

# SEALING AN UNDERSTANDING OF DIFFERENTIATION AND INTEGRATION, BY VISUALISATION

Douglas Butler explores the geometry of calculus using *Autograph*

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## Introduction

It is well established now that students can enhance their understanding of concepts by watching *and controlling* animations on screen. This article, and the associated web activities, will be using *Autograph*, but it should be possible to follow this approach in other graphing tools.

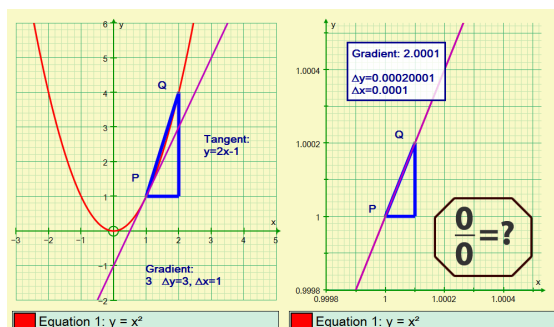
### 1. Differentiation: $0 \div 0$ can $\Rightarrow 2$

This unlikely, but fundamentally important result can be explored using the classical approach of the gradient of a chord PQ as Q tends to P.

Draw the parabola  $y = x^2$  and place points P (1, 1) and Q (2, 4) on it. Draw the gradient triangle and display the gradient = 3 together with  $\Delta x = 1$  and  $\Delta y = 3$ . Also draw the tangent at P, noting that its gradient = 2.

This is the moment where dynamic software can really add new understanding. Move Q closer and closer to P, noting the decreasing gradient of the chord, and zoom in. After a few cycles of this, the all-important local straightness of the parabola is evident, and the chord gradient may have reached 0.000040001/0.00002.

You could have a million zeros before the '4' and a million zeros before the '2', and the result can still approach 2. If the numerator and denominator both tend towards zero, the fraction can be finite. We also see that the limit of the gradient of a chord PQ is the gradient of the tangent at P.



## 2. Max, Min and Point of Inflexion

A useful function to explore here is the cubic  $y = x^3 - 3x - 1$  as it has turning points at nice values of x, at  $x = -1, 0$  and  $+1$ . Add a point anywhere on the graph and add a tangent to that point.

Move it to the local maximum at  $x = -1$  (with zero gradient), and mark  $(-1, 0)$  as one point on the first derivative.

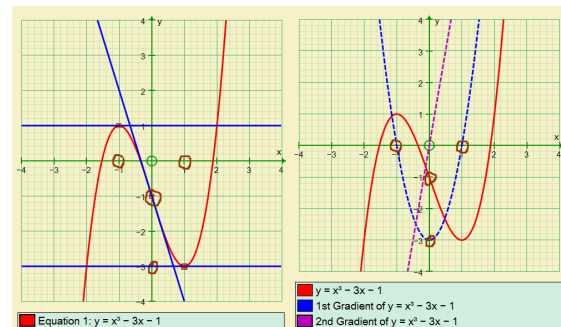
Move it to  $(0, -1)$ . Note this is a very special point where the tangent comes in one side and out the other. This is referred to as a point of inflexion. Note also this is the point of minimum gradient of the tangent  $(-3)$ . Mark  $(0, -3)$  as the second point on the first derivative.

Move to the local minimum at  $x = +1$ , and mark  $(1, 0)$  as the third point on the first derivative.

With 'Slow plot' on, use the 'Gradient Function' option to confirm that the first derivative is parabolic and includes the three points above. Enter  $y = ax^2 - 3$  and use the constant controller to show that the first derivative is when  $a = 3$ . Repeat this to confirm that the second derivative is  $6x$ .

A further extension:

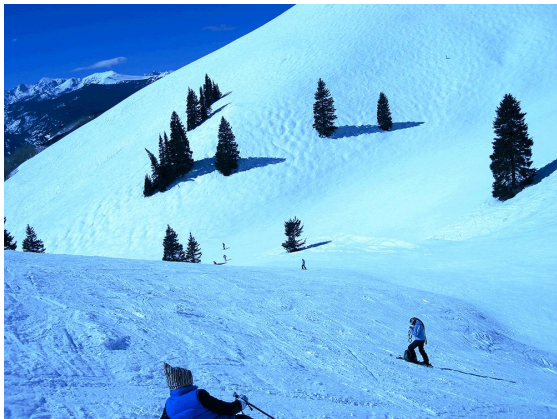
Delete  $y = ax^2 - 3$ , and edit  $y = x^3 - 3x - 1$  to be  $y = x^3 - bx - 1$  with 'b' initially set to 3. Vary 'b' using the constant controller and observe the impact on the first and second derivatives.



### 3. A nice extension to 3D

It is useful to encourage students to transfer these principles to 3D. In 2D the gradient of a curve at a point is measured by the gradient of a tangent at that point.

On a hillside how would you measure how steep it is?



On a 3D page in *Autograph*, a good 3D function to explore is  $z = \text{asinxcosy}$ .

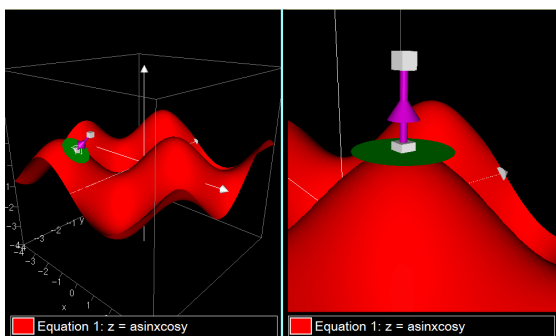
Can you predict what it will look like? Think of slices of constant 'x' then constant 'y'. Does it remind you of anything? An egg box? A ski-slope?

The slope of a 3D surface at a point is determined by a 'tangent plane', whose direction is in turn defined by a normal vector at that point.

To illustrate this, place a point somewhere on the surface, and add a tangent plane and a normal unit vector.

This point has 2 degrees of freedom and can have its position varied by altering its 'x' and 'y' coordinates use the arrow keys.

It should therefore be possible to move the point to the top of a hill to a local maximum.



### 4. Integration: $0 \times \infty$ can $\Rightarrow 7/3$

Differentiation is concerned with zero divide by zero. Integration involves summing an infinite number of zero width strips to form an area.

So it is perfectly possible to find that the limit of a product, as one term tends to infinity and one tends to 0, can be finite. This remarkable result is obtained by integrating  $y = x^2$  from  $x = 1$  to  $x = 2$  from first principles. Let's explore this.

Plot  $y = x^2$ ; add points at  $x = 1$  and  $x = 2$  and find the area under the curve between these points using *Autograph's* default setting: 5 rectangles below the curve. Show the result: "Area = 2.04".

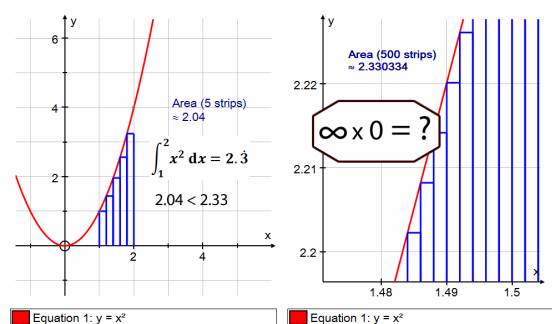
Using calculus, the area is in fact  $7/3$ , and of course 2.04 is clearly an under-estimate, as confirmed by the gaps above the strips.

What would happen to the estimate if you increased the number of strips from 5 to 6? .. from 6 to 500?

What now? I used to get this far in the old days using a spreadsheet. Dynamic software has a fabulous weapon up its sleeve –the zoom.

If you zoom repeatedly, centring on the curve itself, you can observe both local straightness (again!) and the re-emergence of the rectangles.

This process can (and must!) clearly go on forever. If you have an infinite number of zero-width rectangles, the area will become exactly  $7/3$ .



Hold it there! If, in this zoomed state, the 'curve' is now so straight, why don't we forget all this calculus stuff and use trapezia? That should get a perfect result! Good plan. Try it!

Double click on the area and select 'trapezium'. BEFORE you press 'OK', ask yourself will this be an over-estimate, and under-estimate or spot on?

## 5. What about volume elements?

First of all, what do 2D graphs look like in 3D? On a 3D page in *Autograph*, enter  $y = x(3 - x)$ . Wait! Think back to 2D functions with a variable missing, eg  $x = 3$ , which draws  $x = 3$  for all 'y'. Likewise we expect  $y = x(3 - x)$  to draw a parabolic tunnel, for all 'z'. Now press 'OK'!

Use the option to "Plot as 2D", zoom in a bit and enter point on the graph at  $x = a$ . ['a' will be a parameter, with initial value = 1]. Try to visualise where this point is before pressing 'OK'. Now enter a second point at  $x = a + 0.2$ .

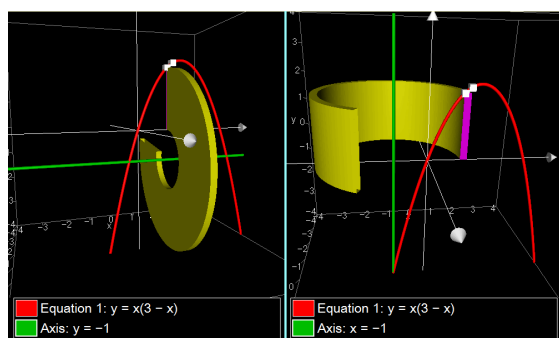
Select the two points, use 'find area', and set it to draw just a single rectangle. This will draw the classic rectangular 'element of area' between the curve and the x-axis.

Using the "Plot as 2D" feature again, draw the line  $y = -1$ . Make sure 'Slow Plot' is on; select the area and the axis of revolution, and use 'Find Volume'. BEFORE you press 'OK' ask yourself what shape to expect as your 'element of volume'?

After it has plotted, use the constant controller to vary 'a' from 0 to 3 to get a feel for all such elements of volume that will, in the limit, give you an exact answer.

Now delete the volume and double-click on the axis of revolution to edit it, and change it from  $y = -1$  to  $x = -1$ .

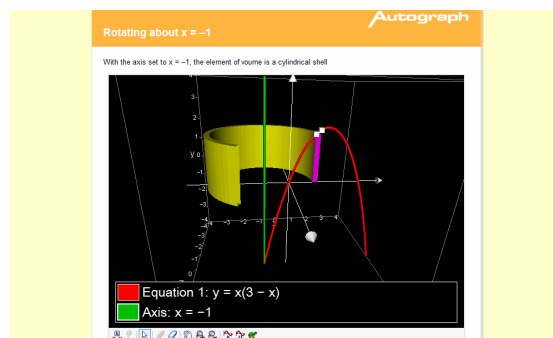
Select the old element of area and the new axis of revolution and 'Find Volume'. Again, BEFORE you press 'OK', what shape are you expecting for your 'element of volume'?



In each the above cases, using axes  $y = -1$  and  $x = -1$ , it is useful to vary the value of 'a' using the constant controller, to get a feel for the total area and total volume generated as 'a' moves from 0 to 3.

## 6. How to create all this on a web page?

Autograph has a useful feature to "Export to Web Page", so that any statistics, 2D or 3D page can be recreated in HTML. When a web page opens that includes an Autograph activity, it automatically invokes the 'Autograph player' appropriate to the browser you are using. This is installed the first time you need it, and the activity looks like this:



The page is 'live': objects can be dragged, and constants can be animated. Any object can be edited by double-clicking.







Alternatively you can paste the 'object code' into a web editor, for example on a Moodle system, so that the activity can be embedded. The only requirement here is that the path for the Autograph '.agg.' file is correctly stated, relative to the web page in which it is embedded.

To accompany this article, a number of **videos** and **online Autograph activities** have been created.

Just go to

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to find find the following:

-  Differentiation:  $0 \div 0$  can  $\Rightarrow 2$
-  Max, min and point of Inflection
-  A nice extension to 3D
-  Integration:  $0 \times \infty$  can  $\Rightarrow 7/3$
-  A look at are and volume elements
-  How to embed an activity in Moodle

These have been composed using a new method for generating Autograph Online Activities. This is being used extensively now by the publishing industry to bring mathematical texts to life.

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