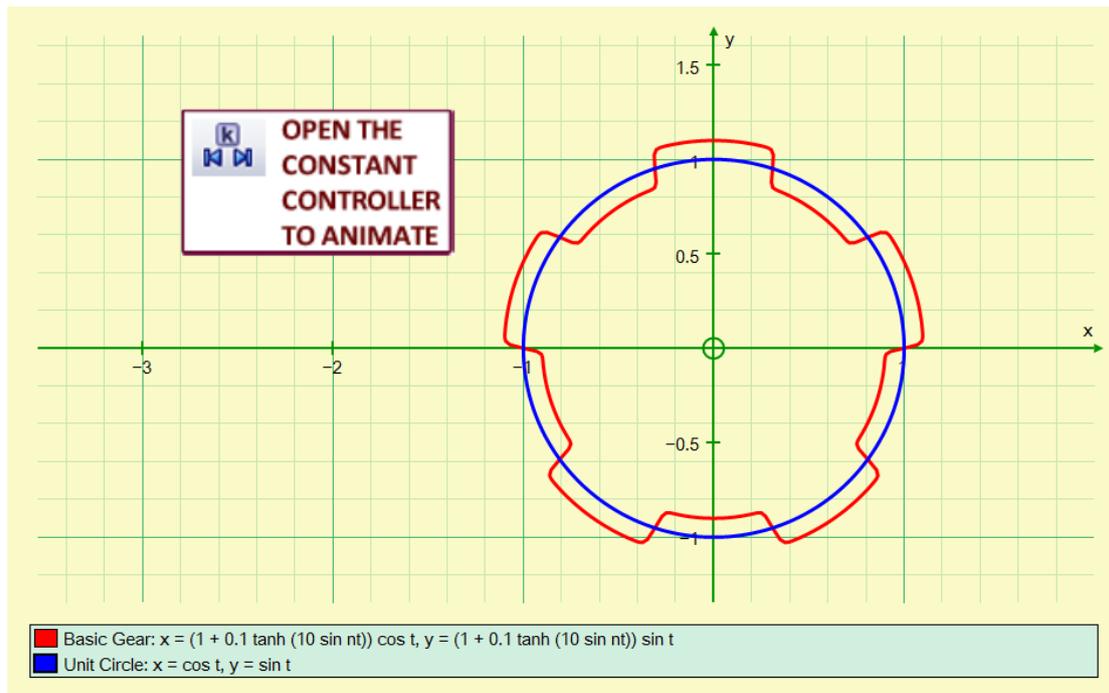


Fig 1: a simple gear with $n = 5$ [Autograph file](#)

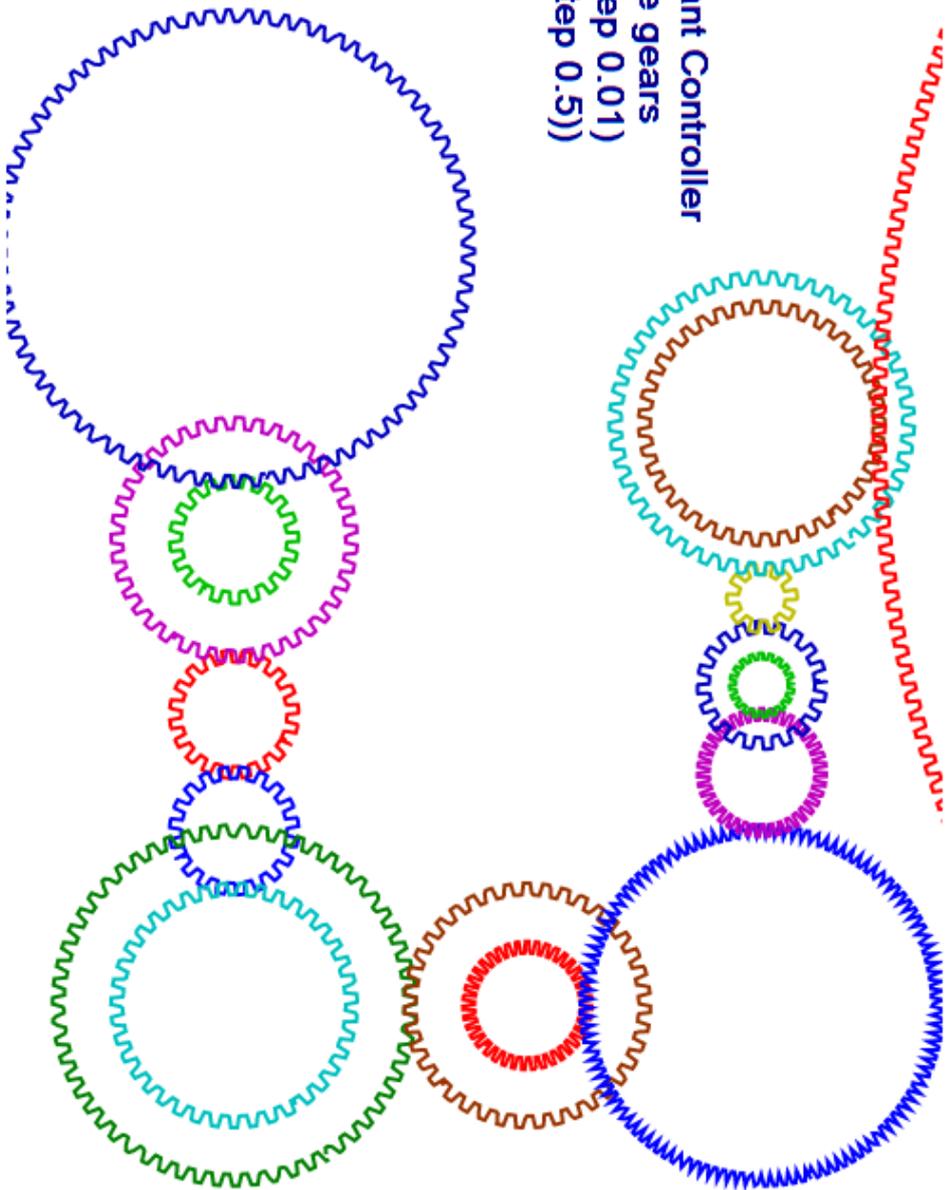
The most obvious starting point for investigation was to vary the parameter n and see what happened. In actual fact, n is equal to the number of teeth on the gear, and the students were soon experimenting with gears that had hundreds of teeth. The next challenge was to create two interlocking gears.

This challenge required the students to recognise how the Cartesian function transformations taught at the top end of GCSE mathematics could be generalised to parametric equations. The third challenge was to devise a way to enlarge the curve, and to make three gears of different sizes that were all touching.

The opportunities afforded by Autograph meant that all of the students were able to construct impressive looking systems, including those who were less comfortable with parametric equations.

Naturally, many of the students went above and beyond our expectations, and some devised a way to animate the gears. This in turn led to further problems that needed to be overcome; how could a single animated parameter be made to turn some gears clockwise, and others anticlockwise? An exemplary piece of work produced by Phil Crockford, a former PGCE student, is offered here as an example of how one student tackled the task.

Click on Constant Controller
to ANIMATE the gears
n (-20 to 20, step 0.01)
and k (0 to 2, step 0.5))



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Equation 1: x=(1 + 0.1tanh(10 sin ((n)(t))))cos(t+kπ), y=(1 + 0.1tanh(10 sin ((n)(t))))sin(t+kπ)
Equation 2: x=(1 + 0.1tanh(10 sin ((n)(t))))cos(t-kπ)+2, y=(1 + 0.1tanh(10 sin ((n)(t))))sin(t-kπ)
Equation 3: x=2(1 + 0.05tanh(10 sin ((2n)(t))))cos(t-(kπ/2))-3, y=2(1 + 0.05tanh(10 sin ((2n)(t))))sin(t-(kπ/2))

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Fig 2 Multiple interlocking gears [Autograph file](#)

Rose Curves

The second curve family that we explored was the **rose curves**. These are a family of curves that were named by the Italian mathematician Guido Grandi in the 18th Century. The polar equation of these curves is $r = \cos(k\theta)$, where k is a constant.

The value of k is particularly critical for this family of curves, as different types of number lead to different types of curve. When k is an odd integer, the 'flower' formed has k petals. However, when k is an even integer, the 'flower' has $2k$ petals. Particularly interesting things happen when k is a fraction, or when k is irrational.

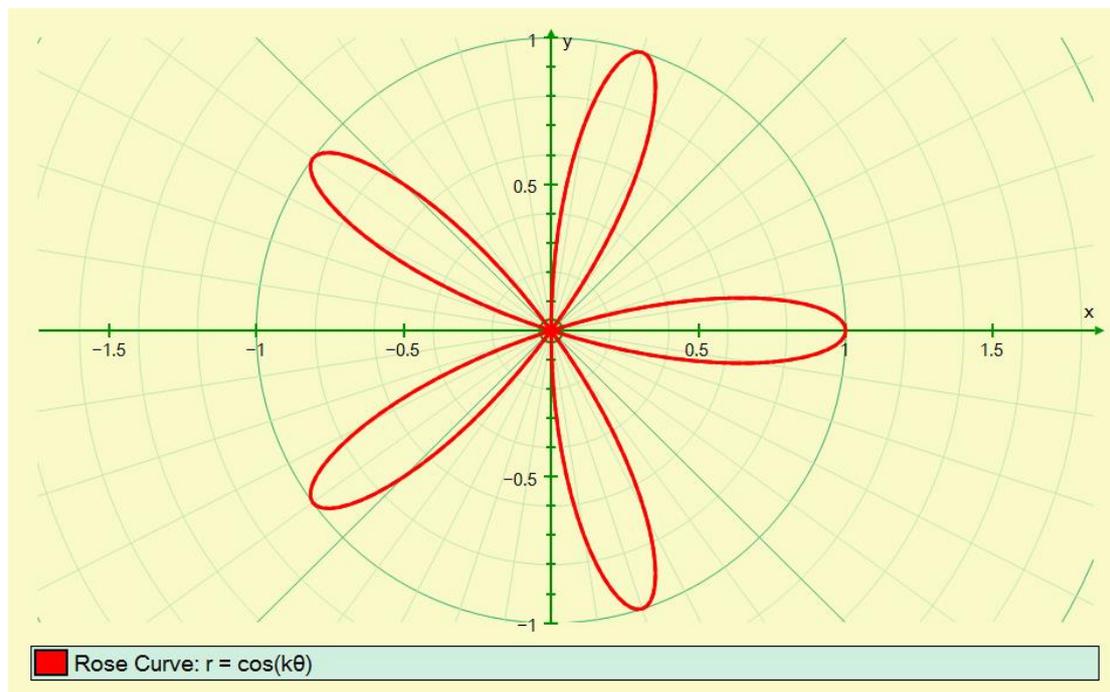


Fig 3: A Rose Curve where $k=\pi$ [Autograph file](#)

The rose curves can give rise to some interesting pattern spotting exercises, and the slow plot function of Autograph can help learners recognise why these patterns come about.

Graphical exploration can also give rise to questions which require a more algebraic approach; for example, what happens to the area of the rose curve as k changes? In fact, the area has one fixed value whenever k is an odd integer, and another fixed value when k is even; it is left as an exercise for the reader to determine these values!

In each of these cases, many students developed their understanding of the co-ordinate system being used through exploration. In addition to this, the use of the graphing package allowed the students to meet curves beyond the standard set of 'quadratic, cubic and reciprocal'. The potential for play is an important quality here, one that can help get learners of all age geared up for some motivating mathematics.

Equations for the multiple gears:

Equation 1:

$$x=(1+0.1\tanh(10\sin((n)(t))))\cos(t+k\pi), y=(1+0.1\tanh(10\sin((n)(t))))\sin(t+k\pi)$$

Equation 2:

$$x=(1+0.1\tanh(10\sin((n)(t))))\cos(t-k\pi)+2, y=(1+0.1\tanh(10\sin((n)(t))))\sin(t-k\pi)$$

Equation 3:

$$x=2(1+0.05\tanh(10\sin((2n)(t))))\cos(t-(k\pi/2))-3, y=2(1+0.05\tanh(10\sin((2n)(t))))\sin(t-(k\pi/2))$$

Equation 4:

$$x=(1+0.1\tanh(10\sin((n)(t))))\cos(t-(\pi k/2))-3, y=(1+0.1\tanh(10\sin((n)(t))))\sin(t-(k\pi/2))$$

Equation 5:

$$x=4(1+0.025\tanh(10\sin((4n)(t))))\cos(t+(k\pi/8))-8, y=4(1+0.025\tanh(10\sin((4n)(t))))\sin(t+(k\pi/8))$$

Equation 7:

$$x=2(1+0.05\tanh(10\sin((2n)(t))))\cos(t+(k\pi/2))+5, y=2(1+0.05\tanh(10\sin((2n)(t))))\sin(t+(k\pi/2))$$

Equation 8:

$$x=3(1+0.1(1/3)\tanh(10\sin((3n)(t))))\cos(t+(k\pi/2))+5, y=3(1+0.1(1/3)\tanh(10\sin((3n)(t))))\sin(t+(k\pi/2))$$

Equation 9:

$$x=2(1+0.05\tanh(10\sin((2n)(t))))\cos(t-(3k\pi/4))+5, y=2(1+0.05\tanh(10\sin((2n)(t))))\sin(t-(3k\pi/4))+5$$

Equation 10:

$$x=(1+0.1\tanh(10\sin((2n)(t))))\cos(t-(3k\pi/4))+5, y=(1+0.1\tanh(10\sin((2n)(t))))\sin(t-(3k\pi/4))+5$$

Equation 11:

$$x=3(1+0.1(1/3)\tanh(10\sin((6n)(t))))\cos(t+(k\pi/4))+5, y=3(1+0.1(1/3)\tanh(10\sin((6n)(t))))\sin(t+(k\pi/4))+9$$

Equation 12:

$$x=(1+0.1\tanh(10\sin((2n)(t))))\cos(t-(3k\pi/4))+1, y=(1+0.1\tanh(10\sin((2n)(t))))\sin(t-(3k\pi/4))+9$$

Equation 13:

$$x=0.5(1+0.1\tanh(10\sin((n)(t))))\cos(t+(3k\pi/2))-0.5, y=0.5(1+0.1\tanh(10\sin((n)(t))))\sin(t+(3k\pi/2))+9$$

Equation 14:

$$x=(1+0.1\tanh(10\sin((n)(t))))\cos(t+(3k\pi/2))-0.5, y=(1+0.1\tanh(10\sin((n)(t))))\sin(t+(3k\pi/2))+9$$

Equation 15:

$$x=0.5(1+0.2\tanh(10\sin((n/2)(t))))\cos(t-(3k\pi))-2, y=0.5(1+0.2\tanh(10\sin((n/2)(t))))\sin(t-(3k\pi))+9$$

Equation 16:

$$x=2.5(1+0.04\tanh(10\sin((5n/2)(t))))\cos(t+(3k\pi/5))-5, y=2.5(1+0.04\tanh(10\sin((5n/2)(t))))\sin(t+(3k\pi/5))+9$$

Equation 17:

$$x=2(1+0.05\tanh(10\sin((2n)(t))))\cos(t+(3k\pi/5))-5, y=2(1+0.05\tanh(10\sin((2n)(t))))\sin(t+(3k\pi/5))+9$$

Equation 18:

$$x=20(1+0.005\tanh(10\sin((20n)(t))))\cos(t-(3k\pi/50))-5, y=20(1+0.005\tanh(10\sin((20n)(t))))\sin(t-(3k\pi/50))+31$$